04)

$2.(3+4)+5.6 = 2.(3+4)+30$

$= 2.3 + 2.4 + 5.6$

$= 6 + 2.4 + 5.6$

$= 6 + 8 + 5.6$

$= 6 + 8 + 30$

$= 14 + 30$

$= 44$

$= 2.7 + 30$

$= 14 + 30$

$= 44$

$$\begin{tikzcd}
2.(3+4)+5.6 \arrow[r] \arrow[d] & 2.(3+4)+30 \arrow[d] \\
2.7+5.6 \arrow[r] \arrow[d] & 2.7+30 \arrow[d] \\
14+5.6 \arrow[r] & 14+30 \arrow[r] & 44
\end{tikzcd}$$

$2.(3+4)+5.6$

$= 2.7 + 5.6$

$= 14 + 5.6$

$= 14 + 30$

$= 44$

$$\begin{tikzcd}
2.(3+4)+5.6 \arrow[d] & & & \\
2.3+2.4+5.6 \arrow[rr] \arrow[d] \arrow[dr] & & 2.3+2.4+30 \arrow[d] & \\
6+2.4+5.6 \arrow[dr] & 2.3+8+5.6 \arrow[r] \arrow[d] & 2.3+8+30 \arrow[d] & \\
& 6+8+5.6 \arrow[r] \arrow[d] & 6+8+30 \arrow[r] \arrow[d] & 6+38 \arrow[d] \\
& 14+5.6 \arrow[r] & 14+30 \arrow[r] & 44
\end{tikzcd}$$

a) $2\cdot(3+4)+5\cdot6 = 2\cdot(3+4)+30$

$= 2(7)+30$

$= 14+30$

$= 44$

$= 2\cdot(7)+5\cdot6$

$= 14+5\cdot6$

$= 14+30$

$= 44$

$$\underbrace{\underbrace{2\cdot\underbrace{(3+4)}_{7}}_{14}+\underbrace{5\cdot6}_{30}}_{44}$$

$= 6+2(4)+5\cdot6$

$= 6+8+5\cdot6$

$= 6+8+30$

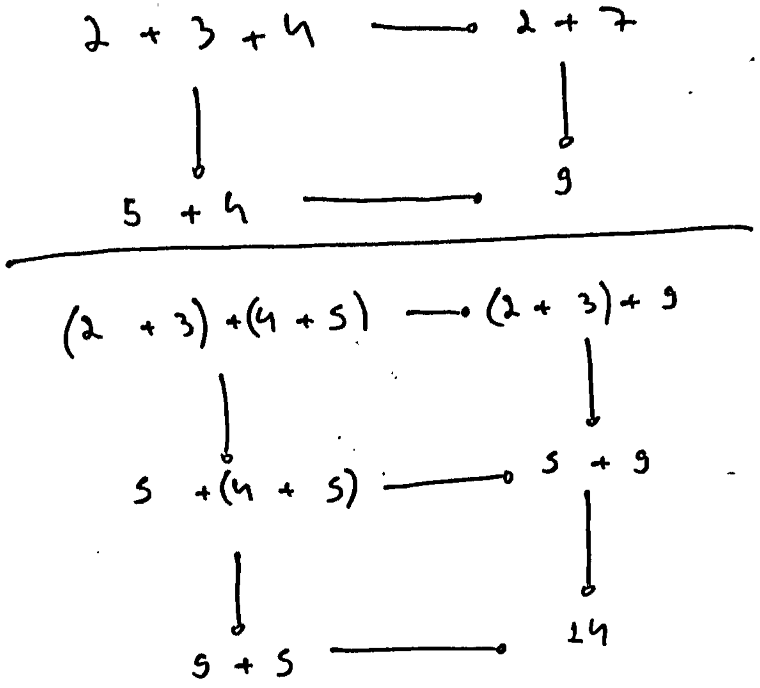
$= 14+30$

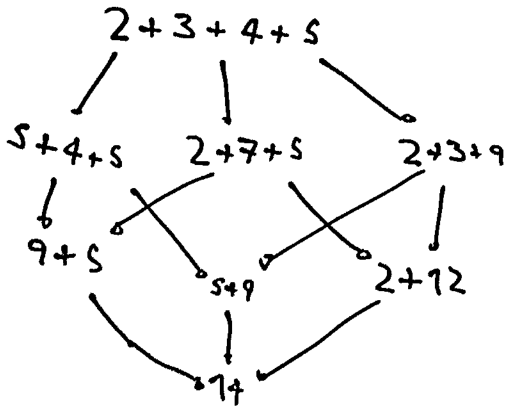
$= 44$

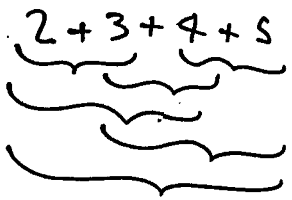
$\to D$

$\downarrow E$

$$\begin{tikzcd}
2\cdot(3+4)+5\cdot6 \arrow[r] \arrow[d] & 2\cdot(3+4)+30 \arrow[d] & \\
2(7)+5\cdot6 \arrow[r] & 2(7)+30 \arrow[d] & \\
& 14+30 \arrow[r] & 44
\end{tikzcd}$$







NOTAÇÃO PARA SUBSTITUIÇÃO (SIMULTÂNEA):

$$((\underbrace{x}_{y+1}+\underbrace{y}_{z\cdot 3})\cdot\underbrace{z}_{x-1})\begin{bmatrix}x:=y+1\\ y:=z\cdot 3\\ z:=x-1\end{bmatrix}$$

$\downarrow$

$((y+1)+(z\cdot 3))\cdot(x-1)$

$(\lambda x.\,expr)\,val$

$\downarrow$

$expr\,[x:=val]$

$\downarrow$

...

1.28:

a) $(\lambda x.xy)(\lambda u.vuu)$

$\downarrow$

$(vuu)y$

EXERCÍCIO 1.4:

OBS: $f\,a\,b\,c = ((f\,a)\,b)\,c$
$= ((f(a))(b))(c)$

1.4

a) $xyz(yx)$

$((x(y))z)(yx)$

$\underline{\underline{xy}z}(yx)$

1.4 b) $\lambda x.uxy$

HIPÓTESES:

$\lambda x.uxy = ((\lambda x.u)x)(y)$ (VER "NOTATION 1.3", p.4) NÃO, PORQUE $\lambda x.PQ = \lambda x.(PQ)$

$\lambda x.uxy = \lambda x.((ux)(y))$ SIM

$\lambda x.uxy = (\lambda x.(u(x)))(y)$ NÃO

$\lambda x.uxy = \lambda x.(u(x(y)))$ NÃO, PORQUE $MNP=(MN)P$

a) $A \times B = \left\{ \begin{matrix} (1,3), (2,3) \\ (1,4), (2,4) \end{matrix} \right\}$

d)

e) V

f) $p=(2,3)$ $\left(\underbrace{g(\underbrace{\pi \underbrace{p}_{(2,3)}}_{2})}_{30}, \underbrace{f(\underbrace{\pi' \underbrace{p}_{(2,3)}}_{3})}_{30} \right)$

$(30,30)$

g) $\left(\underbrace{g(\underbrace{\pi \underbrace{p}_{(1,3)}}_{1})}_{10}, \underbrace{f(\underbrace{\pi' \underbrace{p}_{(1,3)}}_{3})}_{30} \right)$

$(10,30)$ ✓

$\left(\underbrace{g(\underbrace{\pi p}_{2})}_{20}, \underbrace{f(\underbrace{\pi' p}_{3})}_{30} \right)$

$(20,30)$

$\underbrace{\left(\underbrace{g(\underbrace{\pi \underbrace{p}_{:A\times B}}_{:A})}_{:D}, \underbrace{f(\underbrace{\pi' \underbrace{p}_{:A\times B}}_{:B})}_{:C} \right)}_{:D\times C}$

b) $A \to D$ $\left\{ \left\{ \begin{matrix} (1,10) \\ (2,10) \end{matrix} \right\}, \left\{ \begin{matrix} (1,20) \\ (2,20) \end{matrix} \right\}, \left\{ \begin{matrix} (1,10) \\ (2,20) \end{matrix} \right\}, \left\{ \begin{matrix} (1,20) \\ (2,10) \end{matrix} \right\} \right\}$

c) $\underbrace{\left(\underbrace{\pi \underbrace{p}_{(1,3)}}_{1}, \underbrace{f(\underbrace{\pi' \underbrace{p}_{(1,3)}}_{3})}_{30} \right)}_{(1,30)}$ | $\underbrace{\left(\underbrace{\pi \underbrace{p}_{(2,3)}}_{2}, \underbrace{f(\underbrace{\pi' \underbrace{p}_{(2,3)}}_{3})}_{30} \right)}_{(2,30)}$ | $\underbrace{\left(\underbrace{\pi \underbrace{p}_{(1,4)}}_{1}, \underbrace{f(\underbrace{\pi' \underbrace{p}_{(1,4)}}_{4})}_{40} \right)}_{(1,40)}$ | $\underbrace{\left(\underbrace{\pi \underbrace{p}_{(2,4)}}_{2}, \underbrace{f(\underbrace{\pi' \underbrace{p}_{(2,4)}}_{4})}_{40} \right)}_{(2,40)}$

$D \times C = \left\{ \begin{matrix} (10,30), (20,30) \\ (10,40), (20,40) \end{matrix} \right\}$

a) $A \times B = \left\{ \begin{matrix} (1,3), (1,4) \\ (2,3), (2,4) \end{matrix} \right\}$

$A=(1,2)$

$B=(3,4)$

b) $A \to D = \left\{ \left\{ \begin{matrix} (1,10) \\ (2,10) \end{matrix} \right\}, \left\{ \begin{matrix} (1,10) \\ (2,20) \end{matrix} \right\}, \left\{ \begin{matrix} (1,20) \\ (2,10) \end{matrix} \right\}, \left\{ \begin{matrix} (1,20) \\ (2,20) \end{matrix} \right\} \right\}$

$D=(10,20)$

c) Para $p:(2,3)$

$\underbrace{\left(\underbrace{\pi p}_{2}, \underbrace{f(\underbrace{\pi' p}_{3})}_{30} \right)}_{(2,30)}$

$P: A \times B$

Para $p:(1,3)$

$\underbrace{\left(\underbrace{\pi p}_{1}, \underbrace{f(\underbrace{\pi' p}_{3})}_{30} \right)}_{(1,30)}$

Para $p:(1,4)$

$\underbrace{\left(\underbrace{\pi p}_{1}, \underbrace{f(\underbrace{\pi' p}_{4})}_{40} \right)}_{(1,40)}$

$f: \left\{ \begin{matrix} (3,30) \\ (4,40) \end{matrix} \right\}$

Para $p:(2,4)$

$\underbrace{\left(\underbrace{\pi p}_{2}, \underbrace{f(\underbrace{\pi' p}_{4})}_{40} \right)}_{(2,40)}$

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a) $$\underbrace{(\underbrace{\lambda p}_{:A\times C} : A\times C\ .\ \underbrace{(\underbrace{\underbrace{f}_{:A\to B}(\underbrace{\pi \underbrace{p}_{:A\times C}}_{:A})}_{:B},\ \underbrace{\pi' \underbrace{p}_{:A\times C}}_{:C})}_{B\times C})}_{A\times C\to B\times C}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{p:A\times C}{\pi p:A}\quad f:A\to B}{f(\pi p):B}\qquad \dfrac{p:A\times C}{\pi' p:C}}{(f(\pi p),\pi' p)\ :B\times C}}{(\lambda p:A\times C\ .\ (f(\pi p),\pi' p))\ :A\times C\to B\times C}$$

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b) $$\underbrace{\lambda \underbrace{q}_{:B\times C} : B\times C \,.\, \underbrace{\underbrace{\big(\underbrace{h}_{:B\to(C\to D)}\underbrace{(\pi \underbrace{q}_{:B\times C})}_{:B}\big)}_{:C\to D}\underbrace{(\pi' \underbrace{q}_{:B\times C})}_{:C}}_{:D}}_{B\times C\to D}$$

$$\dfrac{\dfrac{\dfrac{q : B\times C}{(\pi q) : B} \quad h : B\to(C\to D)}{(h(\pi q)) : C\to D} \quad \dfrac{q : B\times C}{(\pi' q) : C}}{\dfrac{(h(\pi q))(\pi' q) : D}{\lambda q : B\times C \,.\, (h(\pi q))(\pi' q) : B\times C\to D}}$$

c) $$\underbrace{\lambda \underbrace{b}_{:B} : B \,.\, \underbrace{\lambda \underbrace{c}_{:C} : C \,.\, \underbrace{\underbrace{g}_{B\times C\to D}\underbrace{(\underbrace{b}_{:B},\underbrace{c}_{:C})}_{:B\times C}}_{:D}}_{:C\to D}}_{:B\to(C\to D)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{b : B \quad c : C}{g : B\times C\to D \quad (b,c) : B\times C}}{g(b,c) : D}}{\lambda c : C \,.\, g(b,c) : C\to D}}{\lambda b : B \,.\, \lambda c : C \,.\, g(b,c) : B\to(C\to D)}$$

d) $$\underbrace{\lambda \underbrace{\varphi}_{:C\to D} : C\to D \,.\, \underbrace{\lambda \underbrace{c}_{:C} : C \,.\, \underbrace{\underbrace{k}_{D\to E}(\underbrace{\underbrace{\varphi}_{:C\to D}\underbrace{c}_{:C}}_{:D})}_{:E}}_{:C\to E}}_{(C\to D)\to(C\to E)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{c : C \quad \varphi : C\to D}{\varphi c : D} \quad k : D\to E}{k(\varphi c) : E}}{\lambda c : C \,.\, k(\varphi c) : C\to E}}{\lambda \varphi : C\to D \,.\, \lambda c : C \,.\, k(\varphi c) : (C\to D)\to(C\to E)}$$

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Árvores em $ND_{\to}$ são chamadas de "deduções" (ou "derivações").

Toda vez que temos uma introdução seguida de uma eliminação nós podemos "reduzir" a dedução.

$$(A)\quad \dfrac{\dfrac{\dfrac{\dfrac{P \quad P\to Q}{Q}{\to}E \qquad \dfrac{P \quad P\to R}{R}{\to}E}{Q\&R}\&I}{Q}\&E_1 \qquad Q\to S}{S}{\to}E \quad \rightsquigarrow \quad \dfrac{\dfrac{P \quad P\to Q}{Q}{\to}E \qquad Q\to S}{S}{\to}E \quad (A')$$

$$(B)\quad \dfrac{\dfrac{\dfrac{\dfrac{P \quad P\to Q}{Q}{\to}E \qquad \dfrac{P \quad P\to R}{R}{\to}E}{Q\&R}}{R}\&E_2 \qquad R\to S}{S}{\to}E \quad \rightsquigarrow \quad \dfrac{\dfrac{P \quad P\to R}{R}{\to}E \qquad R\to S}{S}{\to}E \quad (B')$$

Exercício 2: Reescreva as árvores Ⓐ, Ⓐ', Ⓑ, Ⓑ' incluindo "$\vdash$"s.

$$\dfrac{P\vdash P \quad P\to Q\vdash P\to Q}{P, P\to Q\vdash Q}$$

Exercícios:

1) Escreva a árvore Ⓑ'
2) Reescreva elas como árvores de tipos em λ-cálculo. Ex:

$$(A'')\quad \dfrac{\dfrac{\dfrac{\dfrac{a{:}A \quad f{:}A\to B}{f(a):B}\text{app} \qquad \dfrac{a{:}A \quad g{:}A\to C}{g(a):C}\text{app}}{(f(a),g(a)):B\times C}\text{pair}}{\pi(f(a),g(a)):B}\pi \qquad h{:}B\to D}{h(\pi(f(a),g(a))):D}\text{app}$$

$$\Uparrow$$

$$h(\underbrace{\pi(fa,ga)}_{fa}):D \quad \rightsquigarrow \quad h(fa):D$$

$$(A''')\quad \dfrac{\dfrac{a{:}A \quad f{:}A\to B}{f(a):B} \qquad h{:}B\to D}{h(f(a)):D}$$

$$\Uparrow$$

$$h(fa):D$$

$$(B'')\quad \dfrac{\dfrac{\dfrac{\dfrac{a{:}A \quad f{:}A\to B}{f(a):B}\text{APP} \qquad \dfrac{a{:}A \quad g{:}A\to C}{g(a):C}\text{APP}}{(f(a),g(a)):B\times C}\text{PAIR}}{\pi'(f(a),g(a)):C}\pi' \qquad h{:}C\to D}{h(\pi'(f(a),g(a))):D}\text{APP}$$

$$h(\underbrace{\pi'(fa,ga)}_{ga}):D \quad \rightsquigarrow \quad h(ga):D$$

$$(B''')\quad \dfrac{\dfrac{a{:}A \quad g{:}A\to C}{ga:C} \qquad h{:}C\to D}{h(g(a)):D}$$

$$h(ga):D$$

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ÁRVORES EM $ND_{\to}$ SÃO CHAMADAS DE "DEDUÇÕES" (OU "DERIVAÇÕES").

TODA VEZ QUE TEMOS UMA INTRODUÇÃO SEGUIDA DE UMA ELIMINAÇÃO NÓS PODEMOS "REDUZIR" A DEDUÇÃO.

(A)
$$\dfrac{\dfrac{\dfrac{\dfrac{P \quad P\to Q}{Q}{\scriptstyle \to E} \quad \dfrac{P \quad P\to R}{R}{\scriptstyle \to E}}{Q\&R}{\scriptstyle \&I}}{Q}{\scriptstyle \&E_1} \quad Q\to S}{S}{\scriptstyle \to E}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{P \quad P\to Q}{Q}{\scriptstyle \to E} \quad Q\to S}{S}{\scriptstyle \to E} \ \text{(A')}$$

(B)
$$\dfrac{\dfrac{\dfrac{\dfrac{P \quad P\to Q}{Q}{\scriptstyle \to E} \quad \dfrac{P \quad P\to R}{R}{\scriptstyle \to E}}{Q\&R}{\scriptstyle \&I}}{R}{\scriptstyle \&E_2} \quad R\to S}{S}{\scriptstyle \to E}
\quad \rightsquigarrow \quad
\dfrac{\dfrac{P \quad P\to R}{R}{\scriptstyle \to E} \quad R\to S}{S}{\scriptstyle \to E} \ \text{(B')}$$

EXERCÍCIO 2:
REESCREVA AS ÁRVORES (A), (A'), (B), (B') INCLUINDO "$\vdash$"s.

$$\dfrac{P\vdash P \quad P\to Q\vdash P\to Q}{P, P\to Q\vdash Q}$$

EXERCÍCIOS:
1) ESCREVA A ÁRVORE (B')
2) REESCREVA ELAS COMO ÁRVORES DE TIPOS EM λ-CÁLCULO. EX:

(A'')
$$\dfrac{\dfrac{\dfrac{\dfrac{a:A \quad f:A\to B}{f(a):B}{\scriptstyle app} \quad \dfrac{a:A \quad g:A\to C}{g(a):C}{\scriptstyle app}}{(f(a),g(a)):B\times C}{\scriptstyle pair}}{\pi(f(a),g(a)):B}{\scriptstyle \pi} \quad h:B\to D}{h(\pi(f(a),g(a))):D}{\scriptstyle app}$$

$$\Uparrow$$

$$h(\pi(\underbrace{fa,ga}_{fa})):D \rightsquigarrow h(fa):D$$

(B')
$$\dfrac{\dfrac{\dfrac{\dfrac{a:A \quad f:A\to B}{f(a):B}{\scriptstyle APP} \quad \dfrac{a:A \quad g:A\to C}{g(a):C}{\scriptstyle APP}}{(f(a),g(a)):B\times C}{\scriptstyle PAIR}}{\pi'(f(a),g(a)):C}{\scriptstyle \pi'} \quad h:C\to D}{h(\pi'(f(a),g(a))):D}{\scriptstyle APP}$$

$$h(\pi'(\underbrace{fa,ga}_{ga})):D \rightsquigarrow$$

(A)
$$\dfrac{\dfrac{\dfrac{P\vdash P \quad P\to Q\vdash P\to Q}{P,P\to Q\vdash Q} \quad \dfrac{P\vdash P \quad P\to R\vdash P\to R}{P,P\to R\vdash R}}{P,P\to Q,P\to R\vdash Q\&R}}{P,P\to Q,P\to R\vdash Q} \quad Q\to S\vdash Q\to S$$
$$\overline{P,P\to Q,P\to R,Q\to S\vdash S}$$

(A')
$$\dfrac{\dfrac{P\vdash P \quad P\to Q\vdash P\to Q}{P,P\to Q\vdash Q} \quad Q\to S\vdash Q\to S}{P,P\to Q,Q\to S\vdash S}$$

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Árvores em $ND_{\to}$ são chamadas de "deduções" (ou "derivações").

Toda vez que temos uma introdução seguida de uma eliminação nós podemos "reduzir" a dedução.

(A)
$$\dfrac{\dfrac{\dfrac{\dfrac{P \quad P\to Q}{Q}{\to E} \quad \dfrac{P \quad P\to R}{R}{\to E}}{Q\&R}{\&I}}{Q}{\&E_1} \quad Q\to S}{S}{\to E} \quad \rightsquigarrow \quad \dfrac{\dfrac{P \quad P\to Q}{Q}{\to E} \quad Q\to S}{S}{\to E} \quad (A')$$

(B)
$$\dfrac{\dfrac{\dfrac{\dfrac{P \quad P\to Q}{Q}{\to E} \quad \dfrac{P \quad P\to R}{R}{\to E}}{Q\&R}{\&I}}{R}{\&E_2} \quad R\to S}{S}{\to E} \quad \rightsquigarrow \quad \dfrac{\dfrac{P \quad P\to R}{R}{\to E} \quad R\to S}{S}{\to E} \quad (B')$$

Exercício 2:
Reescreva as árvores (A), (A'), (B), (B') incluindo "⊢"s.

$$\dfrac{P\vdash P \quad P\to Q\vdash P\to Q}{P, P\to Q\vdash Q}$$

Exercícios:
1) Escreva a árvore (B')
2) Reescreva elas como árvores de tipos em λ-cálculo. Ex:

(A'')
$$\dfrac{\dfrac{\dfrac{\dfrac{a{:}A \quad f{:}A\to B}{f(a):B}{app} \quad \dfrac{a{:}A \quad g{:}A\to C}{g(a):C}{app}}{(f(a),g(a)):B\times C}{pair}}{\pi(f(a),g(a)):B}{\pi} \quad h{:}B\to D}{h(\pi(f(a),g(a))):D}{app}$$

$$h(\pi(\underbrace{fa,ga}_{fa})):D \rightsquigarrow h(fa):D$$

(B'')
$$\dfrac{\dfrac{\dfrac{\dfrac{a{:}A \quad f{:}A\to B}{f(a):B}{APP} \quad \dfrac{a{:}A \quad g{:}A\to C}{g(a):C}{APP}}{(f(a),g(a)):B\times C}{PAIR}}{\pi'(f(a),g(a)):C}{\pi'} \quad h{:}C\to D}{h(\pi'(f(a),g(a))):D}{APP}$$

$$h(\underbrace{\pi'(fa,ga)}_{ga}):D \rightsquigarrow$$

(A)
$$\dfrac{\dfrac{\dfrac{\dfrac{P\vdash P \quad P\to Q\vdash P\to Q}{P,P\to Q\vdash Q} \quad \dfrac{P\vdash P \quad P\to R\vdash P\to R}{P,P\to R\vdash R}}{P,P\to Q,P\to R\vdash Q\&R}}{P,P\to Q,P\to R\vdash Q} \quad Q\to S\vdash Q\to S}{P,P\to Q,P\to R,Q\to S\vdash S}$$

(A')
$$\dfrac{\dfrac{P\vdash P \quad P\to Q\vdash P\to Q}{P,P\to Q\vdash Q} \quad Q\to S\vdash Q\to S}{P,P\to Q,Q\to S\vdash S}$$

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SLOGAN:

QUASE TODAS AS OPERAÇÕES QUE A GENTE VAI PRECISAR CORRESPONDEM A DERIVAÇÕES EM DEDUÇÃO NATURAL "TRADUZIDAS"!

$$\dfrac{\dfrac{\dfrac{[p:A\times B]}{:A} \qquad \dfrac{\dfrac{[p:A\times B]^1}{:B} \quad f:B\to C}{:C}}{:A\times C}}{:A\times B\to A\times C}\,1$$

$$\dfrac{\dfrac{\dfrac{[P\&Q]^1}{P}\&E_1 \qquad \dfrac{\dfrac{[P\&Q]^1}{Q}\&E_2 \quad Q\to R}{R}\to E}{P\&R}\&I}{P\&Q\to P\&R}\to I;1 \qquad \dfrac{P\&Q \quad Q\to R}{P\&R}$$

"$\vdash$":

$$\dfrac{Q\to R,\ P\&Q\vdash P\&R}{Q\to R\vdash P\&Q\to P\&R}$$

A FIGURA DA P.8 VIRA ISSO AQUI, SE A GENTE TRADUZ ELA PRA NOTAÇÃO LÓGICA:

$$\begin{tikzcd}
P\&R \arrow[d] & P \arrow[l, mapsto] \arrow[d] \\
Q\&R \arrow[d] & Q \arrow[l, mapsto] \arrow[d] \\
S \arrow[d] \arrow[r, mapsto] & R\to S \arrow[d] \\
T \arrow[r, mapsto] & R\to T
\end{tikzcd}$$

$$\dfrac{P\to Q}{P\&R\to Q\&R}=?_a \qquad \dfrac{Q\to(R\to S)}{Q\&R\to S}=?_b \qquad \dfrac{Q\&R\to S}{Q\to(R\to S)}=?_c \qquad \dfrac{S\to T}{(R\to S)\to(R\to T)}=?_d$$

a) $$\dfrac{\dfrac{\dfrac{\dfrac{[P\&R]^1}{P}\&E_1 \quad P\to Q}{Q}\to E \qquad \dfrac{[P\&R]^1}{R}\&E_2}{Q\&R}\&I}{P\&R\to Q\&R}\to I;1$$

b) $$\dfrac{\dfrac{\dfrac{[Q\&R]^1}{R}\&E_2 \qquad \dfrac{\dfrac{[Q\&R]^1}{Q}\&E_1 \quad Q\to(R\to S)}{R\to S}\to E}{S}\to E}{Q\&R\to S}\to I;1$$

c) $$\dfrac{\dfrac{\dfrac{\dfrac{[Q]^1\ [R]^2}{Q\&R}\&I \quad Q\&R\to S}{S}\to E}{R\to S}\to I;2}{Q\to(R\to S)}\to I;1$$

d) $$\dfrac{\dfrac{\dfrac{\dfrac{[R]^1\ [R\to S]^2}{S}\to E \quad S\to T}{T}\to E}{R\to T}\to I;1}{(R\to S)\to(R\to T)}\to I;2$$

TRUQUES:

① UMA "DERIVAÇÃO NORMAL" NÃO TEM &I SEGUIDO DE &E E NEM →I SEGUIDO DE →E

② PRINCÍPIO DA SUBFÓRMULA: NUMA "DERIVAÇÃO NORMAL" DE

$$\dfrac{P\&Q \quad Q\to R}{P\&R}$$

AS "FÓRMULAS" QUE APARECEM NOS NÓS DA ÁRVORE SÃO SÓ ESTAS AQUI:

$$\underbrace{P\&Q}_{P\quad Q} \qquad \underbrace{Q\to R}_{Q\quad R} \qquad \underbrace{P\&Q}_{P\quad Q}$$

E NESTA:

$$\dfrac{Q\to R}{P\&Q\to P\&R} \qquad \underbrace{Q\to R}_{Q\quad R} \qquad \underbrace{\underbrace{P\&Q}_{P\quad Q}\to \underbrace{P\&R}_{P\quad R}}$$

OUTRO DIAGRAMA:

$$\begin{tikzcd}
 & P \arrow[dl] \arrow[d] \arrow[dr] & \\
Q & Q\&R \arrow[l] \arrow[r] & R
\end{tikzcd} \Rightarrow \begin{tikzcd} P \arrow[d] \\ P\&Q \end{tikzcd} \rightleftarrows \left(\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ Q & & R \end{tikzcd}\right)$$

$$\Rightarrow\ Q \leftarrow Q\&R \to R$$

$$\dfrac{P\to Q\&R}{P\to Q}=?_e \qquad \dfrac{P\to Q \quad P\to R}{P\to Q\&R}=?_g$$

$$\dfrac{P\to Q\&R}{P\to R}=?_f \qquad \dfrac{}{Q\&R\to Q}=?_h \qquad \dfrac{}{Q\&R\to R}=?_i$$

REGRAS DE DN QUE NÓS VIMOS ATÉ AGORA:

$$\dfrac{P \quad Q}{P\&Q}\&I \qquad \dfrac{P\&Q}{P}\&E_1 \qquad \dfrac{P\&Q}{Q}\&E_2$$

$$\dfrac{\begin{matrix}P\ [Q]^1\\ \vdots\\ R\end{matrix}}{Q\to R}\to I;1 \qquad \dfrac{Q \quad Q\to R}{R}\to E$$

REGRAS NOVAS:

$$\dfrac{P}{P\vee Q}\vee I_1 \qquad \dfrac{Q}{P\vee Q}\vee I_2 \qquad \dfrac{Q\vee R \quad \begin{matrix}P\ [Q]^1\\ \vdots\\ S\end{matrix} \quad \begin{matrix}P\ [R]^1\\ \vdots\\ S\end{matrix}}{S}\vee E;1 \quad\Rightarrow\quad \dfrac{P,Q\vdash S \quad P,R\vdash S}{P,Q\vee R\vdash S}$$

$$\dfrac{}{\top}\top I \qquad \dfrac{\bot}{P}\bot E$$

E O "NÃO"?

$\neg P := P\to\bot$

$\neg\neg P = (P\to\bot)\to\bot$

$\underbrace{\neg\underbrace{\neg P}_{P\to\bot}}_{(P\to\bot)\to\bot}$

FATO: EM DN DÁ PRA PROVAR $P\to\neg\neg P$ MAS NÃO $\neg\neg P\to P$... PORQUÊ

$$\dfrac{P}{\neg\neg P} \Rightarrow \dfrac{P}{(P\to\bot)\to\bot} \Rightarrow \dfrac{\dfrac{P \quad [P\to\bot]^1}{\bot}\to E}{(P\to\bot)\to\bot}\to I;1$$

QUANDO A GENTE "TRADUZIR" ISSO PRA λ-CÁLCULO VAMOS TER:

$P\vee Q \Rightarrow A+B$ ("+" É UNIÃO DISJUNTA!)

$\top \Rightarrow$ UM CONJUNTO COM UM ELEMENTO

$\bot \Rightarrow$ O CONJUNTO VAZIO.

Lambda

A named function: $g(a) = a \cdot a + 4$
An unnamed function: $\lambda a.\, a \cdot a + 4$
Let $h = \lambda a.\, a \cdot a + 4$.
Then:

$$\begin{tikzcd}
g(2+3) \arrow[rr] \arrow[d] & & g(5) \arrow[ddd] \\
(2+3)\cdot(2+3)+4 \arrow[dr] \arrow[dd] & & \\
& (2+3)\cdot 5+4 \arrow[dr] & \\
5\cdot(2+3)+4 \arrow[rr] & & 5\cdot 5+4 \arrow[d] \\
& & 25+4 \arrow[d] \\
& & 29
\end{tikzcd}
\qquad
\begin{tikzcd}
h(2+3) \arrow[rr] \arrow[d] & & h(5) \arrow[d] & \\
(\lambda a.\, a\cdot a+4)(2+3) \arrow[rr] \arrow[d] & & (\lambda a.\, a\cdot a+4)(5) \arrow[ddd] \arrow[dr] & \\
(2+3)\cdot(2+3)+4 \arrow[dr] \arrow[dd] & & & (a\cdot a+4)[a:=5] \arrow[ddl] \\
& (2+3)\cdot 5+4 \arrow[dr] & & \\
5\cdot(2+3)+4 \arrow[rr] & & 5\cdot 5+4 \arrow[d] & \\
& & 25+4 \arrow[d] & \\
& & 29 &
\end{tikzcd}$$

The usual notation for defining functions is like this:

$$\begin{array}{rccc} f: & \mathbb{N} & \to & \mathbb{R} \\ & n & \mapsto & 2+\sqrt{n} \end{array}$$

$$\begin{array}{rccc} (\text{name}): & (\text{domain}) & \to & (\text{codomain}) \\ & (\text{variable}) & \mapsto & (\text{expression}) \end{array}$$

It creates *named* functions
(with domains and codomains).

The usual notation for creating named functions
without specifying their domains and codomains
is just $f(n) = 2 + \sqrt{n}$.
Note that this is:

$$\begin{array}{cccc} f & (n) & = & 2+\sqrt{n} \\ (\text{name}) & ((\text{variable})) & = & (\text{expression}) \end{array}$$

LA 25/ABRIL/2017

NO FINAL DA AULA EU CONTEI PRA VOCÊS QUE A LÓGICA QUE NÓS ESTAMOS VENDO PROVA

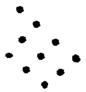
$P \to \neg\neg P$

MAS NÃO

$\neg\neg P \to P$...

HOJE NÓS VAMOS COMEÇAR A VER UMA FERRAMENTA PRA ENTENDER ESSA LÓGICA NOVA

UM "MODELO"

$B =$  ⇐ A NOSSA ZHA PREFERIDA (POR ENQUANTO)

$$= \left\{\begin{array}{ccccc} & (-1,5) & & & \\ & & (0,4) & & \\ & (-1,3) & & (1,3) & \\ (-2,2) & & (0,2) & & (2,2) \\ & (-1,1) & & (1,1) & \\ & & (0,0) & & \end{array}\right\}$$

$$= \left\{\begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array}\right\}$$

EXERCÍCIOS:

$\langle 2,0\rangle = (-2,2)$

$\langle 2,1\rangle = (-1,3)$

$\langle 3,3\rangle = (0,6)$

$\langle 0,3\rangle = (3,3)$

$\langle 1,1\rangle = (0,2)$

$\langle 2,2\rangle = (0,4)$

$\langle 2,0\rangle = (-2,2)$

EXERCÍCIOS:
REPRESENTEM, USANDO A NOTAÇÃO POSICIONAL DA P.1:

a) $\lambda lr{:}B.\ l$

b) $\lambda lr{:}B.\ r$

c) $\lambda lr{:}B.\ (l \leq 1)$

d) $\lambda lr{:}B.\ (r \geq 1)$

e) $\lambda lr{:}B.\ lr \leq 11$

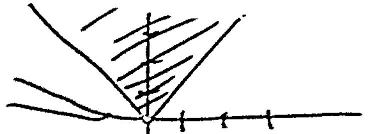
f) $\lambda lr{:}B.\ lr \,\&\, 12$

g) $\lambda lr{:}B.\ \mathrm{valid}(\langle l+1, r\rangle)$

h) $\lambda lr{:}B.\ lr \text{ leftof } 11$

i) $\lambda lr{:}B.\ lr \text{ leftof } 12$

j) $\lambda lr{:}B.\ lr \text{ above } 11$



$$lr = \langle l,r\rangle = (0,0) + l\,(-1,1) + r\,(1,1)$$

"angle brackets"

a) $\left\{\begin{array}{ccccc} & 3 & & & \\ & & 2 & & \\ & 2 & & 1 & \\ 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array}\right\}$

b) $\left\{\begin{array}{ccccc} & 2 & & & \\ & & 2 & & \\ & 1 & & 2 & \\ 0 & & 1 & & 2 \\ & 0 & & 1 & \\ & & 0 & & \end{array}\right\}$

c) $\left\{\begin{array}{ccccc} & 0 & & & \\ & & 0 & & \\ & 0 & & 1 & \\ 0 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{array}\right\}$

d) $\left\{\begin{array}{ccccc} & 1 & & & \\ & & 1 & & \\ & 1 & & 1 & \\ 0 & & 1 & & 1 \\ & 0 & & 1 & \\ & & 0 & & \end{array}\right\}$

e) $\left\{\begin{array}{ccccc} & 0 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}\right\}$

f) $\left\{\begin{array}{ccccc} & 12 & & & \\ & & 12 & & \\ & 11 & & 12 & \\ 10 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array}\right\}$

g) $\left\{\begin{array}{ccccc} & 0 & & & \\ & & 1 & & \\ & 0 & & 1 & \\ 0 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{array}\right\}$

h) $\left\{\begin{array}{ccccc} & 0 & & & \\ & & 0 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array}\right\}$

i) $\left\{\begin{array}{ccccc} & 1 & & & \\ & & 1 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array}\right\}$

j) $\left\{\begin{array}{ccccc} & 1 & & & \\ & & 1 & & \\ & 1 & & 1 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array}\right\}$

LA 25/ABRIL/2017

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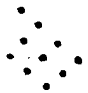
MAS NÃO

$\neg\neg P \to P$ ...

HOJE NÓS VAMOS COMEÇAR A VER UMA FERRAMENTA PRA ENTENDER ESSA LÓGICA NOVA

UM "MODELO"

B =  ⇐ A NOSSA ZHA PREFERIDA (POR ENQUANTO)

$= \{(-1,5), (0,4), (-1,3), (1,3), (-2,2), (0,2), (2,2), (-1,1), (1,1), (0,0)\}$

$= \{32, 22, 21, 12, 20, 11, 02, 10, 01, 00\}$

EXERCÍCIOS:

$\langle 2,0\rangle = (-2,2)$

$\langle 2,1\rangle = (-1,3)$

$\langle 3,3\rangle = (0,6)$

$\langle 0,3\rangle = (3,3)$

$\langle 1,1\rangle = (0,2)$

$\langle 2,2\rangle = (0,4)$

$\langle 2,0\rangle = (-2,2)$

MAIS EXERCÍCIOS:

k) $\lambda \ell r{:}B.\ ne(\ell r)$

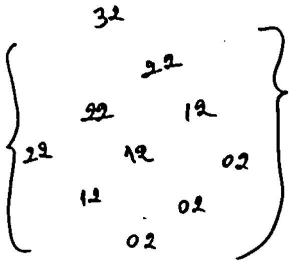
l) $\lambda \ell r{:}B.\ nw(\ell r)$

m) $20 \to 11 = 12$

n) $02 \to 11 = 21$

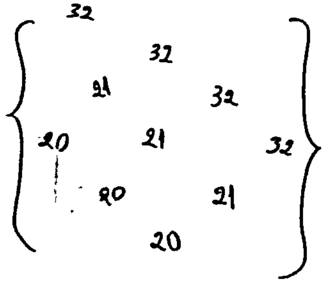
o) $22 \to 11 = 11$

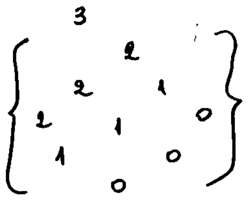
p) $00 \to 11 = 32$

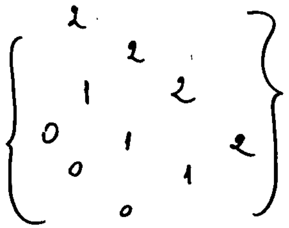
k) 

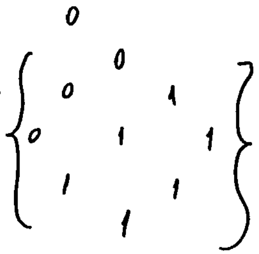
Se a=0 e b=0,

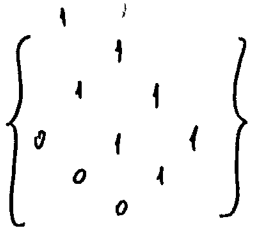
```
ne(00) := if valid(01)
          then ne(01)
          else 00
          end
```

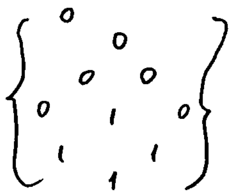
l) 

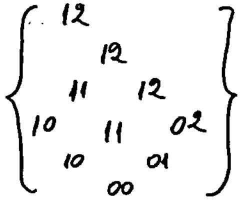
a) 

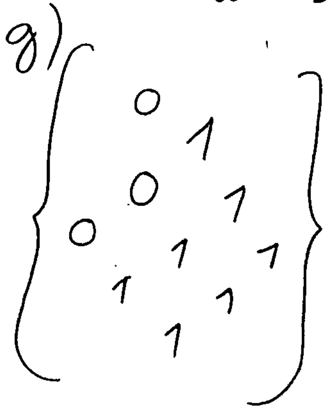
b) 

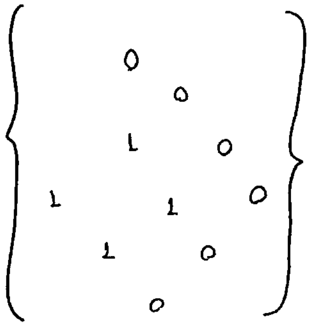
c) 

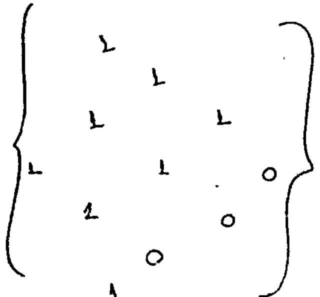
d) 

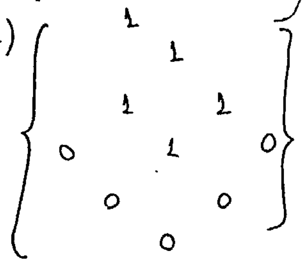
e) 

f) 

g) 

h) 

i) 

j) 

LA 25/ABRIL/2017

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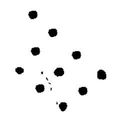
$P \to \neg\neg P$
MAS NÃO
$\neg\neg P \to P$...
HOJE NÓS VAMOS COMEÇAR
A VER UMA FERRAMENTA
PRA ENTENDER ESSA
LÓGICA NOVA

UM "MODELO"

B =  ⇐ A NOSSA ZHA PREFERIDA (POR ENQUANTO)

$$= \left\{ \begin{array}{ccccccc} & (-1,5) & & & & \\ & & (0,4) & & & \\ & (-1,3) & & (1,3) & & \\ (-2,2) & & (0,2) & & (2,2) \\ & (-1,1) & & (1,1) & \\ & & (0,0) & & \end{array} \right\}$$

$$= \left\{ \begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array} \right\}$$

EXERCÍCIOS:

$\langle 2,0\rangle = (-2,2)$
$\langle 2,1\rangle = (-1,3)$
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$\langle 0,3\rangle = (3,3)$
$\langle 1,1\rangle = (0,2)$
$\langle 2,2\rangle = (0,4)$
$\langle 2,0\rangle = (-2,2)$

MAIS EXERCÍCIOS:

k) $\lambda lr{:}B.\ \mathrm{ne}(lr)$
l) $\lambda lr{:}B.\ \mathrm{nw}(lr)$
m) $20 \to 11 = 12$
n) $02 \to 11 = 21$
o) $22 \to 11 = 11$
p) $00 \to 11 = 32$
q) $\lambda lr{:}B.\ \underbrace{\neg lr}_{lr \to 00}$
r) $\lambda lr{:}B.\ \neg(\neg lr)$
s) $\lambda lr{:}B.\ lr = \neg\neg lr$

q)
$$\left\{ \begin{array}{ccccc} & 00 & & & \\ & & 00 & & \\ & 00 & & 00 & \\ 02 & & 00 & & 20 \\ & 02 & & 20 & \\ & & 32 & & \end{array} \right\}$$

r)
$$\left\{ \begin{array}{ccccc} & 32 & & & \\ & & 32 & & \\ & 32 & & 32 & \\ 20 & & 32 & & 02 \\ & 20 & & 02 & \\ & & 00 & & \end{array} \right\}$$

s)
$$\left\{ \begin{array}{ccccc} & 1 & & & \\ & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \right\}$$

SE $P = 10$ E $Q = 01$,

$$\underbrace{\underbrace{\neg(\underbrace{\underbrace{P}_{10} \,\&\, \underbrace{Q}_{01}}_{00})}_{32} \to \underbrace{(\underbrace{\neg \underbrace{P}_{10}}_{02} \vee \underbrace{\neg \underbrace{Q}_{01}}_{20})}_{22}}_{22}$$

LA 25/ABRIL/2017

$$B = \begin{array}{ccccccc} & & 32 & & & & \\ & & & 22 & & & \\ & & 21 & & 12 & & \\ & 20 & & 11 & & 02 & \\ & & 10 & & 01 & & \\ & & & 00 & & & \end{array}$$

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MAS NÃO

$\neg\neg P \to P$ ...

HOJE NÓS VAMOS COMEÇAR
A VER UMA FERRAMENTA
PRA ENTENDER ESSA
LÓGICA NOVA

UM "MODELO"

DE ONDE VÊM ESSAS
DEFINIÇÕES PARA $ab \,\&\, cd$,
$ab \vee cd$,
$ab \to cd$?

LEMBREM QUE:

$$\begin{tikzcd} A \arrow[d] \\ 1 \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\ B & B \times C \arrow[l, "\pi"] \arrow[r, "\pi'"'] & C \end{tikzcd} \qquad \begin{tikzcd} A \times B \arrow[d, "h^\flat"'] \\ C \end{tikzcd} \underset{\mapsto}{\leftrightarrow} \begin{tikzcd} A \arrow[d, "h"] \\ B \to C \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & A+B \arrow[d] & B \arrow[l] \arrow[dl] \\ & C & \end{tikzcd}$$

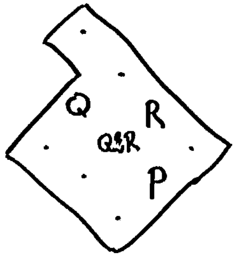
$\{1,2,3\} \to \{4,5,6\}$ TEM 27 ELEMENTOS
$\{1,2,3\} \to \{4\}$ TEM 1 ELEMENTO
$\{1,2,3\} \to \{\}$ TEM 0 ELEMENTOS

$$\begin{tikzcd} P \arrow[d] \\ T \end{tikzcd} \qquad \begin{tikzcd} & P \arrow[dl] \arrow[d] \arrow[dr] & \\ Q & Q\&R \arrow[l] \arrow[r] & R \end{tikzcd} \qquad \begin{tikzcd} P\&Q \arrow[d] \\ R \end{tikzcd} \underset{\mapsto}{\leftrightarrow} \begin{tikzcd} P \arrow[d] \\ Q \to R \end{tikzcd}$$

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & P \vee Q \arrow[d] & Q \arrow[l] \arrow[dl] \\ & R & \end{tikzcd}$$

$01 \to 12$ TEM 1 ELEMENTO (PORQUE $01 \leq 12$)

$20 \to 11$ TEM 0 ELEMENTOS (PORQUE $20 \not\leq 11$)

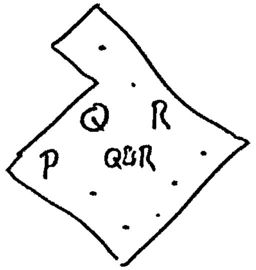


EXISTE UMA "FUNÇÃO" $P \to Q$? SIM
$P \to R$? SIM
$P \to Q\&R$? SIM
$Q\&R \to Q$? SIM
$Q\&R \to R$? SIM

$Q\&R = 11$

$$\begin{tikzcd} & 10 \arrow[dl, "\text{sim}"'] \arrow[d, "\text{sim}"] \arrow[dr, "\text{sim}"] & \\ 21 & 21\&12 = 11 \arrow[l, "\text{sim}"] \arrow[r, "\text{sim}"'] & 12 \end{tikzcd}$$

OUTRO CASO:



$$\begin{tikzcd} & 20 \arrow[dl, "\text{sim}"'] \arrow[d] \arrow[dr, "\text{não}"] & \\ 21 & 21\&12 = 11 \arrow[l, "\text{sim}"] \arrow[r, "\text{sim}"'] & 12 \end{tikzcd}$$

LA 2/maio/2017

Alguns anéis familiares:

$$\mathbb{Z}_6 = (\{0,1,2,3,4,5\},\ 0,\ 1,\ +_{\bmod 6},\ -_{\bmod 6},\ \cdot_{\bmod 6})$$

$\uparrow$ R

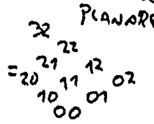
$(-_{\bmod 6})(1) = 5$

$\underbrace{1 +_{\bmod 6} \underbrace{(-_{\bmod 6})(1)}_{5}} = 0$

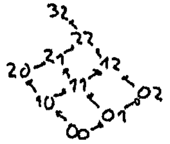
[Lemos o Awodey das págs 3 a 8]

As nossas categorias preferidas (por enquanto!) vão ser coisas bem concretas:

- Set
- ZHAs (as álgebras de Heyting planares da aula passada)

$B =$ 

$\mathbf{B}$ (B como categoria):



$\mathbf{B} = (\mathbf{B}_0, Hom_{\mathbf{B}}, id_{\mathbf{B}}, \circ_{\mathbf{B}})$

$\mathbf{B}_0 = B$ (10 elementos)

Um truque pra entender o Hom:

$f: A \to B$ é "formalizado" como $f \in Hom_{\mathbf{C}}(A,B)$

$$Hom_{\mathbf{B}}(P,Q) = \begin{cases} \{(P,Q)\} & \text{se } P \leq_B Q, \\ \emptyset & \text{se } P \not\leq_B Q. \end{cases}$$

Exemplos:

$Hom_{\mathbf{B}}(11,22) = \{(11,22)\}$

$Hom_{\mathbf{B}}(22,22) = \{(22,22)\}$

$Hom_{\mathbf{B}}(11,02) = \{\}$

Composição:

$$\begin{tikzcd} 00 \arrow[r, "(00{,}12)"] \arrow[rr, bend right, "(00{,}32)"'] & 12 \arrow[r, "(12{,}32)"] & 32 \end{tikzcd}$$

(não tem muita escolha)

$12 \xrightarrow{id_{12}} 12 \qquad id_{12} = (12,12)$

Que que a gente pode provar numa categoria qualquer?

Muito pouca coisa...

1) Se $A \xrightarrow{g} B$ tem inversa à esquerda e à direita elas são iguais.

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr, "id_B"'] & A \arrow[d, "g"] \arrow[dr, "id_A"] & \\ & B \arrow[r, "h"'] & A \end{tikzcd}$$

$g \circ f = f;g$

$f;g = id_B$

$g;h = id_A$

$f;(g;h) = f;id_A = f$

$(f;g);h = id_B;h = h$

2) Um diagrama "comuta" se (definição informal!) para cada dois objetos X e Y dele todos os jeitos de obter uma seta de X pra Y compondo setas do diagrama dão o mesmo resultado.

Isto aqui comuta?

$$\begin{tikzcd} \{1,2\} \arrow[r, "{1\mapsto 3,\ 2\mapsto 4}"] \arrow[d, "{1\mapsto 5,\ 2\mapsto 6}"'] & \{3,4\} \arrow[d, "{3\mapsto 7,\ 4\mapsto 7}"] \\ \{5,6\} \arrow[r, "{5\mapsto 7,\ 6\mapsto 8}"'] & \{7,8\} \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, shift left, "f;h"] \arrow[r, shift right, "g;k"'] & D \end{tikzcd}$$

$f;h \neq g;k$... Não!

Teorema: num diagrama planar em que cada célula tem exatamente uma "fonte" e um "ralo", se cada célula comuta então o diagrama todo comuta.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "c"'] & B \arrow[r, "b"] \arrow[d, "d"] & C \arrow[d, "e"] \\ D \arrow[r, "f"'] & E \arrow[r, "g"'] & F \end{tikzcd}$$

Sabemos que $a;d = c;f$ e $b;e = d;g$

Então:

$$\begin{aligned} a;b;e &= a;(b;e) \\ &= a;(d;g) \\ &= (a;d);g \\ &= (c;f);g \\ &= c;f;g \end{aligned}$$

Reparem que o 1 e o 2 precisam que a gente esteja numa categoria, não numa proto-categoria...

Uma coisa muito legal que os livros não contam é que todos os teoremas bacanas de categorias podem ser entendidos primeiro em proto-categorias...

$\boxed{\text{LA } 2/\text{maio}/2017}$

Alguns anéis familiares:

$$\mathbb{Z}_6 = (\underset{\mathbb{R}}{\underbrace{\{0,1,2,3,4,5\}}}, 0, 1, +_{\text{mod }6}, -_{\text{mod }6}, \cdot_{\text{mod }6})$$

$$(-_{\text{mod }6})(1) = 5$$

$$\underbrace{1 +_{\text{mod }6} \underbrace{(-_{\text{mod }6})(1)}_{5}} = 0$$

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- Set
- ZHAs (as álgebras de Heyting planares da aula passada)

$$B = \begin{array}{ccccccc} & & 32 & & & & \\ & & & 22 & & & \\ & & 21 & & 12 & & \\ & 20 & & 11 & & 02 & \\ & & 10 & & 01 & & \\ & & & 00 & & & \end{array}$$

B (B como categoria):

$$\begin{tikzcd}[column sep=tiny, row sep=tiny]
& 32 & & & \\
& & 22 \arrow[ul] & & \\
& 21 \arrow[ur] & & 12 \arrow[ul] & \\
20 \arrow[ur] & & 11 \arrow[ul] \arrow[ur] & & 02 \arrow[ul] \\
& 10 \arrow[ul] \arrow[ur] & & 01 \arrow[ul] \arrow[ur] & \\
& & 00 \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

$$\mathbf{B} = (\underset{\substack{\| \\ B \\ (10 \text{ elementos})}}{\mathbf{B}_0}, \text{Hom}_{\mathbf{B}}, \text{id}_{\mathbf{B}}, \circ_{\mathbf{B}})$$

Idéia:

"Categorias" são sem graça, "categorias com" são legais.

Primeiro exemplo: categorias com produtos. (proto-categorias com produtos)

Exemplo:

$$(\mathbf{B}_0, \text{Hom}_{\mathbf{B}}, \text{id}_{\mathbf{B}}, \circ_{\mathbf{B}}, \underset{\text{exemplo: }\&}{\times}, \pi, \pi', \langle , \rangle)$$

$\times : \mathbf{B}_0 \times \mathbf{B}_0 \to \mathbf{B}_0$

$\pi_{QR} \in \text{Hom}_{\mathbf{B}}(Q\times R, Q)$ $\quad (Q \xleftarrow{\pi_{QR}} Q\&R)$

$\pi'_{QR} \in \text{Hom}_{\mathbf{B}}(Q\times R, R)$ $\quad (Q\&R \xrightarrow{\pi'_{QR}} R)$

$\langle , \rangle_{PQR} \in \ldots$

$$P \xrightarrow{\langle P \xrightarrow{f} Q,\ P \xrightarrow{g} R \rangle_{PQR}} Q\&R$$

$$\begin{tikzcd}
& P \arrow[dl, "f"'] \arrow[d, "\langle f,g \rangle"] \arrow[dr, "g"] & \\
Q & Q\times R \arrow[l, "\pi_{QR}"] \arrow[r, "\pi'_{QR}"'] & R
\end{tikzcd}$$

E no caso mais abstrato? Em dois passos:

1) $Q \xleftarrow{\pi} Q\times R \xrightarrow{\pi'} R$ é um "product diagram" se e só se:

$\forall P.$
$\forall f: P \to Q,\ \forall g: P \to R.$
$\exists!\, h: P \to Q\times R.$
$(h;\pi = f) \,\&\, (h;\pi' = g)$

Exercício: Mostre que isto aqui (em B) não é um product diagram:

$$21 \xleftarrow{\pi} 01 \xrightarrow{\pi'} 12$$

(Dica: procurem um P!)

$P = 11$

Em Set a gente também tem produtos...

Exercício: Mostre que isto aqui não é um product diagram em Set:

$$\begin{tikzcd}
& \{35, 46\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\underset{\substack{\| \\ Q}}{\{3,4\}} & \underset{\substack{\| \\ Q\times R}}{\{35, 36, 45\}} \arrow[l, "\pi"', "\substack{35\mapsto 3 \\ 36 \mapsto 3 \\ 45 \mapsto 4}"] \arrow[r, "\pi'", "\substack{35\mapsto 5 \\ 36 \mapsto 6 \\ 45 \mapsto 5}"'] & \underset{\substack{\| \\ R}}{\{5,6\}}
\end{tikzcd}$$

$$\begin{tikzcd}
& \{46\} \arrow[dl] \arrow[d, dotted] \arrow[dr] & \\
\{3,4\} & \{35, 36, 45\} \arrow[l] \arrow[r] & \{5,6\}
\end{tikzcd}$$

Isto aqui é um coproduct diagram em Set: (Exercício: complete)

$$\begin{tikzcd}
& \{0, 45, 67, 8, 9\} & \\
\{4,5,6\} \arrow[ur] \arrow[r, "i"'] & \{4,5,6,7,8,9\} \arrow[u, dotted] & \{7,8,9\} \arrow[ul] \arrow[l, "i'"]
\end{tikzcd}$$