10: VECTORS

This chapter introduces a new mathematical object, the **vector**. Defined in Section 10.2, we will see that vectors provide a powerful language for describing quantities that have magnitude and direction aspects. A simple example of such a quantity is force: when applying a force, one is generally interested in how much force is applied (i.e., the magnitude of the force) and the direction in which the force was applied. Vectors will play an important role in many of the subsequent chapters in this text.

This chapter begins with moving our mathematics out of the plane and into "space." That is, we begin to think mathematically not only in two dimensions, but in three. With this foundation, we can explore vectors both in the plane and in space.

10.1 Introduction to Cartesian Coordinates in Space

Up to this point in this text we have considered mathematics in a 2–dimensional world. We have plotted graphs on the *x*-*y* plane using rectangular and polar coordinates and found the area of regions in the plane. We have considered properties of *solid* objects, such as volume and surface area, but only by first defining a curve in the plane and then rotating it out of the plane.

While there is wonderful mathematics to explore in "2D," we live in a "3D" world and eventually we will want to apply mathematics involving this third dimension. In this section we introduce Cartesian coordinates in space and explore basic surfaces. This will lay a foundation for much of what we do in the remainder of the text.

Each point P in space can be represented with an ordered triple, $P = (a, b, c)$, where a, b and c represent the relative position of P along the x-, y- and z-axes, respectively. Each axis is perpendicular to the other two.

Visualizing points in space on paper can be problematic, as we are trying to represent a 3-dimensional concept on a 2–dimensional medium. We cannot draw three lines representing the three axes in which each line is perpendicular to the other two. Despite this issue, standard conventions exist for plotting shapes in space that we will discuss that are more than adequate.

One convention is that the axes must conform to the **right hand rule**. This rule states that when the index finger of the right hand is extended in the direction of the positive x-axis, and the middle finger (bent "inward" so it is perpendicular to the palm) points along the positive y-axis, then the extended thumb will point in the direction of the positive z-axis. (It may take some thought to

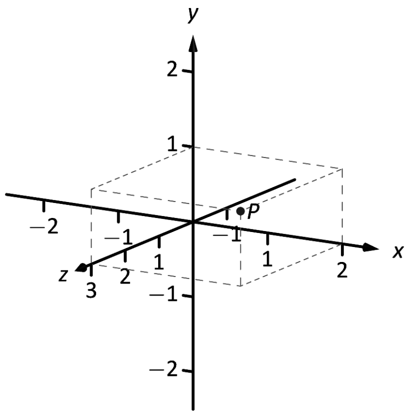

Figure 10.1.1: Plotting the point $P = (2, 1, 3)$ in space.

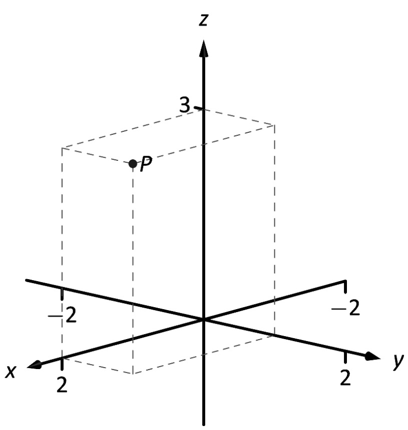

Figure 10.1.2: Plotting the point $P = (2, 1, 3)$ in space with a perspective used in this text.

verify this, but this system is inherently different from the one created by using the "left hand rule.")

As long as the coordinate axes are positioned so that they follow this rule, it does not matter how the axes are drawn on paper. There are two popular methods that we briefly discuss.

In Figure 10.1.1 we see the point $P = (2, 1, 3)$ plotted on a set of axes. The basic convention here is that the *x*-*y* plane is drawn in its standard way, with the *z*-axis down to the left. The perspective is that the paper represents the *x*-*y* plane and the positive *z* axis is coming up, off the page. This method is preferred by many engineers. Because it can be hard to tell where a single point lies in relation to all the axes, dashed lines have been added to let one see how far along each axis the point lies.

One can also consider the *x*-*y* plane as being a horizontal plane in, say, a room, where the positive *z*-axis is pointing up. When one steps back and looks at this room, one might draw the axes as shown in Figure 10.1.2. The same point *P* is drawn, again with dashed lines. This point of view is preferred by most mathematicians, and is the convention adopted by this text.

Just as the *x*- and *y*-axes divide the plane into four *quadrants*, the *x*-, *y*-, and *z*-coordinate planes divide space into eight *octants*. The octant in which *x*, *y*, and *z* are positive is called the **first octant**. We do not name the other seven octants in this text.

Measuring Distances

It is of critical importance to know how to measure distances between points in space. The formula for doing so is based on measuring distance in the plane, and is known (in both contexts) as the Euclidean measure of distance.

Definition 10.1.1 Distance In Space

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be points in space. The distance D between P and Q is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

We refer to the line segment that connects points P and Q in space as $\overline{PQ}$, and refer to the length of this segment as $||\overline{PQ}||$. The above distance formula allows us to compute the length of this segment.

Example 10.1.1 Length of a line segment

Let $P = (1, 4, -1)$ and let $Q = (2, 1, 1)$. Draw the line segment $\overline{PQ}$ and find its length.

Notes:

Solution The points P and Q are plotted in Figure 10.1.3; no special consideration need be made to draw the line segment connecting these two points; simply connect them with a straight line. One *cannot* actually measure this line on the page and deduce anything meaningful; its true length must be measured analytically. Applying Definition 10.1.1, we have

$$||\overline{PQ}|| = \sqrt{(2-1)^2 + (1-4)^2 + (1-(-1))^2} = \sqrt{14} \approx 3.74.$$

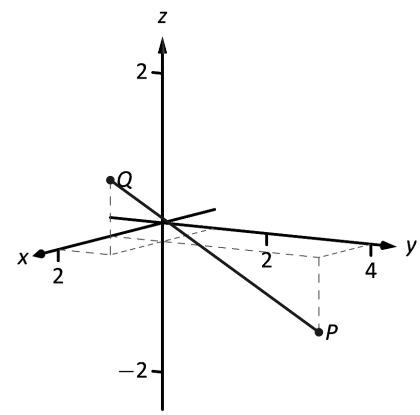

Figure 10.1.3: Plotting points P and Q in Example 10.1.1.

Spheres

Just as a circle is the set of all points in the *plane* equidistant from a given point (its center), a sphere is the set of all points in *space* that are equidistant from a given point. Definition 10.1.1 allows us to write an equation of the sphere.

We start with a point $C = (a, b, c)$ which is to be the center of a sphere with radius r. If a point $P = (x, y, z)$ lies on the sphere, then P is r units from C; that is,

$$||\overline{PC}|| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r.$$

Squaring both sides, we get the standard equation of a sphere in space with center at $C = (a, b, c)$ with radius r, as given in the following Key Idea.

Key Idea 10.1.1 Standard Equation of a Sphere in Space

The standard equation of the sphere with radius r, centered at $C = (a, b, c)$, is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

Example 10.1.2 Equation of a sphere

Find the center and radius of the sphere defined by $x^2+2x+y^2-4y+z^2-6z = 2$.

Solution To determine the center and radius, we must put the equation in standard form. This requires us to complete the square (three times).

$$\begin{aligned} x^2 + 2x + y^2 - 4y + z^2 - 6z &= 2 \\ (x^2 + 2x + 1) + (y^2 - 4y + 4) + (z^2 - 6z + 9) - 14 &= 2 \\ (x+1)^2 + (y-2)^2 + (z-3)^2 &= 16 \end{aligned}$$

The sphere is centered at $(-1, 2, 3)$ and has a radius of 4.

The equation of a sphere is an example of an implicit function defining a surface in space. In the case of a sphere, the variables x, y and z are all used. We

Notes:

now consider situations where surfaces are defined where one or two of these variables are absent.

Introduction to Planes in Space

The coordinate axes naturally define three planes (shown in Figure 10.1.4), the **coordinate planes**: the *x*-*y* plane, the *y*-*z* plane and the *x*-*z* plane. The *x*-*y* plane is characterized as the set of all points in space where the z-value is 0. This, in fact, gives us an equation that describes this plane: $z = 0$. Likewise, the *x*-*z* plane is all points where the y-value is 0, characterized by $y = 0$.

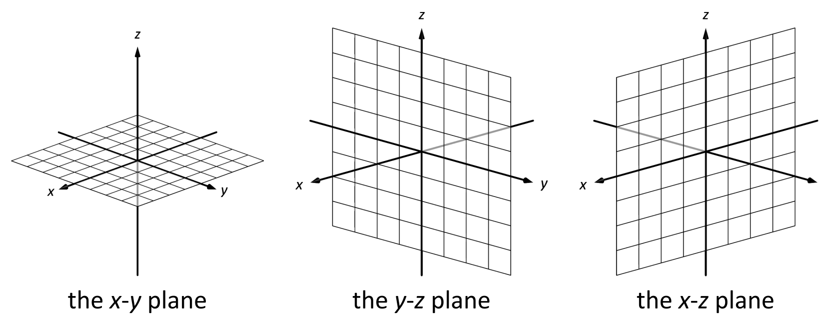

Figure 10.1.4: The coordinate planes.

The equation $x = 2$ describes all points in space where the x-value is 2. This is a plane, parallel to the *y*-*z* coordinate plane, shown in Figure 10.1.5.

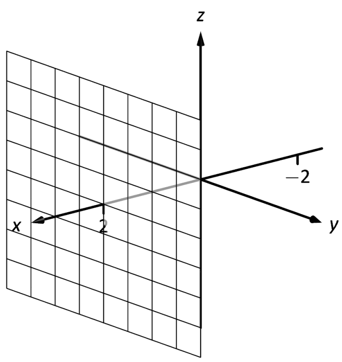

Figure 10.1.5: The plane $x = 2$.

Example 10.1.3 Regions defined by planes

Sketch the region defined by the inequalities $-1 \leq y \leq 2$.

Solution The region is all points between the planes $y = -1$ and $y = 2$. These planes are sketched in Figure 10.1.6, which are parallel to the *x*-*z* plane. Thus the region extends infinitely in the x and z directions, and is bounded by planes in the y direction.

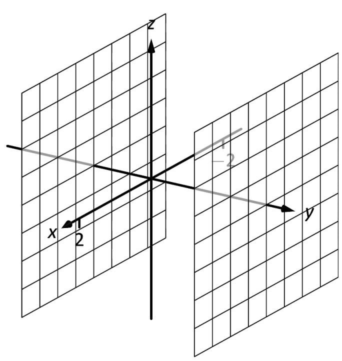

Figure 10.1.6: Sketching the boundaries of a region in Example 10.1.3.

Cylinders

The equation $x = 1$ obviously lacks the y and z variables, meaning it defines points where the y and z coordinates can take on any value. Now consider the equation $x^2 + y^2 = 1$ *in space.* In *the plane*, this equation describes a circle of radius 1, centered at the origin. In space, the z coordinate is not specified, meaning it can take on any value. In Figure 10.1.8 (a), we show part of the graph of the equation $x^2 + y^2 = 1$ by sketching 3 circles: the bottom one has a constant z-value of -1.5, the middle one has a z-value of 0 and the top circle has a z-value of 1. By plotting *all* possible z-values, we get the surface shown in Figure

Notes:

10.1.8(b). This surface looks like a "tube," or a "cylinder"; mathematicians call this surface a **cylinder** for an entirely different reason.

Definition 10.1.2 Cylinder

Let C be a curve in a plane and let L be a line not parallel to C. A **cylinder** is the set of all lines parallel to L that pass through C. The curve C is the **directrix** of the cylinder, and the lines are the **rulings**.

In this text, we consider curves C that lie in planes parallel to one of the coordinate planes, and lines L that are perpendicular to these planes, forming **right cylinders**. Thus the directrix can be defined using equations involving 2 variables, and the rulings will be parallel to the axis of the 3rd variable.

In the example preceding the definition, the curve $x^2 + y^2 = 1$ in the x-y plane is the directrix and the rulings are lines parallel to the z-axis. (Any circle shown in Figure 10.1.8 can be considered a directrix; we simply choose the one where $z = 0$.) Sample rulings can also be viewed in part (b) of the figure. More examples will help us understand this definition.

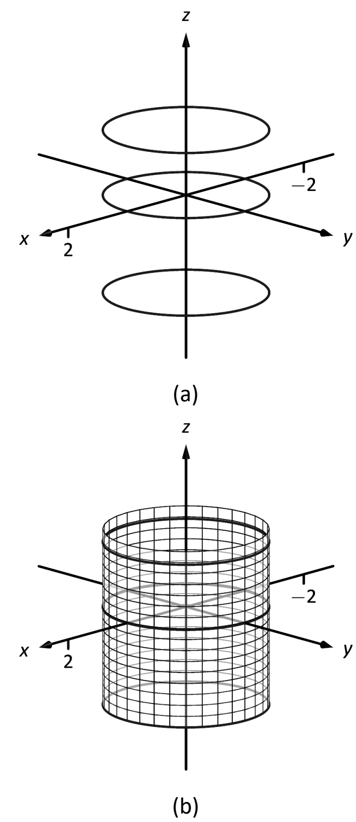

Figure 10.1.8: Sketching $x^2 + y^2 = 1$.

Example 10.1.4 Graphing cylinders

Graph the following cylinders.

1. $z = y^2$ 2. $x = \sin z$

Solution

1. We can view the equation $z = y^2$ as a parabola in the y-z plane, as illustrated in Figure 10.1.7(a). As x does not appear in the equation, the rulings are lines through this parabola parallel to the x-axis, shown in (b). These rulings give an idea as to what the surface looks like, drawn in (c).

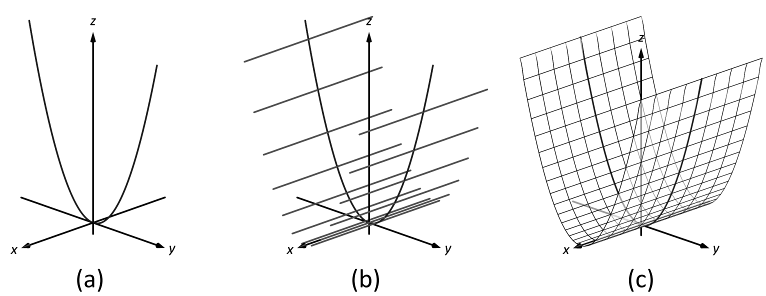

Figure 10.1.7: Sketching the cylinder defined by $z = y^2$.

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2. We can view the equation $x = \sin z$ as a sine curve that exists in the x-z plane, as shown in Figure 10.1.9 (a). The rules are parallel to the y axis as the variable y does not appear in the equation $x = \sin z$; some of these are shown in part (b). The surface is shown in part (c) of the figure.

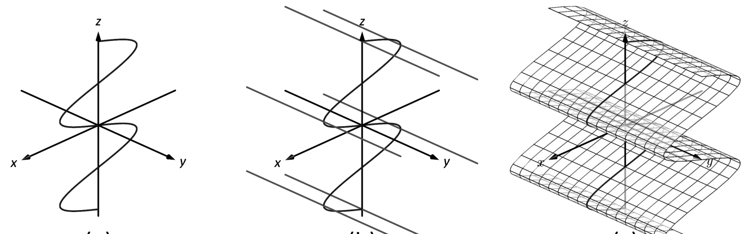

Figure 10.1.9: Sketching the cylinder defined by $x = \sin z$.

Surfaces of Revolution

One of the applications of integration we learned previously was to find the volume of solids of revolution – solids formed by revolving a curve about a horizontal or vertical axis. We now consider how to find the equation of the surface of such a solid.

Consider the surface formed by revolving $y = \sqrt{x}$ about the x-axis. Cross–sections of this surface parallel to the y-z plane are circles, as shown in Figure 10.1.10(a). Each circle has equation of the form $y^2 + z^2 = r^2$ for some radius r. The radius is a function of x; in fact, it is $r(x) = \sqrt{x}$. Thus the equation of the surface shown in Figure 10.1.10b is $y^2 + z^2 = (\sqrt{x})^2$.

We generalize the above principles to give the equations of surfaces formed by revolving curves about the coordinate axes.

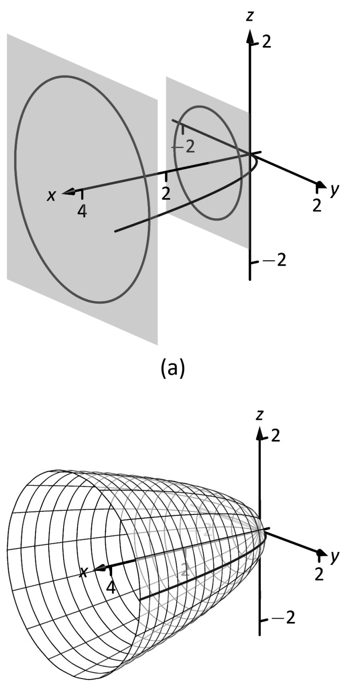

Figure 10.1.10: Introducing surfaces of revolution.

Key Idea 10.1.2 Surfaces of Revolution, Part 1

Let r be a radius function.

1. The equation of the surface formed by revolving $y = r(x)$ or $z = r(x)$ about the x-axis is $y^2 + z^2 = r(x)^2$.
2. The equation of the surface formed by revolving $x = r(y)$ or $z = r(y)$ about the y-axis is $x^2 + z^2 = r(y)^2$.
3. The equation of the surface formed by revolving $x = r(z)$ or $y = r(z)$ about the z-axis is $x^2 + y^2 = r(z)^2$.

Notes:

Example 10.1.5 Finding equation of a surface of revolution

Let $y = \sin z$ on $[0, \pi]$. Find the equation of the surface of revolution formed by revolving $y = \sin z$ about the z-axis.

Solution Using Key Idea 10.1.2, we find the surface has equation $x^2 + y^2 = \sin^2 z$. The curve is sketched in Figure 10.1.11(a) and the surface is drawn in Figure 10.1.11(b).

Note how the surface (and hence the resulting equation) is the same if we began with the curve $x = \sin z$, which is also drawn in Figure 10.1.11(a).

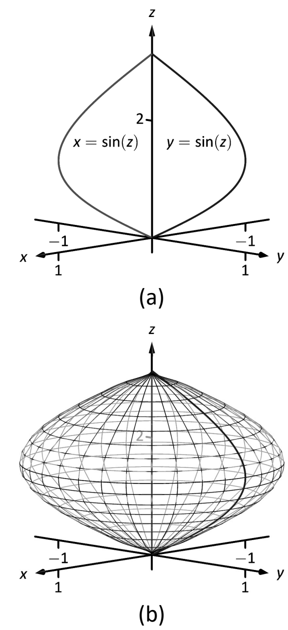

Figure 10.1.11: Revolving $y = \sin z$ about the z-axis in Example 10.1.5.

This particular method of creating surfaces of revolution is limited. For instance, in Example 7.3.4 of Section 7.3 we found the volume of the solid formed by revolving $y = \sin x$ about the y-axis. Our current method of forming surfaces can only rotate $y = \sin x$ about the x-axis. Trying to rewrite $y = \sin x$ as a function of y is not trivial, as simply writing $x = \sin^{-1} y$ only gives part of the region we desire.

What we desire is a way of writing the surface of revolution formed by rotating $y = f(x)$ about the y-axis. We start by first recognizing this surface is the same as revolving $z = f(x)$ about the z-axis. This will give us a more natural way of viewing the surface.

A value of x is a measurement of distance from the z-axis. At the distance r, we plot a z-height of $f(r)$. When rotating $f(x)$ about the z-axis, we want all points a distance of r from the z-axis in the x-y plane to have a z-height of $f(r)$. All such points satisfy the equation $r^2 = x^2 + y^2$; hence $r = \sqrt{x^2 + y^2}$. Replacing r with $\sqrt{x^2 + y^2}$ in $f(r)$ gives $z = f(\sqrt{x^2 + y^2})$. This is the equation of the surface.

Key Idea 10.1.3 Surfaces of Revolution, Part 2

Let $z = f(x)$, $x \geq 0$, be a curve in the x-z plane. The surface formed by revolving this curve about the z-axis has equation $z = f\big(\sqrt{x^2 + y^2}\big)$.

Example 10.1.6 Finding equation of surface of revolution

Find the equation of the surface found by revolving $z = \sin x$ about the z-axis.

Solution Using Key Idea 10.1.3, the surface has equation $z = \sin\left(\sqrt{x^2 + y^2}\right)$. The curve and surface are graphed in Figure 10.1.12.

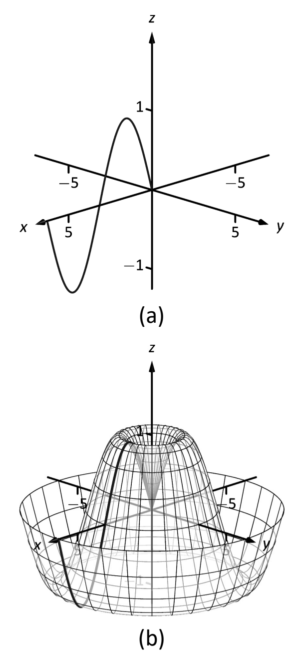

Figure 10.1.12: Revolving $z = \sin x$ about the z-axis in Example 10.1.6.

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Quadric Surfaces

Spheres, planes and cylinders are important surfaces to understand. We now consider one last type of surface, a **quadric surface**. The definition may look intimidating, but we will show how to analyze these surfaces in an illuminating way.

Definition 10.1.3 Quadric Surface

A **quadric surface** is the graph of the general second–degree equation in three variables:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

When the coefficients D, E or F are not zero, the basic shapes of the quadric surfaces are rotated in space. We will focus on quadric surfaces where these coefficients are 0; we will not consider rotations. There are six basic quadric surfaces: the elliptic paraboloid, elliptic cone, ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, and the hyperbolic paraboloid.

We study each shape by considering **traces**, that is, intersections of each surface with a plane parallel to a coordinate plane. For instance, consider the elliptic paraboloid $z = x^2/4 + y^2$, shown in Figure 10.1.13. If we intersect this shape with the plane $z = d$ (i.e., replace z with d), we have the equation:

$$d = \frac{x^2}{4} + y^2.$$

Divide both sides by d:

$$1 = \frac{x^2}{4d} + \frac{y^2}{d}.$$

This describes an ellipse – so cross sections parallel to the x-y coordinate plane are ellipses. This ellipse is drawn in the figure.

Now consider cross sections parallel to the x-z plane. For instance, letting $y = 0$ gives the equation $z = x^2/4$, clearly a parabola. Intersecting with the plane $x = 0$ gives a cross section defined by $z = y^2$, another parabola. These parabolas are also sketched in the figure.

Thus we see where the elliptic paraboloid gets its name: some cross sections are ellipses, and others are parabolas.

Such an analysis can be made with each of the quadric surfaces. We give a sample equation of each, provide a sketch with representative traces, and describe these traces.

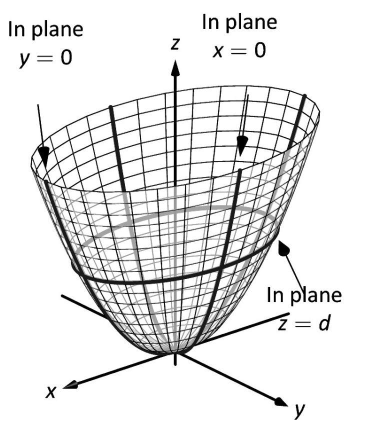

Figure 10.1.13: The elliptic paraboloid $z = x^2/4 + y^2$.

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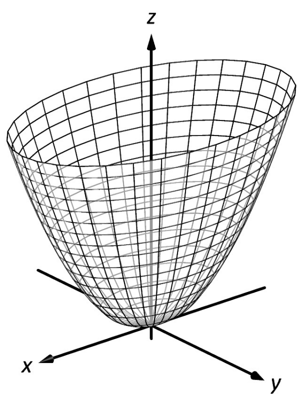
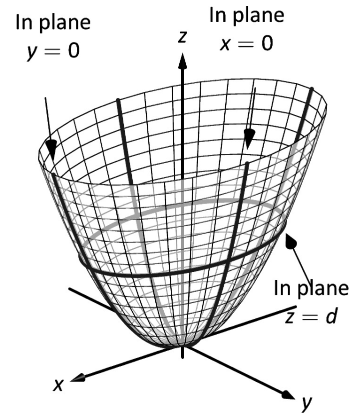
Elliptic Paraboloid, $\quad z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$



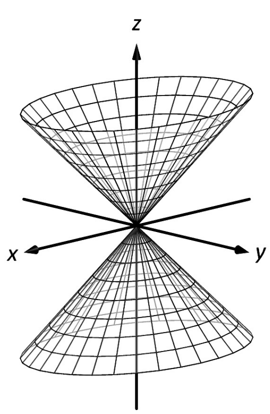
Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Ellipse



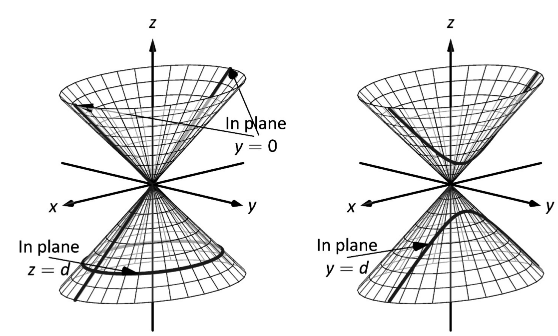
One variable in the equation of the elliptic paraboloid will be raised to the first power; above, this is the z variable. The paraboloid will "open" in the direction of this variable's axis. Thus $x = y^2/a^2 + z^2/b^2$ is an elliptic paraboloid that opens along the x-axis.

Multiplying the right hand side by (-1) defines an elliptic paraboloid that "opens" in the opposite direction.

Elliptic Cone, $\quad z^2 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

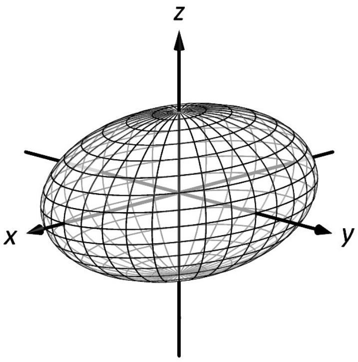


Plane	Trace
$x = 0$	Crossed Lines
$y = 0$	Crossed Lines
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

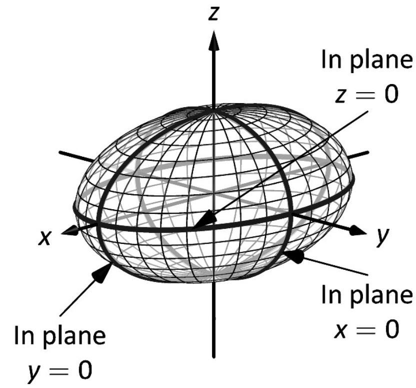


One can rewrite the equation as $z^2 - x^2/a^2 - y^2/b^2 = 0$. The one variable with a positive coefficient corresponds to the axis that the cones "open" along.

Ellipsoid, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

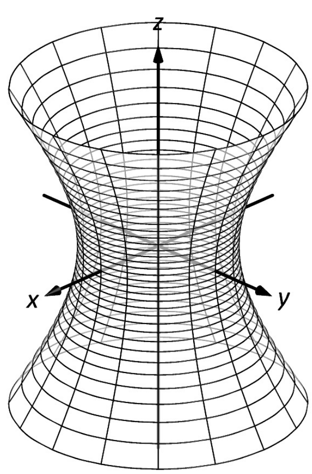


Plane	Trace
$x = d$	Ellipse
$y = d$	Ellipse
$z = d$	Ellipse

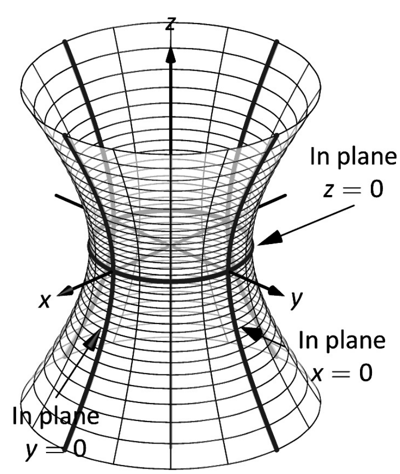


If $a = b = c \neq 0$, the ellipsoid is a sphere with radius a; compare to Key Idea 10.1.1.

Hyperboloid of One Sheet, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

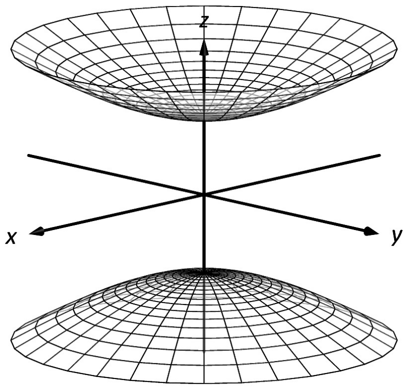


Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

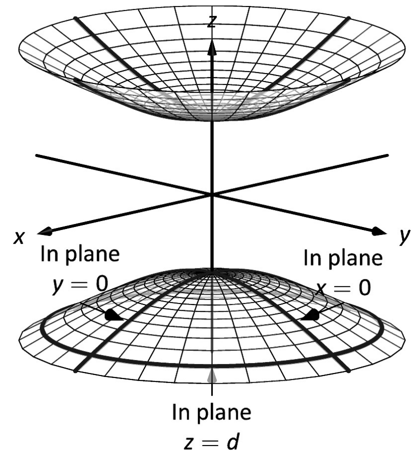


The one variable with a negative coefficient corresponds to the axis that the hyperboloid "opens" along.

Hyperboloid of Two Sheets, $\dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

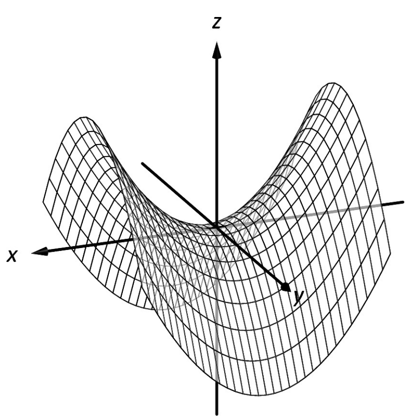


Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

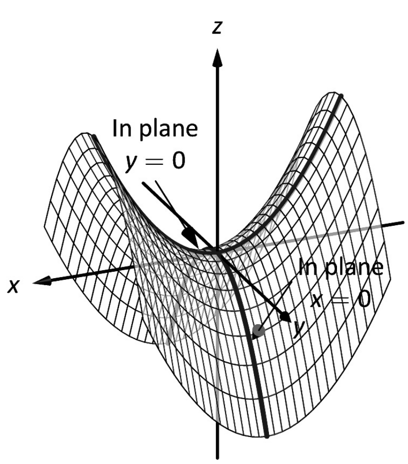


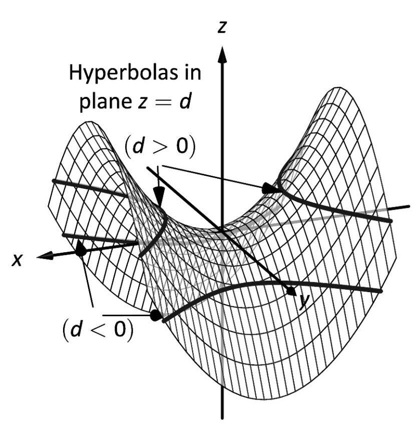
The one variable with a positive coefficient corresponds to the axis that the hyperboloid "opens" along. In the case illustrated, when $|d| < |c|$, there is no trace.

Hyperbolic Paraboloid, $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$



Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Hyperbola





The parabolic traces will open along the axis of the one variable that is raised to the first power.

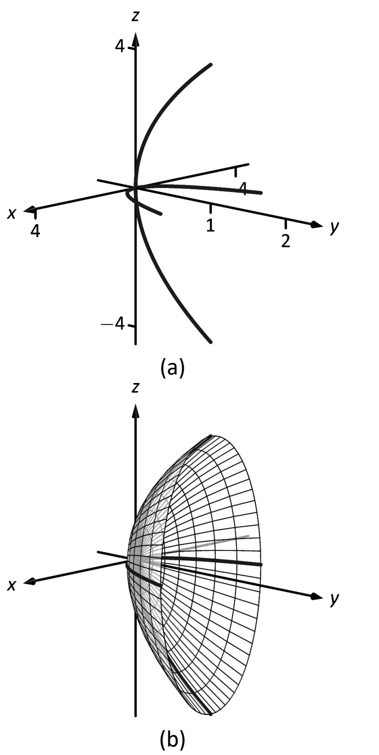


Figure 10.1.14: Sketching an elliptic paraboloid.

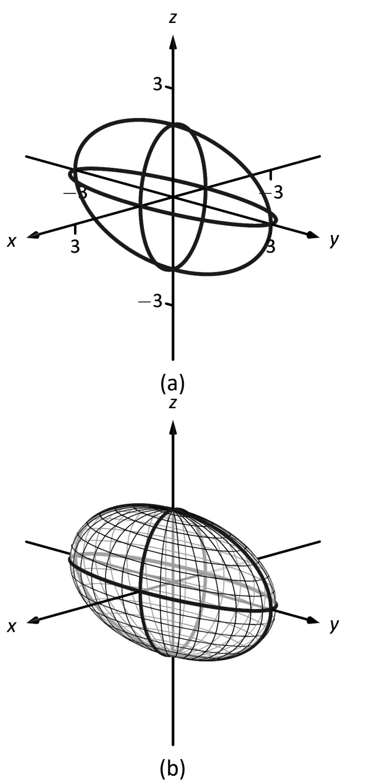


Figure 10.1.15: Sketching an ellipsoid.

Example 10.1.7 Sketching quadric surfaces

Sketch the quadric surface defined by the given equation.

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$ 2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1.$ 3. $z = y^2 - x^2.$

Solution

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$:

   We first identify the quadric by pattern–matching with the equations given previously. Only two surfaces have equations where one variable is raised to the first power, the elliptic paraboloid and the hyperbolic paraboloid. In the latter case, the other variables have different signs, so we conclude that this describes a hyperbolic paraboloid. As the variable with the first power is y, we note the paraboloid opens along the y-axis.

   To make a decent sketch by hand, we need only draw a few traces. In this case, the traces $x = 0$ and $z = 0$ form parabolas that outline the shape.

   $x = 0$: The trace is the parabola $y = z^2/16$

   $z = 0$: The trace is the parabola $y = x^2/4$.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 10.1.14(a). This is enough to give an idea of what the paraboloid looks like. The surface is filled in in (b).

2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$:

   This is an ellipsoid. We can get a good idea of its shape by drawing the traces in the coordinate planes.

   $x = 0$: The trace is the ellipse $\dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$. The major axis is along the y–axis with length 6 (as $b = 3$, the length of the axis is 6); the minor axis is along the z-axis with length 4.

   $y = 0$: The trace is the ellipse $x^2 + \dfrac{z^2}{4} = 1$. The major axis is along the z-axis, and the minor axis has length 2 along the x-axis.

   $z = 0$: The trace is the ellipse $x^2 + \dfrac{y^2}{9} = 1$, with major axis along the y-axis.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 10.1.15(a). Filling in the surface gives Figure 10.1.15(b).

3. $z = y^2 - x^2$:

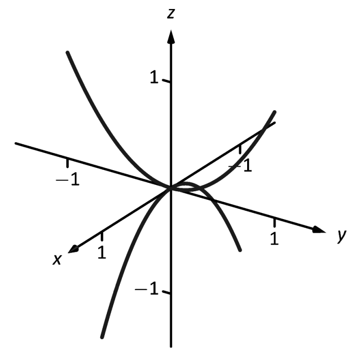
Notes:

This defines a hyperbolic paraboloid, very similar to the one shown in the gallery of quadric sections. Consider the traces in the $y-z$ and $x-z$ planes:

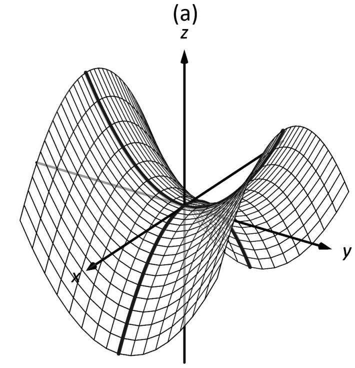
$x = 0$: The trace is $z = y^2$, a parabola opening up in the $y - z$ plane.

$y = 0$: The trace is $z = -x^2$, a parabola opening down in the $x - z$ plane.

Sketching these two parabolas gives a sketch like that in Figure 10.1.16(a), and filling in the surface gives a sketch like (b).



(a)



(b)

Figure 10.1.16: Sketching a hyperbolic paraboloid.

Example 10.1.8 Identifying quadric surfaces

Consider the quadric surface shown in Figure 10.1.17. Which of the following equations best fits this surface?

(a) $x^2 - y^2 - \dfrac{z^2}{9} = 0$ (c) $z^2 - x^2 - y^2 = 1$

(b) $x^2 - y^2 - z^2 = 1$ (d) $4x^2 - y^2 - \dfrac{z^2}{9} = 1$

Solution The image clearly displays a hyperboloid of two sheets. The gallery informs us that the equation will have a form similar to $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

We can immediately eliminate option (a), as the constant in that equation is not 1.

The hyperboloid "opens" along the x-axis, meaning x must be the only variable with a positive coefficient, eliminating (c).

The hyperboloid is wider in the z-direction than in the y-direction, so we need an equation where $c > b$. This eliminates (b), leaving us with (d). We should verify that the equation given in (d), $4x^2 - y^2 - \frac{z^2}{9} = 1$, fits.

We already established that this equation describes a hyperboloid of two sheets that opens in the x-direction and is wider in the z-direction than in the y. Now note the coefficient of the x-term. Rewriting $4x^2$ in standard form, we have: $4x^2 = \dfrac{x^2}{(1/2)^2}$. Thus when $y = 0$ and $z = 0$, x must be $1/2$; i.e., each hyperboloid "starts" at $x = 1/2$. This matches our figure.

We conclude that $4x^2 - y^2 - \dfrac{z^2}{9} = 1$ best fits the graph.

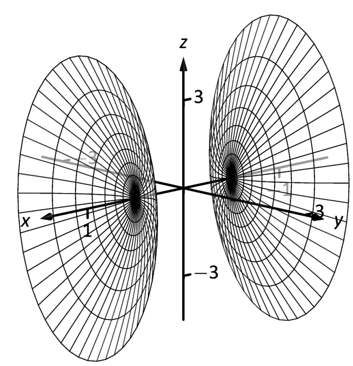


Figure 10.1.17: A possible equation of this quadric surface is found in Example 10.1.8.

This section has introduced points in space and shown how equations can describe surfaces. The next sections explore *vectors*, an important mathematical object that we'll use to explore curves in space.

Notes:

Exercises 10.1

Terms and Concepts

1. Axes drawn in space must conform to the ________ ________ rule.

2. In the plane, the equation $x = 2$ defines a ________; in space, $x = 2$ defines a ________.

3. In the plane, the equation $y = x^2$ defines a ________; in space, $y = x^2$ defines a ________.

4. Which quadric surface looks like a Pringles® chip?

5. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the x-axis, what quadric surface is formed?

6. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the y-axis, what quadric surface is formed?

Problems

7. The points $A = (1, 4, 2)$, $B = (2, 6, 3)$ and $C = (4, 3, 1)$ form a triangle in space. Find the distances between each pair of points and determine if the triangle is a right triangle.

8. The points $A = (1, 1, 3)$, $B = (3, 2, 7)$, $C = (2, 0, 8)$ and $D = (0, -1, 4)$ form a quadrilateral $ABCD$ in space. Is this a parallelogram?

9. Find the center and radius of the sphere defined by $x^2 - 8x + y^2 + 2y + z^2 + 8 = 0$.

10. Find the center and radius of the sphere defined by $x^2 + y^2 + z^2 + 4x - 2y - 4z + 4 = 0$.

In Exercises 11 – 14, describe the region in space defined by the inequalities.

11. $x^2 + y^2 + z^2 < 1$

12. $0 \le x \le 3$

13. $x \ge 0,\ y \ge 0,\ z \ge 0$

14. $y \ge 3$

In Exercises 15 – 18, sketch the cylinder in space.

15. $z = x^3$

16. $y = \cos z$

17. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

18. $y = \dfrac{1}{x}$

In Exercises 19 – 22, give the equation of the surface of revolution described.

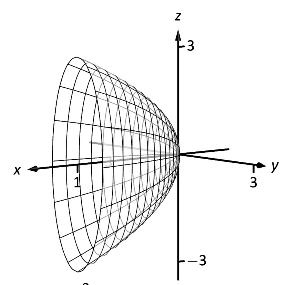
19. Revolve $z = \dfrac{1}{1 + y^2}$ about the y-axis.

20. Revolve $y = x^2$ about the x-axis.

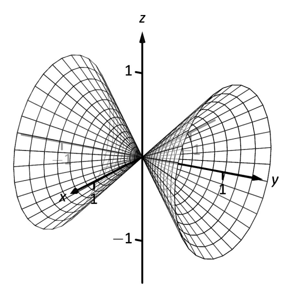
21. Revolve $z = x^2$ about the z-axis.

22. Revolve $z = 1/x$ about the z-axis.

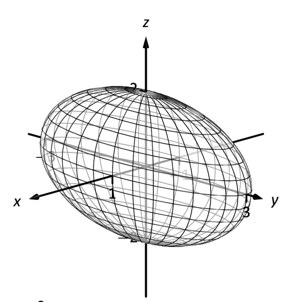
In Exercises 23 – 26, a quadric surface is sketched. Determine which of the given equations best fits the graph.

23. 

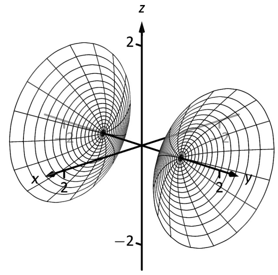
(a) $x = y^2 + \dfrac{z^2}{9}$ (b) $x = y^2 + \dfrac{z^2}{3}$

24. 

(a) $x^2 - y^2 - z^2 = 0$ (b) $x^2 - y^2 + z^2 = 0$

25. 

(a) $x^2 + \dfrac{y^2}{3} + \dfrac{z^2}{2} = 1$ (b) $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

26. 

(a) $y^2 - x^2 - z^2 = 1$

(b) $y^2 + x^2 - z^2 = 1$

In Exercises 27 – 32, sketch the quadric surface.

27. $z - y^2 + x^2 = 0$

28. $z^2 = x^2 + \frac{y^2}{4}$

29. $x = -y^2 - z^2$

30. $16x^2 - 16y^2 - 16z^2 = 1$

31. $\frac{x^2}{9} - y^2 + \frac{z^2}{25} = 1$

32. $4x^2 + 2y^2 + z^2 = 4$

10.2 An Introduction to Vectors

Many quantities we think about daily can be described by a single number: temperature, speed, cost, weight and height. There are also many other concepts we encounter daily that cannot be described with just one number. For instance, a weather forecaster often describes wind with its speed and its direction ("... with winds from the southeast gusting up to 30 mph ..."). When applying a force, we are concerned with both the magnitude and direction of that force. In both of these examples, *direction* is important. Because of this, we study *vectors*, mathematical objects that convey both magnitude and direction information.

One "bare–bones" definition of a vector is based on what we wrote above: "a vector is a mathematical object with magnitude and direction parameters." This definition leaves much to be desired, as it gives no indication as to how such an object is to be used. Several other definitions exist; we choose here a definition rooted in a geometric visualization of vectors. It is very simplistic but readily permits further investigation.

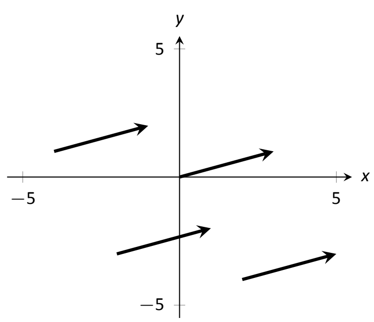


Figure 10.2.1: Drawing the same vector with different initial points.

Definition 10.2.1 Vector

A **vector** is a directed line segment.

Given points P and Q (either in the plane or in space), we denote with $\overrightarrow{PQ}$ the vector *from P to Q*. The point P is said to be the **initial point** of the vector, and the point Q is the **terminal point**.

The **magnitude**, **length** or **norm** of $\overrightarrow{PQ}$ is the length of the line segment $\overline{PQ}$: $\| \overrightarrow{PQ} \| = \| \overline{PQ} \|$.

Two vectors are **equal** if they have the same magnitude and direction.

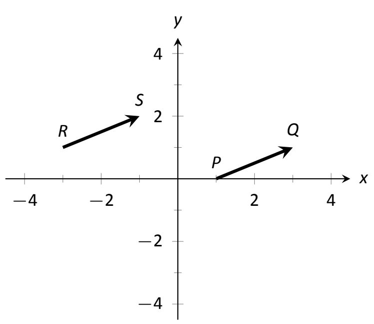


Figure 10.2.2: Illustrating how equal vectors have the same displacement.

Figure 10.2.1 shows multiple instances of the same vector. Each directed line segment has the same direction and length (magnitude), hence each is the same vector.

We use $\mathbb{R}^2$ (pronounced "r two") to represent all the vectors in the plane, and use $\mathbb{R}^3$ (pronounced "r three") to represent all the vectors in space.

Consider the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ as shown in Figure 10.2.2. The vectors look to be equal; that is, they seem to have the same length and direction. Indeed, they are. Both vectors move 2 units to the right and 1 unit up from the initial point to reach the terminal point. One can analyze this movement to measure the

Notes:

magnitude of the vector, and the movement itself gives direction information (one could also measure the slope of the line passing through P and Q or R and S). Since they have the same length and direction, these two vectors are equal.

This demonstrates that inherently all we care about is *displacement*; that is, how far in the x, y and possibly z directions the terminal point is from the initial point. Both the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in Figure 10.2.2 have an x-displacement of 2 and a y-displacement of 1. This suggests a standard way of describing vectors in the plane. A vector whose x-displacement is a and whose y-displacement is b will have terminal point (a, b) when the initial point is the origin, $(0, 0)$. This leads us to a definition of a standard and concise way of referring to vectors.

Definition 10.2.2 Component Form of a Vector

1. The **component form** of a vector $\vec{v}$ in $\mathbb{R}^2$, whose terminal point is (a, b) when its initial point is $(0, 0)$, is $\langle a, b \rangle$.

2. The **component form** of a vector $\vec{v}$ in $\mathbb{R}^3$, whose terminal point is (a, b, c) when its initial point is $(0, 0, 0)$, is $\langle a, b, c \rangle$.

The numbers a, b (and c, respectively) are the **components** of $\vec{v}$.

It follows from the definition that the component form of the vector $\overrightarrow{PQ}$, where $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle ;$$

in space, where $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$, the component form of $\overrightarrow{PQ}$ is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle .$$

We practice using this notation in the following example.

Example 10.2.1 Using component form notation for vectors

1. Sketch the vector $\vec{v} = \langle 2, -1 \rangle$ starting at $P = (3, 2)$ and find its magnitude.

2. Find the component form of the vector $\vec{w}$ whose initial point is $R = (-3, -2)$ and whose terminal point is $S = (-1, 2)$.

3. Sketch the vector $\vec{u} = \langle 2, -1, 3 \rangle$ starting at the point $Q = (1, 1, 1)$ and find its magnitude.

Notes:

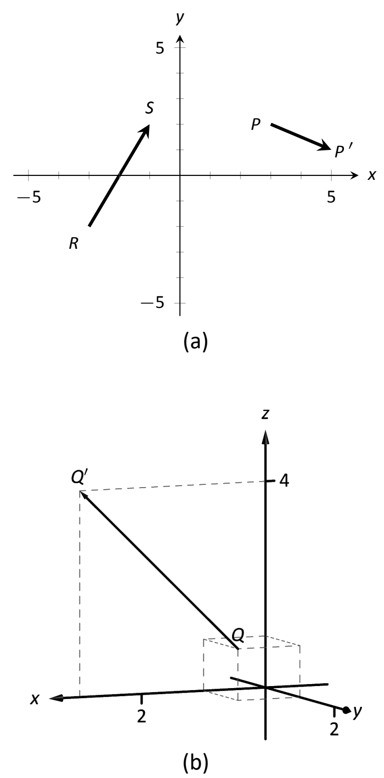


Figure 10.2.3: Graphing vectors in Example 10.2.1.

Solution

1. Using P as the initial point, we move 2 units in the positive x-direction and -1 units in the positive y-direction to arrive at the terminal point $P' = (5,1)$, as drawn in Figure 10.2.3(a).

   The magnitude of $\vec{v}$ is determined directly from the component form:

   $$||\,\vec{v}\,|| = \sqrt{2^2+(-1)^2} = \sqrt{5}.$$

2. Using the note following Definition 10.2.2, we have

   $$\overrightarrow{RS} = \langle -1-(-3), 2-(-2)\rangle = \langle 2,4\rangle\,.$$

   One can readily see from Figure 10.2.3(a) that the x- and y-displacement of $\overrightarrow{RS}$ is 2 and 4, respectively, as the component form suggests.

3. Using Q as the initial point, we move 2 units in the positive x-direction, -1 unit in the positive y-direction, and 3 units in the positive z-direction to arrive at the terminal point $Q' = (3,0,4)$, illustrated in Figure 10.2.3(b).

   The magnitude of $\vec{u}$ is:

   $$||\,\vec{u}\,|| = \sqrt{2^2+(-1)^2+3^2} = \sqrt{14}.$$

Now that we have defined vectors, and have created a nice notation by which to describe them, we start considering how vectors interact with each other. That is, we define an *algebra* on vectors.

Notes:

Definition 10.2.3 Vector Algebra

1. Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ be vectors in $\mathbb{R}^2$, and let c be a scalar.

   (a) The addition, or sum, of the vectors $\vec{u}$ and $\vec{v}$ is the vector

   $$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle .$$

   (b) The scalar product of c and $\vec{v}$ is the vector

   $$c\vec{v} = c\,\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle .$$

2. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in $\mathbb{R}^3$, and let c be a scalar.

   (a) The addition, or sum, of the vectors $\vec{u}$ and $\vec{v}$ is the vector

   $$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle .$$

   (b) The scalar product of c and $\vec{v}$ is the vector

   $$c\vec{v} = c\,\langle v_1, v_2, v_3 \rangle = \langle cv_1, cv_2, cv_3 \rangle .$$

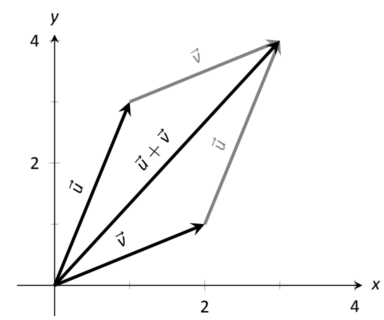


Figure 10.2.5: Illustrating how to add vectors using the Head to Tail Rule and Parallelogram Law.

In short, we say addition and scalar multiplication are computed "component–wise."

Example 10.2.2 Adding vectors

Sketch the vectors $\vec{u} = \langle 1, 3 \rangle$, $\vec{v} = \langle 2, 1 \rangle$ and $\vec{u} + \vec{v}$ all with initial point at the origin.

Solution We first compute $\vec{u} + \vec{v}$.

$$\begin{aligned} \vec{u} + \vec{v} &= \langle 1, 3 \rangle + \langle 2, 1 \rangle \\ &= \langle 3, 4 \rangle . \end{aligned}$$

These are all sketched in Figure 10.2.4.

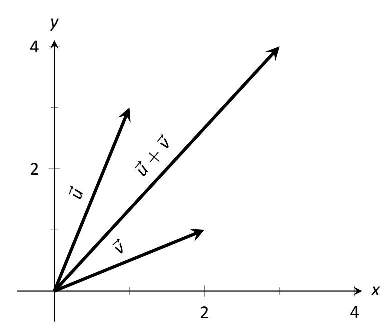


Figure 10.2.4: Graphing the sum of vectors in Example 10.2.2.

As vectors convey magnitude and direction information, the sum of vectors also convey length and magnitude information. Adding $\vec{u} + \vec{v}$ suggests the following idea:

"Starting at an initial point, go out $\vec{u}$, then go out $\vec{v}$."

Notes:

This idea is sketched in Figure 10.2.5, where the initial point of $\vec{v}$ is the terminal point of $\vec{u}$. This is known as the "Head to Tail Rule" of adding vectors. Vector addition is very important. For instance, if the vectors $\vec{u}$ and $\vec{v}$ represent forces acting on a body, the sum $\vec{u}+\vec{v}$ gives the resulting force. Because of various physical applications of vector addition, the sum $\vec{u}+\vec{v}$ is often referred to as the **resultant vector**, or just the "resultant."

Analytically, it is easy to see that $\vec{u}+\vec{v}=\vec{v}+\vec{u}$. Figure 10.2.5 also gives a graphical representation of this, using gray vectors. Note that the vectors $\vec{u}$ and $\vec{v}$, when arranged as in the figure, form a parallelogram. Because of this, the Head to Tail Rule is also known as the Parallelogram Law: the vector $\vec{u}+\vec{v}$ is defined by forming the parallelogram defined by the vectors $\vec{u}$ and $\vec{v}$; the initial point of $\vec{u}+\vec{v}$ is the common initial point of parallelogram, and the terminal point of the sum is the common terminal point of the parallelogram.

While not illustrated here, the Head to Tail Rule and Parallelogram Law hold for vectors in $\mathbb{R}^3$ as well.

It follows from the properties of the real numbers and Definition 10.2.3 that

$$\vec{u}-\vec{v}=\vec{u}+(-1)\vec{v}.$$

The Parallelogram Law gives us a good way to visualize this subtraction. We demonstrate this in the following example.

Example 10.2.3 Vector Subtraction

Let $\vec{u}=\langle 3,1\rangle$ and $\vec{v}=\langle 1,2\rangle$. Compute and sketch $\vec{u}-\vec{v}$.

Solution The computation of $\vec{u}-\vec{v}$ is straightforward, and we show all steps below. Usually the formal step of multiplying by (-1) is omitted and we "just subtract."

$$\begin{aligned}\vec{u}-\vec{v}&=\vec{u}+(-1)\vec{v}\\&=\langle 3,1\rangle+\langle -1,-2\rangle\\&=\langle 2,-1\rangle .\end{aligned}$$

Figure 10.2.6 illustrates, using the Head to Tail Rule, how the subtraction can be viewed as the sum $\vec{u}+(-\vec{v})$. The figure also illustrates how $\vec{u}-\vec{v}$ can be obtained by looking only at the terminal points of $\vec{u}$ and $\vec{v}$ (when their initial points are the same).

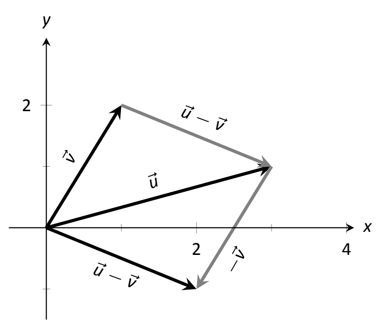


Figure 10.2.6: Illustrating how to subtract vectors graphically.

Example 10.2.4 Scaling vectors

1. Sketch the vectors $\vec{v}=\langle 2,1\rangle$ and $2\vec{v}$ with initial point at the origin.
2. Compute the magnitudes of $\vec{v}$ and $2\vec{v}$.

Notes:

Solution

1. We compute $2\vec{v}$:

$$\begin{aligned} 2\vec{v} &= 2\langle 2,1\rangle \\ &= \langle 4,2\rangle . \end{aligned}$$

Both $\vec{v}$ and $2\vec{v}$ are sketched in Figure 10.2.7. Make note that $2\vec{v}$ does not start at the terminal point of $\vec{v}$; rather, its initial point is also the origin.

2. The figure suggests that $2\vec{v}$ is twice as long as $\vec{v}$. We compute their magnitudes to confirm this.

$$\begin{aligned} ||\,\vec{v}\,|| &= \sqrt{2^2+1^2} \\ &= \sqrt{5}. \\ ||\,2\vec{v}\,|| &= \sqrt{4^2+2^2} \\ &= \sqrt{20} \\ &= \sqrt{4\cdot 5} = 2\sqrt{5}. \end{aligned}$$

As we suspected, $2\vec{v}$ is twice as long as $\vec{v}$.

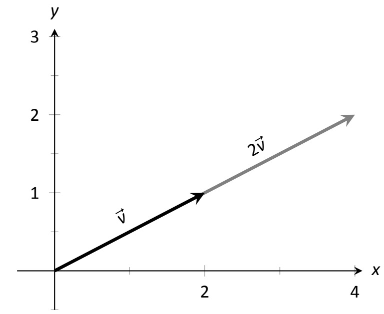


Figure 10.2.7: Graphing vectors $\vec{v}$ and $2\vec{v}$ in Example 10.2.4.

The **zero vector** is the vector whose initial point is also its terminal point. It is denoted by $\vec{0}$. Its component form, in $\mathbb{R}^2$, is $\langle 0,0\rangle$; in $\mathbb{R}^3$, it is $\langle 0,0,0\rangle$. Usually the context makes is clear whether $\vec{0}$ is referring to a vector in the plane or in space.

Our examples have illustrated key principles in vector algebra: how to add and subtract vectors and how to multiply vectors by a scalar. The following theorem states formally the properties of these operations.

Notes:

Theorem 10.2.1 Properties of Vector Operations

The following are true for all scalars c and d, and for all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all in $\mathbb{R}^2$ or where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all in $\mathbb{R}^3$:

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ — Commutative Property
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$ — Associative Property
3. $\vec{v} + \vec{0} = \vec{v}$ — Additive Identity
4. $(cd)\vec{v} = c(d\vec{v})$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$ — Distributive Property
6. $(c + d)\vec{v} = c\vec{v} + d\vec{v}$ — Distributive Property
7. $0\vec{v} = \vec{0}$
8. $||\, c\vec{v}\, || = |c| \cdot ||\, \vec{v}\, ||$
9. $||\, \vec{u}\, || = 0$ if, and only if, $\vec{u} = \vec{0}$.

As stated before, each nonvector $\vec{v}$ conveys magnitude and direction information. We have a method of extracting the magnitude, which we write as $||\, \vec{v}\, ||$. *Unit vectors* are a way of extracting just the direction information from a vector.

Definition 10.2.4 Unit Vector

A **unit vector** is a vector $\vec{v}$ with a magnitude of 1; that is,

$$||\, \vec{v}\, || = 1.$$

Consider this scenario: you are given a vector $\vec{v}$ and are told to create a vector of length 10 in the direction of $\vec{v}$. How does one do that? If we knew that $\vec{u}$ was the unit vector in the direction of $\vec{v}$, the answer would be easy: $10\vec{u}$. So how do we find $\vec{u}$?

Property 8 of Theorem 10.2.1 holds the key. If we divide $\vec{v}$ by its magnitude, it becomes a vector of length 1. Consider:

$$\begin{aligned} \left|\left| \frac{1}{||\, \vec{v}\, ||}\vec{v} \right|\right| &= \frac{1}{||\, \vec{v}\, ||}||\, \vec{v}\, || \qquad \text{(we can pull out } \tfrac{1}{||\, \vec{v}\, ||} \text{ as it is a positive scalar)} \\ &= 1. \end{aligned}$$

Notes:

So the vector of length 10 in the direction of $\vec{v}$ is $10\dfrac{1}{||\,\vec{v}\,||}\vec{v}$. An example will make this more clear.

Example 10.2.5 Using Unit Vectors

Let $\vec{v} = \langle 3, 1\rangle$ and let $\vec{w} = \langle 1, 2, 2\rangle$.

1. Find the unit vector in the direction of $\vec{v}$.
2. Find the unit vector in the direction of $\vec{w}$.
3. Find the vector in the direction of $\vec{v}$ with magnitude 5.

Solution

1. We find $||\,\vec{v}\,|| = \sqrt{10}$. So the unit vector $\vec{u}$ in the direction of $\vec{v}$ is

$$\vec{u} = \frac{1}{\sqrt{10}}\vec{v} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right\rangle.$$

2. We find $||\,\vec{w}\,|| = 3$, so the unit vector $\vec{z}$ in the direction of $\vec{w}$ is

$$\vec{u} = \frac{1}{3}\vec{w} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle.$$

3. To create a vector with magnitude 5 in the direction of $\vec{v}$, we multiply the unit vector $\vec{u}$ by 5. Thus $5\vec{u} = \langle 15/\sqrt{10}, 5/\sqrt{10}\rangle$ is the vector we seek. This is sketched in Figure 10.2.8.

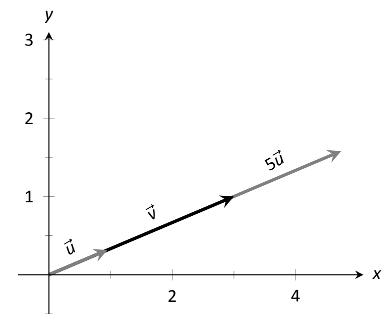


Figure 10.2.8: Graphing vectors in Example 10.2.5. All vectors shown have their initial point at the origin.

The basic formation of the unit vector $\vec{u}$ in the direction of a vector $\vec{v}$ leads to a interesting equation. It is:

$$\vec{v} = ||\,\vec{v}\,||\frac{1}{||\,\vec{v}\,||}\vec{v}.$$

We rewrite the equation with parentheses to make a point:

$$\vec{v} = \underbrace{||\,\vec{v}\,||}_{\text{magnitude}} \cdot \underbrace{\left(\frac{1}{||\,\vec{v}\,||}\vec{v}\right)}_{\text{direction}}.$$

This equation illustrates the fact that a nonzero vector has both magnitude and direction, where we view a unit vector as supplying *only* direction information. Identifying unit vectors with direction allows us to define **parallel vectors**.

Notes:

Definition 10.2.5 Parallel Vectors

1. Unit vectors $\vec{u}_1$ and $\vec{u}_2$ are **parallel** if $\vec{u}_1 = \pm\vec{u}_2$.
2. Nonzero vectors $\vec{v}_1$ and $\vec{v}_2$ are **parallel** if their respective unit vectors are parallel.

It is equivalent to say that vectors $\vec{v}_1$ and $\vec{v}_2$ are parallel if there is a scalar $c \neq 0$ such that $\vec{v}_1 = c\vec{v}_2$ (see marginal note).

If one graphed all unit vectors in $\mathbb{R}^2$ with the initial point at the origin, then the terminal points would all lie on the unit circle. Based on what we know from trigonometry, we can then say that the component form of all unit vectors in $\mathbb{R}^2$ is $\langle \cos\theta, \sin\theta \rangle$ for some angle θ.

A similar construction in $\mathbb{R}^3$ shows that the terminal points all lie on the unit sphere. These vectors also have a particular component form, but its derivation is not as straightforward as the one for unit vectors in $\mathbb{R}^2$. Important concepts about unit vectors are given in the following Key Idea.

Note: $\vec{0}$ is directionless; because $|| \vec{0} || = 0$, there is no unit vector in the "direction" of $\vec{0}$.

Some texts define two vectors as being parallel if one is a scalar multiple of the other. By this definition, $\vec{0}$ is parallel to all vectors as $\vec{0} = 0\vec{v}$ for all $\vec{v}$.

We define what it means for two vectors to be perpendicular in Definition 10.3.2, which is written to exclude $\vec{0}$. It could be written to include $\vec{0}$; by such a definition, $\vec{0}$ is perpendicular to all vectors. While counter-intuitive, it is mathematically sound to allow $\vec{0}$ to be both parallel and perpendicular to all vectors.

We prefer the given definition of parallel as it is grounded in the fact that unit vectors provide direction information. One may adopt the convention that $\vec{0}$ is parallel to all vectors if they desire. (See also the marginal note on page 604.)

Key Idea 10.2.1 Unit Vectors

1. The unit vector in the direction of a nonzero vector $\vec{v}$ is
$$\vec{u} = \frac{1}{|| \vec{v} ||}\vec{v}.$$
2. A vector $\vec{u}$ in $\mathbb{R}^2$ is a unit vector if, and only if, its component form is $\langle \cos\theta, \sin\theta \rangle$ for some angle θ.
3. A vector $\vec{u}$ in $\mathbb{R}^3$ is a unit vector if, and only if, its component form is $\langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta \rangle$ for some angles θ and φ.

These formulas can come in handy in a variety of situations, especially the formula for unit vectors in the plane.

Example 10.2.6 Finding Component Forces

Consider a weight of 50lb hanging from two chains, as shown in Figure 10.2.9. One chain makes an angle of 30° with the vertical, and the other an angle of 45°. Find the force applied to each chain.

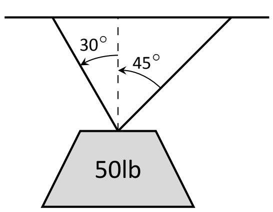


Figure 10.2.9: A diagram of a weight hanging from 2 chains in Example 10.2.6.

Solution Knowing that gravity is pulling the 50lb weight straight down,

Notes:

we can create a vector $\vec{F}$ to represent this force.

$$\vec{F} = 50\langle 0, -1\rangle = \langle 0, -50\rangle .$$

We can view each chain as "pulling" the weight up, preventing it from falling. We can represent the force from each chain with a vector. Let $\vec{F_1}$ represent the force from the chain making an angle of 30° with the vertical, and let $\vec{F_2}$ represent the force form the other chain. Convert all angles to be measured from the horizontal (as shown in Figure 10.2.10), and apply Key Idea 10.2.1. As we do not yet know the magnitudes of these vectors, (that is the problem at hand), we use m_1 and m_2 to represent them.

$$\vec{F_1} = m_1\langle \cos 120^\circ, \sin 120^\circ\rangle$$

$$\vec{F_2} = m_2\langle \cos 45^\circ, \sin 45^\circ\rangle$$

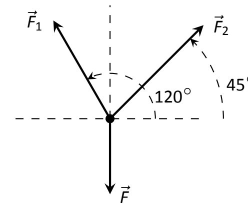


Figure 10.2.10: A diagram of the force vectors from Example 10.2.6.

As the weight is not moving, we know the sum of the forces is $\vec{0}$. This gives:

$$\begin{aligned} \vec{F} + \vec{F_1} + \vec{F_2} &= \vec{0} \\ \langle 0, -50\rangle + m_1\langle \cos 120^\circ, \sin 120^\circ\rangle + m_2\langle \cos 45^\circ, \sin 45^\circ\rangle &= \vec{0} \end{aligned}$$

The sum of the entries in the first component is 0, and the sum of the entries in the second component is also 0. This leads us to the following two equations:

$$\begin{aligned} m_1 \cos 120^\circ + m_2 \cos 45^\circ &= 0 \\ m_1 \sin 120^\circ + m_2 \sin 45^\circ &= 50 \end{aligned}$$

This is a simple 2-equation, 2-unkown system of linear equations. We leave it to the reader to verify that the solution is

$$m_1 = 50(\sqrt{3} - 1) \approx 36.6; \qquad m_2 = \frac{50\sqrt{2}}{1 + \sqrt{3}} \approx 25.88.$$

It might seem odd that the sum of the forces applied to the chains is more than 50lb. We leave it to a physics class to discuss the full details, but offer this short explanation. Our equations were established so that the *vertical* components of each force sums to 50lb, thus supporting the weight. Since the chains are at an angle, they also pull against each other, creating an "additional" horizontal force while holding the weight in place.

Unit vectors were very important in the previous calculation; they allowed us to define a vector in the proper direction but with an unknown magnitude. Our computations were then computed component–wise. Because such calculations are often necessary, the *standard unit vectors* can be useful.

Notes:

Definition 10.2.6 Standard Unit Vectors

1. In $\mathbb{R}^2$, the standard unit vectors are
$$\vec{i} = \langle 1, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1 \rangle .$$
2. In $\mathbb{R}^3$, the standard unit vectors are
$$\vec{i} = \langle 1, 0, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1, 0 \rangle \quad \text{and} \quad \vec{k} = \langle 0, 0, 1 \rangle .$$

Example 10.2.7 Using standard unit vectors

1. Rewrite $\vec{v} = \langle 2, -3 \rangle$ using the standard unit vectors.
2. Rewrite $\vec{w} = 4\vec{i} - 5\vec{j} + 2\vec{k}$ in component form.

Solution

1.
$$\begin{aligned} \vec{v} &= \langle 2, -3 \rangle \\ &= \langle 2, 0 \rangle + \langle 0, -3 \rangle \\ &= 2\langle 1, 0 \rangle - 3\langle 0, 1 \rangle \\ &= 2\vec{i} - 3\vec{j} \end{aligned}$$

2.
$$\begin{aligned} \vec{w} &= 4\vec{i} - 5\vec{j} + 2\vec{k} \\ &= \langle 4, 0, 0 \rangle + \langle 0, -5, 0 \rangle + \langle 0, 0, 2 \rangle \\ &= \langle 4, -5, 2 \rangle \end{aligned}$$

These two examples demonstrate that converting between component form and the standard unit vectors is rather straightforward. Many mathematicians prefer component form, and it is the preferred notation in this text. Many engineers prefer using the standard unit vectors, and many engineering text use that notation.

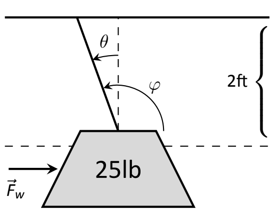


Figure 10.2.11: A figure of a weight being pushed by the wind in Example 10.2.8.

Example 10.2.8 Finding Component Force

A weight of 25lb is suspended from a chain of length 2ft while a wind pushes the weight to the right with constant force of 5lb as shown in Figure 10.2.11. What angle will the chain make with the vertical as a result of the wind's pushing? How much higher will the weight be?

Notes:

Solution The force of the wind is represented by the vector $\vec{F}_w = 5\vec{i}$. The force of gravity on the weight is represented by $\vec{F}_g = -25\vec{j}$. The direction and magnitude of the vector representing the force on the chain are both unknown. We represent this force with

$$\vec{F}_c = m\left\langle \cos\varphi, \sin\varphi\right\rangle = m\cos\varphi\,\vec{i} + m\sin\varphi\,\vec{j}$$

for some magnitude m and some angle with the horizontal φ. (Note: θ is the angle the chain makes with the *vertical*; φ is the angle with the *horizontal*.)

As the weight is at equilibrium, the sum of the forces is $\vec{0}$:

$$\begin{aligned} \vec{F}_c + \vec{F}_w + \vec{F}_g &= \vec{0} \\ m\cos\varphi\,\vec{i} + m\sin\varphi\,\vec{j} + 5\vec{i} - 25\vec{j} &= \vec{0} \end{aligned}$$

Thus the sum of the $\vec{i}$ and $\vec{j}$ components are 0, leading us to the following system of equations:

$$\begin{aligned} 5 + m\cos\varphi &= 0 \\ -25 + m\sin\varphi &= 0 \end{aligned} \qquad (10.1)$$

This is enough to determine $\vec{F}_c$ already, as we know $m\cos\varphi = -5$ and $m\sin\varphi = 25$. Thus $F_c = \langle -5, 25\rangle$. We can use this to find the magnitude m:

$$m = \sqrt{(-5)^2 + 25^2} = 5\sqrt{26} \approx 25.5\text{lb}.$$

We can then use either equality from Equation (10.1) to solve for φ. We choose the first equality as using arccosine will return an angle in the 2$^{\text{nd}}$ quadrant:

$$5 + 5\sqrt{26}\cos\varphi = 0 \quad \Rightarrow \quad \varphi = \cos^{-1}\left(\frac{-5}{5\sqrt{26}}\right) \approx 1.7682 \approx 101.31^\circ.$$

Subtracting 90° from this angle gives us an angle of 11.31° with the vertical.

We can now use trigonometry to find out how high the weight is lifted. The diagram shows that a right triangle is formed with the 2ft chain as the hypotenuse with an interior angle of 11.31°. The length of the adjacent side (in the diagram, the dashed vertical line) is $2\cos 11.31^\circ \approx 1.96$ft. Thus the weight is lifted by about 0.04ft, almost 1/2in.

The algebra we have applied to vectors is already demonstrating itself to be very useful. There are two more fundamental operations we can perform with vectors, the *dot product* and the *cross product*. The next two sections explore each in turn.

Notes:

Exercises 10.2

Terms and Concepts

1. Name two different things that cannot be described with just one number, but rather need 2 or more numbers to fully describe them.

2. What is the difference between $(1, 2)$ and $\langle 1, 2\rangle$?

3. What is a unit vector?

4. Unit vectors can be thought of as conveying what type of information?

5. What does it mean for two vectors to be parallel?

6. What effect does multiplying a vector by -2 have?

Problems

In Exercises 7 – 10, points P and Q are given. Write the vector $\overrightarrow{PQ}$ in component form and using the standard unit vectors.

7. $P = (2, -1), \quad Q = (3, 5)$

8. $P = (3, 2), \quad Q = (7, -2)$

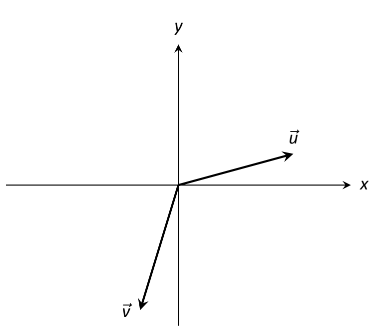
9. $P = (0, 3, -1), \quad Q = (6, 2, 5)$

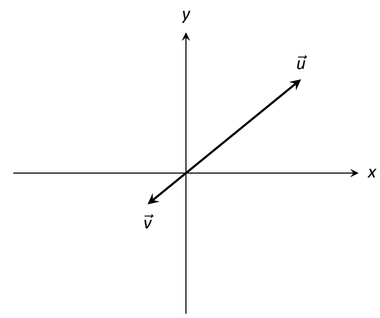
10. $P = (2, 1, 2), \quad Q = (4, 3, 2)$

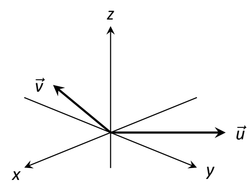
11. Let $\vec{u} = \langle 1, -2\rangle$ and $\vec{v} = \langle 1, 1\rangle$.
    (a) Find $\vec{u}+\vec{v}$, $\vec{u}-\vec{v}$, $2\vec{u}-3\vec{v}$.
    (b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.
    (c) Find $\vec{x}$ where $\vec{u}+\vec{x} = 2\vec{v}-\vec{x}$.

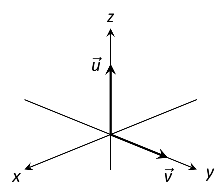
12. Let $\vec{u} = \langle 1, 1, -1\rangle$ and $\vec{v} = \langle 2, 1, 2\rangle$.
    (a) Find $\vec{u}+\vec{v}$, $\vec{u}-\vec{v}$, $\pi\vec{u}-\sqrt{2}\vec{v}$.
    (b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.
    (c) Find $\vec{x}$ where $\vec{u}+\vec{x} = \vec{v}+2\vec{x}$.

In Exercises 13 – 16, sketch $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ on the same axes.

13.


14.


15.


16.


In Exercises 17 – 20, find $\|\vec{u}\|$, $\|\vec{v}\|$, $\|\vec{u}+\vec{v}\|$ and $\|\vec{u}-\vec{v}\|$.

17. $\vec{u} = \langle 2, 1\rangle, \quad \vec{v} = \langle 3, -2\rangle$

18. $\vec{u} = \langle -3, 2, 2\rangle, \quad \vec{v} = \langle 1, -1, 1\rangle$

19. $\vec{u} = \langle 1, 2\rangle, \quad \vec{v} = \langle -3, -6\rangle$

20. $\vec{u} = \langle 2, -3, 6\rangle, \quad \vec{v} = \langle 10, -15, 30\rangle$

21. Under what conditions is $\|\vec{u}\| + \|\vec{v}\| = \|\vec{u}+\vec{v}\|$?

In Exercises 22 – 25, find the unit vector $\vec{u}$ in the direction of $\vec{v}$.

22. $\vec{v} = \langle 3, 7\rangle$

23. $\vec{v} = \langle 6, 8\rangle$

24. $\vec{v} = \langle 1, -2, 2\rangle$

25. $\vec{v} = \langle 2, -2, 2\rangle$

26. Find the unit vector in the first quadrant of $\mathbb{R}^2$ that makes a 50° angle with the x-axis.

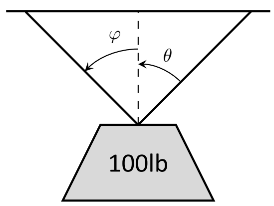
27. Find the unit vector in the second quadrant of $\mathbb{R}^2$ that makes a 30° angle with the y-axis.

28. Verify, from Key Idea 10.2.1, that

$$\vec{u} = \langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\rangle$$

is a unit vector for all angles θ and φ.

A weight of 100lb is suspended from two chains, making angles with the vertical of θ and φ as shown in the figure below.



In Exercises 29 – 32, angles θ and φ are given. Find the magnitude of the force applied to each chain.

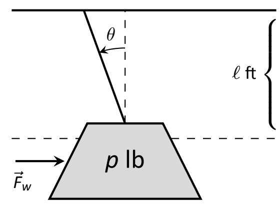
29. $\theta = 30^\circ, \quad \varphi = 30^\circ$

30. $\theta = 60^\circ, \quad \varphi = 60^\circ$

31. $\theta = 20^\circ, \quad \varphi = 15^\circ$

32. $\theta = 0^\circ, \quad \varphi = 0^\circ$

A weight of $p$lb is suspended from a chain of length ℓ while a constant force of $\vec{F}_w$ pushes the weight to the right, making an angle of θ with the vertical, as shown in the figure below.



In Exercises 33 – 36, a force $\vec{F}_w$ and length ℓ are given. Find the angle θ and the height the weight is lifted as it moves to the right.

33. $\vec{F}_w = 1\text{lb}, \quad \ell = 1\text{ft}, \quad p = 1\text{lb}$

34. $\vec{F}_w = 1\text{lb}, \quad \ell = 1\text{ft}, \quad p = 10\text{lb}$

35. $\vec{F}_w = 1\text{lb}, \quad \ell = 10\text{ft}, \quad p = 1\text{lb}$

36. $\vec{F}_w = 10\text{lb}, \quad \ell = 10\text{ft}, \quad p = 1\text{lb}$

10.3 The Dot Product

The previous section introduced vectors and described how to add them together and how to multiply them by scalars. This section introduces *a* multiplication on vectors called the **dot product**.

Definition 10.3.1 Dot Product

1. Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ in $\mathbb{R}^2$. The **dot product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is
$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2.$$
2. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$. The **dot product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is
$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Note how this product of vectors returns a *scalar*, not another vector. We practice evaluating a dot product in the following example, then we will discuss why this product is useful.

Example 10.3.1 Evaluating dot products

1. Let $\vec{u} = \langle 1, 2 \rangle$, $\vec{v} = \langle 3, -1 \rangle$ in $\mathbb{R}^2$. Find $\vec{u} \cdot \vec{v}$.
2. Let $\vec{x} = \langle 2, -2, 5 \rangle$ and $\vec{y} = \langle -1, 0, 3 \rangle$ in $\mathbb{R}^3$. Find $\vec{x} \cdot \vec{y}$.

Solution

1. Using Definition 10.3.1, we have
$$\vec{u} \cdot \vec{v} = 1(3) + 2(-1) = 1.$$
2. Using the definition, we have
$$\vec{x} \cdot \vec{y} = 2(-1) - 2(0) + 5(3) = 13.$$

The dot product, as shown by the preceding example, is very simple to evaluate. It is only the sum of products. While the definition gives no hint as to why

Notes:

we would care about this operation, there is an amazing connection between the dot product and angles formed by the vectors. Before stating this connection, we give a theorem stating some of the properties of the dot product.

Theorem 10.3.1 Properties of the Dot Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let c be a scalar.

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ Commutative Property
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ Distributive Property
3. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
4. $\vec{0} \cdot \vec{v} = 0$
5. $\vec{v} \cdot \vec{v} = || \vec{v} ||^2$

The last statement of the theorem makes a handy connection between the magnitude of a vector and the dot product with itself. Our definition and theorem give properties of the dot product, but we are still likely wondering "What does the dot product *mean*?" It is helpful to understand that the dot product of a vector with itself is connected to its magnitude.

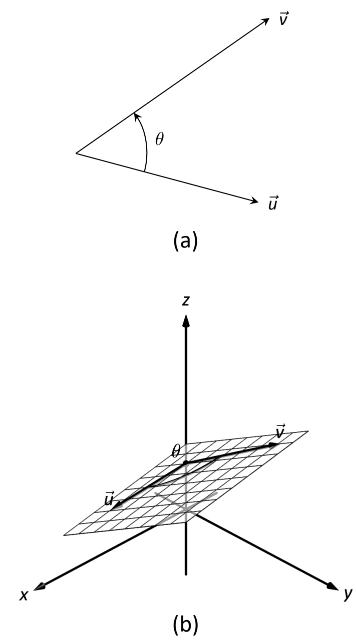


Figure 10.3.1: Illustrating the angle formed by two vectors with the same initial point.

The next theorem extends this understanding by connecting the dot product to magnitudes and angles. Given vectors $\vec{u}$ and $\vec{v}$ in the plane, an angle θ is clearly formed when $\vec{u}$ and $\vec{v}$ are drawn with the same initial point as illustrated in Figure 10.3.1(a). (We always take θ to be the angle in $[0, \pi]$ as two angles are actually created.)

The same is also true of 2 vectors in space: given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with the same initial point, there is a plane that contains both $\vec{u}$ and $\vec{v}$. (When $\vec{u}$ and $\vec{v}$ are co-linear, there are infinitely many planes that contain both vectors.) In that plane, we can again find an angle θ between them (and again, $0 \leq \theta \leq \pi$). This is illustrated in Figure 10.3.1(b).

The following theorem connects this angle θ to the dot product of $\vec{u}$ and $\vec{v}$.

Notes:

Theorem 10.3.2 The Dot Product and Angles

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then

$$\vec{u} \cdot \vec{v} = ||\,\vec{u}\,||\,||\,\vec{v}\,|| \cos\theta,$$

where θ, $0 \leq \theta \leq \pi$, is the angle between $\vec{u}$ and $\vec{v}$.

Using Theorem 10.3.1, we can rewrite this theorem as

$$\frac{\vec{u}}{||\,\vec{u}\,||} \cdot \frac{\vec{v}}{||\,\vec{v}\,||} = \cos\theta.$$

Note how on the left hand side of the equation, we are computing the dot product of two unit vectors. Recalling that unit vectors essentially only provide direction information, we can informally restate Theorem 10.3.2 as saying "The dot product of two directions gives the cosine of the angle between them."

When θ is an acute angle (i.e., $0 \leq \theta < \pi/2$), $\cos\theta$ is positive; when $\theta = \pi/2$, $\cos\theta = 0$; when θ is an obtuse angle ($\pi/2 < \theta \leq \pi$), $\cos\theta$ is negative. Thus the sign of the dot product gives a general indication of the angle between the vectors, illustrated in Figure 10.3.2.

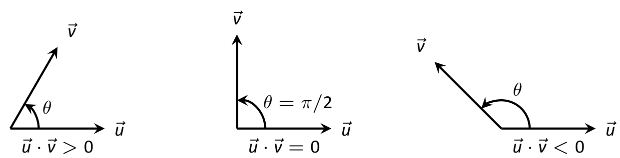


Figure 10.3.2: Illustrating the relationship between the angle between vectors and the sign of their dot product.

We *can* use Theorem 10.3.2 to compute the dot product, but generally this theorem is used to find the angle between known vectors (since the dot product is generally easy to compute). To this end, we rewrite the theorem's equation as

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\,\vec{u}\,||||\,\vec{v}\,||} \quad \Leftrightarrow \quad \theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{||\,\vec{u}\,||||\,\vec{v}\,||}\right).$$

We practice using this theorem in the following example.

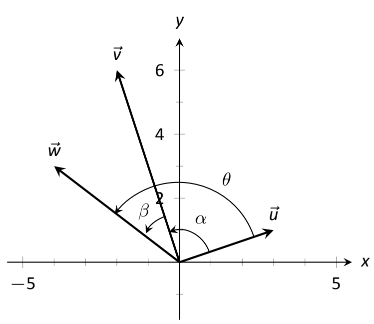


Figure 10.3.3: Vectors used in Example 10.3.2.

Example 10.3.2 Using the dot product to find angles

Let $\vec{u} = \langle 3, 1\rangle$, $\vec{v} = \langle -2, 6\rangle$ and $\vec{w} = \langle -4, 3\rangle$, as shown in Figure 10.3.3. Find the angles α, β and θ.

Notes:

Solution We start by computing the magnitude of each vector.

$$||\vec{u}|| = \sqrt{10}; \quad ||\vec{v}|| = 2\sqrt{10}; \quad ||\vec{w}|| = 5.$$

We now apply Theorem 10.3.2 to find the angles.

$$\begin{aligned}\alpha &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{(\sqrt{10})(2\sqrt{10})}\right)\\ &= \cos^{-1}(0) = \frac{\pi}{2} = 90^\circ.\end{aligned}$$

$$\begin{aligned}\beta &= \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{(2\sqrt{10})(5)}\right)\\ &= \cos^{-1}\left(\frac{26}{10\sqrt{10}}\right)\\ &\approx 0.6055 \approx 34.7^\circ.\end{aligned}$$

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{w}}{(\sqrt{10})(5)}\right)\\ &= \cos^{-1}\left(\frac{-9}{5\sqrt{10}}\right)\\ &\approx 2.1763 \approx 124.7^\circ\end{aligned}$$

We see from our computation that $\alpha+\beta=\theta$, as indicated by Figure 10.3.3. While we knew this should be the case, it is nice to see that this non-intuitive formula indeed returns the results we expected.

We do a similar example next in the context of vectors in space.

Example 10.3.3 Using the dot product to find angles

Let $\vec{u} = \langle 1,1,1\rangle$, $\vec{v} = \langle -1,3,-2\rangle$ and $\vec{w} = \langle -5,1,4\rangle$, as illustrated in Figure 10.3.4. Find the angle between each pair of vectors.

Solution

1. Between $\vec{u}$ and $\vec{v}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||||\vec{v}||}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{3}\sqrt{14}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

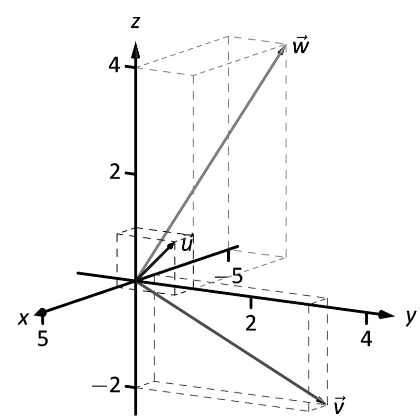


Figure 10.3.4: Vectors used in Example 10.3.3.

Notes:

2. Between $\vec{u}$ and $\vec{w}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{w}}{||\,\vec{u}\,||||\,\vec{w}\,||}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{3}\sqrt{42}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

3. Between $\vec{v}$ and $\vec{w}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{||\,\vec{v}\,||||\,\vec{w}\,||}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{14}\sqrt{42}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

While our work shows that each angle is $\pi/2$, i.e., 90°, none of these angles looks to be a right angle in Figure 10.3.4. Such is the case when drawing three–dimensional objects on the page.

All three angles between these vectors was $\pi/2$, or 90°. We know from geometry and everyday life that 90° angles are "nice" for a variety of reasons, so it should seem significant that these angles are all $\pi/2$. Notice the common feature in each calculation (and also the calculation of α in Example 10.3.2): the dot products of each pair of angles was 0. We use this as a basis for a definition of the term **orthogonal**, which is essentially synonymous to *perpendicular*.

Note: The term *perpendicular* originally referred to lines. As mathematics progressed, the concept of "being at right angles to" was applied to other objects, such as vectors and planes, and the term *orthogonal* was introduced. It is especially used when discussing objects that are hard, or impossible, to visualize: two vectors in 5-dimensional space are orthogonal if their dot product is 0. It is not wrong to say they are *perpendicular*, but common convention gives preference to the word *orthogonal*.

Definition 10.3.2 Orthogonal

Nonzero vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if their dot product is 0.

Example 10.3.4 Finding orthogonal vectors

Let $\vec{u} = \langle 3, 5\rangle$ and $\vec{v} = \langle 1, 2, 3\rangle$.

1. Find two vectors in $\mathbb{R}^2$ that are orthogonal to $\vec{u}$.
2. Find two non–parallel vectors in $\mathbb{R}^3$ that are orthogonal to $\vec{v}$.

Solution

Notes:

1. Recall that a line perpendicular to a line with slope m has slope $-1/m$, the "opposite reciprocal slope." We can think of the slope of $\vec{u}$ as $5/3$, its "rise over run." A vector orthogonal to $\vec{u}$ will have slope $-3/5$. There are many such choices, though all parallel:

$$\langle -5, 3\rangle \quad \text{or} \quad \langle 5, -3\rangle \quad \text{or} \quad \langle -10, 6\rangle \quad \text{or} \quad \langle 15, -9\rangle \text{, etc.}$$

2. There are infinitely many directions in space orthogonal to any given direction, so there are an infinite number of non–parallel vectors orthogonal to $\vec{v}$. Since there are so many, we have great leeway in finding some.

   One way is to arbitrarily pick values for the first two components, leaving the third unknown. For instance, let $\vec{v}_1 = \langle 2, 7, z\rangle$. If $\vec{v}_1$ is to be orthogonal to $\vec{v}$, then $\vec{v}_1 \cdot \vec{v} = 0$, so

$$2 + 14 + 3z = 0 \quad \Rightarrow z = \frac{-16}{3}.$$

   So $\vec{v}_1 = \langle 2, 7, -16/3\rangle$ is orthogonal to $\vec{v}$. We can apply a similar technique by leaving the first or second component unknown.

   Another method of finding a vector orthogonal to $\vec{v}$ mirrors what we did in part 1. Let $\vec{v}_2 = \langle -2, 1, 0\rangle$. Here we switched the first two components of $\vec{v}$, changing the sign of one of them (similar to the "opposite reciprocal" concept before). Letting the third component be 0 effectively ignores the third component of $\vec{v}$, and it is easy to see that

$$\vec{v}_2 \cdot \vec{v} = \langle -2, 1, 0\rangle \cdot \langle 1, 2, 3\rangle = 0.$$

   Clearly $\vec{v}_1$ and $\vec{v}_2$ are not parallel.

An important construction is illustrated in Figure 10.3.5, where vectors $\vec{u}$ and $\vec{v}$ are sketched. In part (a), a dotted line is drawn from the tip of $\vec{u}$ to the line containing $\vec{v}$, where the dotted line is orthogonal to $\vec{v}$. In part (b), the dotted line is replaced with the vector $\vec{z}$ and $\vec{w}$ is formed, parallel to $\vec{v}$. It is clear by the diagram that $\vec{u} = \vec{w} + \vec{z}$. What is important about this construction is this: $\vec{u}$ is *decomposed* as the sum of two vectors, one of which is parallel to $\vec{v}$ and one that is perpendicular to $\vec{v}$. It is hard to overstate the importance of this construction (as we'll see in upcoming examples).

The vectors $\vec{w}$, $\vec{z}$ and $\vec{u}$ as shown in Figure 10.3.5 (b) form a right triangle, where the angle between $\vec{v}$ and $\vec{u}$ is labeled θ. We can find $\vec{w}$ in terms of $\vec{v}$ and $\vec{u}$.

Using trigonometry, we can state that

$$\|\,\vec{w}\,\| = \|\,\vec{u}\,\| \cos\theta. \tag{10.2}$$

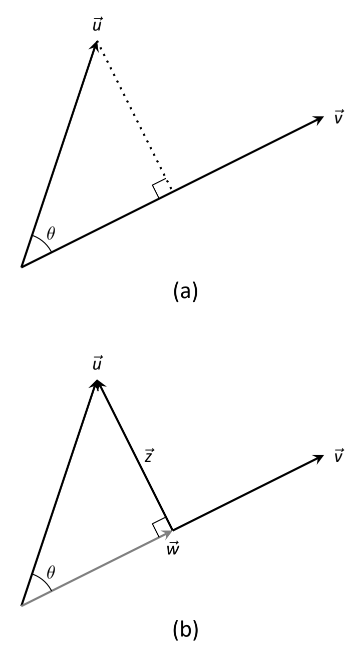


Figure 10.3.5: Developing the construction of the *orthogonal projection*.

Notes:

We also know that $\vec{w}$ is parallel to to $\vec{v}$; that is, the direction of $\vec{w}$ is the direction of $\vec{v}$, described by the unit vector $\vec{v}/\|\,\vec{v}\,\|$. The vector $\vec{w}$ is the vector in the direction $\vec{v}/\|\,\vec{v}\,\|$ with magnitude $\|\,\vec{u}\,\|\cos\theta$:

$$\vec{w} = \Big(\|\,\vec{u}\,\|\cos\theta\Big)\frac{1}{\|\,\vec{v}\,\|}\vec{v}.$$

Replace $\cos\theta$ using Theorem 10.3.2:

$$\begin{aligned} &= \left(\|\,\vec{u}\,\|\frac{\vec{u}\cdot\vec{v}}{\|\,\vec{u}\,\|\|\,\vec{v}\,\|}\right)\frac{1}{\|\,\vec{v}\,\|}\vec{v} \\ &= \frac{\vec{u}\cdot\vec{v}}{\|\,\vec{v}\,\|^2}\vec{v}. \end{aligned}$$

Now apply Theorem 10.3.1.

$$= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

Since this construction is so important, it is given a special name.

Definition 10.3.3 Orthogonal Projection

Let nonzero vectors $\vec{u}$ and $\vec{v}$ be given. The **orthogonal projection of $\vec{u}$ onto $\vec{v}$**, denoted $\text{proj}_{\,\vec{v}}\,\vec{u}$, is

$$\text{proj}_{\,\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

Example 10.3.5 Computing the orthogonal projection

1. Let $\vec{u} = \langle -2, 1\rangle$ and $\vec{v} = \langle 3, 1\rangle$. Find $\text{proj}_{\,\vec{v}}\,\vec{u}$, and sketch all three vectors with initial points at the origin.
2. Let $\vec{w} = \langle 2, 1, 3\rangle$ and $\vec{x} = \langle 1, 1, 1\rangle$. Find $\text{proj}_{\,\vec{x}}\,\vec{w}$, and sketch all three vectors with initial points at the origin.

Solution

Notes:

1. Applying Definition 10.3.3, we have

$$\begin{aligned}\text{proj}_{\vec{v}}\,\vec{u} &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\\ &= \frac{-5}{10}\left\langle 3,1\right\rangle\\ &= \left\langle -\frac{3}{2}, -\frac{1}{2}\right\rangle.\end{aligned}$$

   Vectors $\vec{u}$, $\vec{v}$ and $\text{proj}_{\vec{v}}\,\vec{u}$ are sketched in Figure 10.3.6(a). Note how the projection is parallel to $\vec{v}$; that is, it lies on the same line through the origin as $\vec{v}$, although it points in the opposite direction. That is because the angle between $\vec{u}$ and $\vec{v}$ is obtuse (i.e., greater than 90°).

2. Apply the definition:

$$\begin{aligned}\text{proj}_{\vec{x}}\,\vec{w} &= \frac{\vec{w}\cdot\vec{x}}{\vec{x}\cdot\vec{x}}\vec{x}\\ &= \frac{6}{3}\left\langle 1,1,1\right\rangle\\ &= \left\langle 2,2,2\right\rangle.\end{aligned}$$

   These vectors are sketched in Figure 10.3.6(b), and again in part (c) from a different perspective. Because of the nature of graphing these vectors, the sketch in part (b) makes it difficult to recognize that the drawn projection has the geometric properties it should. The graph shown in part (c) illustrates these properties better.

We can use the properties of the dot product found in Theorem 10.3.1 to rearrange the formula found in Definition 10.3.3:

$$\begin{aligned}\text{proj}_{\vec{v}}\,\vec{u} &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\\ &= \frac{\vec{u}\cdot\vec{v}}{\|\,\vec{v}\,\|^2}\vec{v}\\ &= \left(\vec{u}\cdot\frac{\vec{v}}{\|\,\vec{v}\,\|}\right)\frac{\vec{v}}{\|\,\vec{v}\,\|}.\end{aligned}$$

The above formula shows that the orthogonal projection of $\vec{u}$ onto $\vec{v}$ is only concerned with the *direction* of $\vec{v}$, as both instances of $\vec{v}$ in the formula come in the form $\vec{v}/\|\,\vec{v}\,\|$, the unit vector in the direction of $\vec{v}$.

A special case of orthogonal projection occurs when $\vec{v}$ is a unit vector. In this situation, the formula for the orthogonal projection of a vector $\vec{u}$ onto $\vec{v}$ reduces to just $\text{proj}_{\vec{v}}\,\vec{u} = (\vec{u}\cdot\vec{v})\vec{v}$, as $\vec{v}\cdot\vec{v} = 1$.

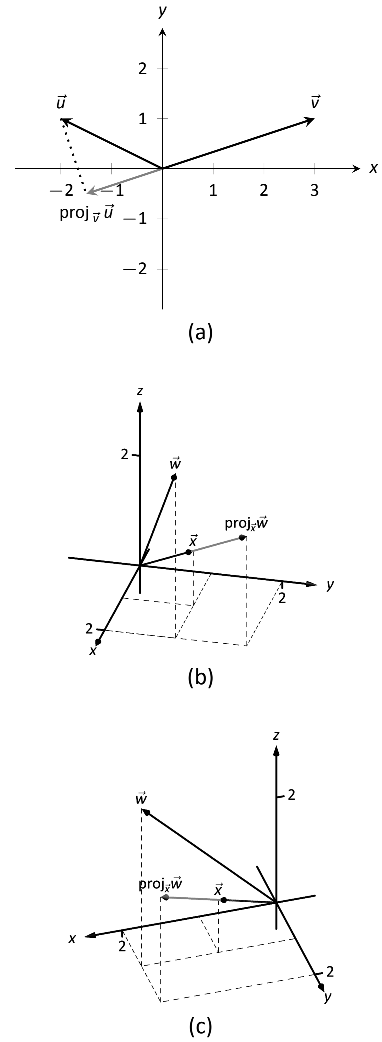


Figure 10.3.6: Graphing the vectors used in Example 10.3.5.

Notes:

This gives us a new understanding of the dot product. When $\vec{v}$ is a unit vector, essentially providing only direction information, the dot product of $\vec{u}$ and $\vec{v}$ gives "how much of $\vec{u}$ is in the direction of $\vec{v}$." This use of the dot product will be very useful in future sections.

Now consider Figure 10.3.7 where the concept of the orthogonal projection is again illustrated. It is clear that

$$\vec{u} = \text{proj}_{\vec{v}}\,\vec{u} + \vec{z}. \tag{10.3}$$

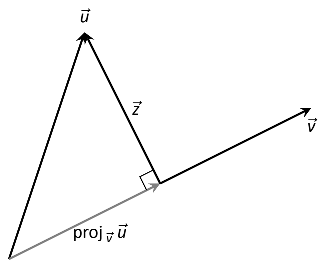


Figure 10.3.7: Illustrating the orthogonal projection.

As we know what $\vec{u}$ and $\text{proj}_{\vec{v}}\,\vec{u}$ are, we can solve for $\vec{z}$ and state that

$$\vec{z} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u}.$$

This leads us to rewrite Equation (10.3) in a seemingly silly way:

$$\vec{u} = \text{proj}_{\vec{v}}\,\vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}).$$

This is not nonsense, as pointed out in the following Key Idea. (Notation note: the expression "$\parallel \vec{y}$" means "is parallel to $\vec{y}$." We can use this notation to state "$\vec{x} \parallel \vec{y}$" which means "$\vec{x}$ is parallel to $\vec{y}$." The expression "$\perp \vec{y}$" means "is orthogonal to $\vec{y}$," and is used similarly.)

Key Idea 10.3.1 Orthogonal Decomposition of Vectors

Let nonzero vectors $\vec{u}$ and $\vec{v}$ be given. Then $\vec{u}$ can be written as the sum of two vectors, one of which is parallel to $\vec{v}$, and one of which is orthogonal to $\vec{v}$:

$$\vec{u} = \underbrace{\text{proj}_{\vec{v}}\,\vec{u}}_{\parallel \vec{v}} + (\underbrace{\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}}_{\perp \vec{v}}).$$

We illustrate the use of this equality in the following example.

Example 10.3.6 Orthogonal decomposition of vectors

1. Let $\vec{u} = \langle -2, 1\rangle$ and $\vec{v} = \langle 3, 1\rangle$ as in Example 10.3.5. Decompose $\vec{u}$ as the sum of a vector parallel to $\vec{v}$ and a vector orthogonal to $\vec{v}$.
2. Let $\vec{w} = \langle 2, 1, 3\rangle$ and $\vec{x} = \langle 1, 1, 1\rangle$ as in Example 10.3.5. Decompose $\vec{w}$ as the sum of a vector parallel to $\vec{x}$ and a vector orthogonal to $\vec{x}$.

Solution

Notes:

1. In Example 10.3.5, we found that $\text{proj}_{\vec v}\,\vec u = \langle -1.5, -0.5\rangle$. Let

$$\vec z = \vec u - \text{proj}_{\vec v}\,\vec u = \langle -2, 1\rangle - \langle -1.5, -0.5\rangle = \langle -0.5, 1.5\rangle\,.$$

   Is $\vec z$ orthogonal to $\vec v$? (I.e, is $\vec z \perp \vec v$?) We check for orthogonality with the dot product:

$$\vec z \cdot \vec v = \langle -0.5, 1.5\rangle \cdot \langle 3, 1\rangle = 0.$$

   Since the dot product is 0, we know $\vec z \perp \vec v$. Thus:

$$\begin{aligned} \vec u &= \text{proj}_{\vec v}\,\vec u \;+\; (\vec u - \text{proj}_{\vec v}\,\vec u)\\ \langle -2, 1\rangle &= \underbrace{\langle -1.5, -0.5\rangle}_{\parallel \vec v} \;+\; \underbrace{\langle -0.5, 1.5\rangle}_{\perp \vec v}. \end{aligned}$$

2. We found in Example 10.3.5 that $\text{proj}_{\vec x}\,\vec w = \langle 2, 2, 2\rangle$. Applying the Key Idea, we have:

$$\vec z = \vec w - \text{proj}_{\vec x}\,\vec w = \langle 2, 1, 3\rangle - \langle 2, 2, 2\rangle = \langle 0, -1, 1\rangle\,.$$

   We check to see if $\vec z \perp \vec x$:

$$\vec z \cdot \vec x = \langle 0, -1, 1\rangle \cdot \langle 1, 1, 1\rangle = 0.$$

   Since the dot product is 0, we know the two vectors are orthogonal. We now write $\vec w$ as the sum of two vectors, one parallel and one orthogonal to $\vec x$:

$$\begin{aligned} \vec w &= \text{proj}_{\vec x}\,\vec w \;+\; (\vec w - \text{proj}_{\vec x}\,\vec w)\\ \langle 2, 1, 3\rangle &= \underbrace{\langle 2, 2, 2\rangle}_{\parallel \vec x} \;+\; \underbrace{\langle 0, -1, 1\rangle}_{\perp \vec x} \end{aligned}$$

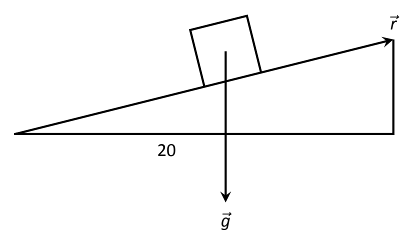
We give an example of where this decomposition is useful.

Example 10.3.7 Orthogonally decomposing a force vector

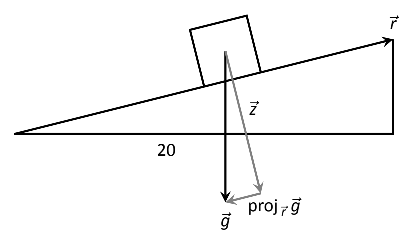
Consider Figure 10.3.8(a), showing a box weighing 50lb on a ramp that rises 5ft over a span of 20ft. Find the components of force, and their magnitudes, acting on the box (as sketched in part (b) of the figure):

1. in the direction of the ramp, and
2. orthogonal to the ramp.

Solution As the ramp rises 5ft over a horizontal distance of 20ft, we can represent the direction of the ramp with the vector $\vec r = \langle 20, 5\rangle$. Gravity pulls down with a force of 50lb, which we represent with $\vec g = \langle 0, -50\rangle$.



(a)



(b)

Figure 10.3.8: Sketching the ramp and box in Example 10.3.7. Note: *The vectors are not drawn to scale.*

Notes:

1. To find the force of gravity in the direction of the ramp, we compute $\text{proj}_{\vec{r}}\,\vec{g}$:

$$\begin{aligned}\text{proj}_{\vec{r}}\,\vec{g} &= \frac{\vec{g}\cdot\vec{r}}{\vec{r}\cdot\vec{r}}\vec{r}\\ &= \frac{-250}{425}\langle 20, 5\rangle\\ &= \left\langle -\frac{200}{17}, -\frac{50}{17}\right\rangle \approx \langle -11.76, -2.94\rangle .\end{aligned}$$

The magnitude of $\text{proj}_{\vec{r}}\,\vec{g}$ is $||\,\text{proj}_{\vec{r}}\,\vec{g}\,|| = 50/\sqrt{17} \approx 12.13\text{lb}$. Though the box weighs 50lb, a force of about 12lb is enough to keep the box from sliding down the ramp.

2. To find the component $\vec{z}$ of gravity orthogonal to the ramp, we use Key Idea 10.3.1.

$$\begin{aligned}\vec{z} &= \vec{g} - \text{proj}_{\vec{r}}\,\vec{g}\\ &= \left\langle \frac{200}{17}, -\frac{800}{17}\right\rangle \approx \langle 11.76, -47.06\rangle .\end{aligned}$$

The magnitude of this force is $||\,\vec{z}\,|| \approx 48.51\text{lb}$. In physics and engineering, knowing this force is important when computing things like static frictional force. (For instance, we could easily compute if the static frictional force alone was enough to keep the box from sliding down the ramp.)

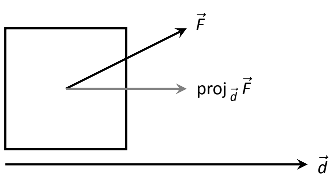


Figure 10.3.9: Finding work when the force and direction of travel are given as vectors.

Application to Work

In physics, the application of a force F to move an object in a straight line a distance d produces *work*; the amount of work W is $W = Fd$, (where F is in the direction of travel). The orthogonal projection allows us to compute work when the force is not in the direction of travel.

Consider Figure 10.3.9, where a force $\vec{F}$ is being applied to an object moving in the direction of $\vec{d}$. (The distance the object travels is the magnitude of $\vec{d}$.) The

Notes:

work done is the amount of force in the direction of $\vec{d}$, $||\text{proj}_{\vec{d}}\vec{F}||$, times $||\vec{d}||$:

$$\begin{aligned}
||\text{proj}_{\vec{d}}\vec{F}|| \cdot ||\vec{d}|| &= \left|\left| \frac{\vec{F}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\vec{d} \right|\right| \cdot ||\vec{d}|| \\
&= \left| \frac{\vec{F}\cdot\vec{d}}{||\vec{d}||^2} \right| \cdot ||\vec{d}|| \cdot ||\vec{d}|| \\
&= \frac{\left|\vec{F}\cdot\vec{d}\right|}{||\vec{d}||^2} ||\vec{d}||^2 \\
&= \left|\vec{F}\cdot\vec{d}\right|.
\end{aligned}$$

The expression $\vec{F}\cdot\vec{d}$ will be positive if the angle between $\vec{F}$ and $\vec{d}$ is acute; when the angle is obtuse (hence $\vec{F}\cdot\vec{d}$ is negative), the force is causing motion in the opposite direction of $\vec{d}$, resulting in "negative work." We want to capture this sign, so we drop the absolute value and find that $W = \vec{F}\cdot\vec{d}$.

Definition 10.3.4 Work

Let $\vec{F}$ be a constant force that moves an object in a straight line from point P to point Q. Let $\vec{d} = \overrightarrow{PQ}$. The **work** W done by $\vec{F}$ along $\vec{d}$ is $W = \vec{F}\cdot\vec{d}$.

Example 10.3.8 Computing work

A man slides a box along a ramp that rises 3ft over a distance of 15ft by applying 50lb of force as shown in Figure 10.3.10. Compute the work done.

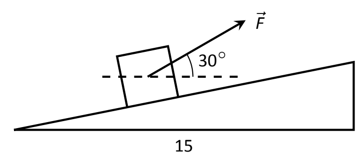


Figure 10.3.10: Computing work when sliding a box up a ramp in Example 10.3.8.

Solution The figure indicates that the force applied makes a 30° angle with the horizontal, so $\vec{F} = 50\langle\cos 30°, \sin 30°\rangle \approx \langle 43.3, 25\rangle$. The ramp is represented by $\vec{d} = \langle 15, 3\rangle$. The work done is simply

$$\vec{F}\cdot\vec{d} = 50\langle\cos 30°, \sin 30°\rangle \cdot \langle 15, 3\rangle \approx 724.5\text{ft–lb}.$$

Note how we did not actually compute the distance the object traveled, nor the magnitude of the force in the direction of travel; this is all inherently computed by the dot product!

The dot product is a powerful way of evaluating computations that depend on angles without actually using angles. The next section explores another "product" on vectors, the *cross product*. Once again, angles play an important role, though in a much different way.

Notes:

Exercises 10.3

Terms and Concepts

1. The dot product of two vectors is a ________, not a vector.

2. How are the concepts of the dot product and vector magnitude related?

3. How can one quickly tell if the angle between two vectors is acute or obtuse?

4. Give a synonym for "orthogonal."

Problems

In Exercises 5 – 10, find the dot product of the given vectors.

5. $\vec{u} = \langle 2, -4\rangle$, $\vec{v} = \langle 3, 7\rangle$

6. $\vec{u} = \langle 5, 3\rangle$, $\vec{v} = \langle 6, 1\rangle$

7. $\vec{u} = \langle 1, -1, 2\rangle$, $\vec{v} = \langle 2, 5, 3\rangle$

8. $\vec{u} = \langle 3, 5, -1\rangle$, $\vec{v} = \langle 4, -1, 7\rangle$

9. $\vec{u} = \langle 1, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

10. $\vec{u} = \langle 1, 2, 3\rangle$, $\vec{v} = \langle 0, 0, 0\rangle$

11. Create your own vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$ and show that $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$.

12. Create your own vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ and scalar c and show that $c(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\vec{v})$.

In Exercises 13 – 16, find the measure of the angle between the two vectors in both radians and degrees.

13. $\vec{u} = \langle 1, 1\rangle$, $\vec{v} = \langle 1, 2\rangle$

14. $\vec{u} = \langle -2, 1\rangle$, $\vec{v} = \langle 3, 5\rangle$

15. $\vec{u} = \langle 8, 1, -4\rangle$, $\vec{v} = \langle 2, 2, 0\rangle$

16. $\vec{u} = \langle 1, 7, 2\rangle$, $\vec{v} = \langle 4, -2, 5\rangle$

In Exercises 17 – 20, a vector $\vec{v}$ is given. Give two vectors that are orthogonal to $\vec{v}$.

17. $\vec{v} = \langle 4, 7\rangle$

18. $\vec{v} = \langle -3, 5\rangle$

19. $\vec{v} = \langle 1, 1, 1\rangle$

20. $\vec{v} = \langle 1, -2, 3\rangle$

In Exercises 21 – 26, vectors $\vec{u}$ and $\vec{v}$ are given. Find $\text{proj}_{\vec{v}}\,\vec{u}$, the orthogonal projection of $\vec{u}$ onto $\vec{v}$, and sketch all three vectors with the same initial point.

21. $\vec{u} = \langle 1, 2\rangle$, $\vec{v} = \langle -1, 3\rangle$

22. $\vec{u} = \langle 5, 5\rangle$, $\vec{v} = \langle 1, 3\rangle$

23. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 1, 1\rangle$

24. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 2, 3\rangle$

25. $\vec{u} = \langle 1, 5, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

26. $\vec{u} = \langle 3, -1, 2\rangle$, $\vec{v} = \langle 2, 2, 1\rangle$

In Exercises 27 – 32, vectors $\vec{u}$ and $\vec{v}$ are given. Write $\vec{u}$ as the sum of two vectors, one of which is parallel to $\vec{v}$ and one of which is perpendicular to $\vec{v}$. Note: these are the same pairs of vectors as found in Exercises 21 – 26.

27. $\vec{u} = \langle 1, 2\rangle$, $\vec{v} = \langle -1, 3\rangle$

28. $\vec{u} = \langle 5, 5\rangle$, $\vec{v} = \langle 1, 3\rangle$

29. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 1, 1\rangle$

30. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 2, 3\rangle$

31. $\vec{u} = \langle 1, 5, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

32. $\vec{u} = \langle 3, -1, 2\rangle$, $\vec{v} = \langle 2, 2, 1\rangle$

33. A 10lb box sits on a ramp that rises 4ft over a distance of 20ft. How much force is required to keep the box from sliding down the ramp?

34. A 10lb box sits on a 15ft ramp that makes a 30° angle with the horizontal. How much force is required to keep the box from sliding down the ramp?

35. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 45° to the horizontal?

36. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 10° to the horizontal?

37. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied horizontally?

38. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied at an angle of 45° to the horizontal?

39. How much work is performed in moving a box up the length of a 10ft ramp that makes a 5° angle with the horizontal, with 50lb of force applied in the direction of the ramp?

10.4 The Cross Product

"Orthogonality" is immensely important. A quick scan of your current environment will undoubtedly reveal numerous surfaces and edges that are perpendicular to each other (including the edges of this page). The dot product provides a quick test for orthogonality: vectors $\vec{u}$ and $\vec{v}$ are perpendicular if, and only if, $\vec{u} \cdot \vec{v} = 0$.

Given two non–parallel, nonzero vectors $\vec{u}$ and $\vec{v}$ in space, it is very useful to find a vector $\vec{w}$ that is perpendicular to both $\vec{u}$ and $\vec{v}$. There is a operation, called the **cross product**, that creates such a vector. This section defines the cross product, then explores its properties and applications.

Definition 10.4.1 Cross Product

Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in $\mathbb{R}^3$. The **cross product of $\vec{u}$ and $\vec{v}$**, denoted $\vec{u} \times \vec{v}$, is the vector

$$\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle .$$

This definition can be a bit cumbersome to remember. After an example we will give a convenient method for computing the cross product. For now, careful examination of the products and differences given in the definition should reveal a pattern that is not too difficult to remember. (For instance, in the first component only 2 and 3 appear as subscripts; in the second component, only 1 and 3 appear as subscripts. Further study reveals the order in which they appear.)

Let's practice using this definition by computing a cross product.

Example 10.4.1 Computing a cross product

Let $\vec{u} = \langle 2, -1, 4 \rangle$ and $\vec{v} = \langle 3, 2, 5 \rangle$. Find $\vec{u} \times \vec{v}$, and verify that it is orthogonal to both $\vec{u}$ and $\vec{v}$.

Solution Using Definition 10.4.1, we have

$$\vec{u} \times \vec{v} = \Big\langle (-1)5 - (4)2, -\big((2)5 - (4)3\big), (2)2 - (-1)3 \Big\rangle = \langle -13, 2, 7 \rangle .$$

(We encourage the reader to compute this product on their own, then verify their result.)

We test whether or not $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$ using the dot product:

$$\big(\vec{u} \times \vec{v}\big) \cdot \vec{u} = \langle -13, 2, 7 \rangle \cdot \langle 2, -1, 4 \rangle = 0,$$

$$\big(\vec{u} \times \vec{v}\big) \cdot \vec{v} = \langle -13, 2, 7 \rangle \cdot \langle 3, 2, 5 \rangle = 0.$$

Since both dot products are zero, $\vec{u} \times \vec{v}$ is indeed orthogonal to both $\vec{u}$ and $\vec{v}$.

Notes:

A convenient method of computing the cross product starts with forming a particular 3×3 *matrix*, or rectangular array. The first row comprises the standard unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$. The second and third rows are the vectors $\vec{u}$ and $\vec{v}$, respectively. Using $\vec{u}$ and $\vec{v}$ from Example 10.4.1, we begin with:

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{array}$$

Now repeat the first two columns after the original three:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2 & -1 & 4 & 2 & -1 \\ 3 & 2 & 5 & 3 & 2 \end{array}$$

This gives three full "upper left to lower right" diagonals, and three full "upper right to lower left" diagonals, as shown. Compute the products along each diagonal, then add the products on the right and subtract the products on the left:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2 & -1 & 4 & 2 & -1 \\ 3 & 2 & 5 & 3 & 2 \end{array}$$

$$-3\vec{k} \quad 8\vec{i} \quad 10\vec{j} \qquad\qquad -5\vec{i} \quad 12\vec{j} \quad 4\vec{k}$$

$$\vec{u} \times \vec{v} = \big(-5\vec{i} + 12\vec{j} + 4\vec{k}\big) - \big(-3\vec{k} + 8\vec{i} + 10\vec{j}\big) = -13\vec{i} + 2\vec{j} + 7\vec{k} = \langle -13, 2, 7 \rangle .$$

We practice using this method.

Example 10.4.2 Computing a cross product

Let $\vec{u} = \langle 1, 3, 6 \rangle$ and $\vec{v} = \langle -1, 2, 1 \rangle$. Compute both $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$.

Solution To compute $\vec{u} \times \vec{v}$, we form the matrix as prescribed above, complete with repeated first columns:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 3 & 6 & 1 & 3 \\ -1 & 2 & 1 & -1 & 2 \end{array}$$

We let the reader compute the products of the diagonals; we give the result:

$$\vec{u} \times \vec{v} = \big(3\vec{i} - 6\vec{j} + 2\vec{k}\big) - \big(-3\vec{k} + 12\vec{i} + \vec{j}\big) = \langle -9, -7, 5 \rangle .$$

Notes:

To compute $\vec{v} \times \vec{u}$, we switch the second and third rows of the above matrix, then multiply along diagonals and subtract:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ -1 & 2 & 1 & -1 & 2 \\ 1 & 3 & 6 & 1 & 3 \end{array}$$

Note how with the rows being switched, the products that once appeared on the right now appear on the left, and vice–versa. Thus the result is:

$$\vec{v} \times \vec{u} = \left(12\vec{i} + \vec{j} - 3\vec{k}\right) - \left(2\vec{k} + 3\vec{i} - 6\vec{j}\right) = \langle 9, 7, -5 \rangle,$$

which is the opposite of $\vec{u} \times \vec{v}$. We leave it to the reader to verify that each of these vectors is orthogonal to $\vec{u}$ and $\vec{v}$.

Properties of the Cross Product

It is not coincidence that $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$ in the preceding example; one can show using Definition 10.4.1 that this will always be the case. The following theorem states several useful properties of the cross product, each of which can be verified by referring to the definition.

Theorem 10.4.1 Properties of the Cross Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$ and let c be a scalar. The following identities hold:

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ — Anticommutative Property
2. (a) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$ — Distributive Properties
   (b) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
3. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
4. (a) $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$ — Orthogonality Properties
   (b) $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
5. $\vec{u} \times \vec{u} = \vec{0}$
6. $\vec{u} \times \vec{0} = \vec{0}$
7. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ — Triple Scalar Product

Notes:

We introduced the cross product as a way to find a vector orthogonal to two given vectors, but we did not give a proof that the construction given in Definition 10.4.1 satisfies this property. Theorem 10.4.1 asserts this property holds; we leave it as a problem in the Exercise section to verify this.

Property 5 from the theorem is also left to the reader to prove in the Exercise section, but it reveals something more interesting than "the cross product of a vector with itself is $\vec{0}$." Let $\vec{u}$ and $\vec{v}$ be parallel vectors; that is, let there be a scalar c such that $\vec{v} = c\vec{u}$. Consider their cross product:

$$\begin{aligned} \vec{u} \times \vec{v} &= \vec{u} \times (c\vec{u}) \\ &= c(\vec{u} \times \vec{u}) && \text{(by Property 3 of Theorem 10.4.1)} \\ &= \vec{0}. && \text{(by Property 5 of Theorem 10.4.1)} \end{aligned}$$

We have just shown that the cross product of parallel vectors is $\vec{0}$. This hints at something deeper. Theorem 10.3.2 related the angle between two vectors and their dot product; there is a similar relationship relating the cross product of two vectors and the angle between them, given by the following theorem.

Note: We could rewrite Definition 10.3.2 and Theorem 10.4.2 to include $\vec{0}$, then define that $\vec{u}$ and $\vec{v}$ are parallel if $\vec{u} \times \vec{v} = \vec{0}$. Since $\vec{0} \cdot \vec{v} = 0$ and $\vec{0} \times \vec{v} = \vec{0}$, this would mean that $\vec{0}$ is both parallel *and* orthogonal to all vectors. Apparent paradoxes such as this are not uncommon in mathematics and can be very useful. (See also the marginal note on page 582.)

Theorem 10.4.2 The Cross Product and Angles

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^3$. Then

$$|| \, \vec{u} \times \vec{v} \, || = || \, \vec{u} \, || \, || \, \vec{v} \, || \sin\theta,$$

where θ, $0 \leq \theta \leq \pi$, is the angle between $\vec{u}$ and $\vec{v}$.

Note that this theorem makes a statement about the *magnitude* of the cross product. When the angle between $\vec{u}$ and $\vec{v}$ is 0 or π (i.e., the vectors are parallel), the magnitude of the cross product is 0. The only vector with a magnitude of 0 is $\vec{0}$ (see Property 9 of Theorem 10.2.1), hence the cross product of parallel vectors is $\vec{0}$.

We demonstrate the truth of this theorem in the following example.

Example 10.4.3 The cross product and angles

Let $\vec{u} = \langle 1, 3, 6 \rangle$ and $\vec{v} = \langle -1, 2, 1 \rangle$ as in Example 10.4.2. Verify Theorem 10.4.2 by finding θ, the angle between $\vec{u}$ and $\vec{v}$, and the magnitude of $\vec{u} \times \vec{v}$.

Notes:

Solution We use Theorem 10.3.2 to find the angle between $\vec{u}$ and $\vec{v}$.

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\,\vec{u}\,\|\,\|\,\vec{v}\,\|}\right)\\ &= \cos^{-1}\left(\frac{11}{\sqrt{46}\sqrt{6}}\right)\\ &\approx 0.8471 = 48.54^\circ.\end{aligned}$$

Our work in Example 10.4.2 showed that $\vec{u}\times\vec{v} = \langle -9, -7, 5\rangle$, hence $\|\,\vec{u}\times\vec{v}\,\| = \sqrt{155}$. Is $\|\,\vec{u}\times\vec{v}\,\| = \|\,\vec{u}\,\|\,\|\,\vec{v}\,\|\sin\theta$? Using numerical approximations, we find:

$$\begin{aligned}\|\,\vec{u}\times\vec{v}\,\| &= \sqrt{155} & \|\,\vec{u}\,\|\,\|\,\vec{v}\,\|\sin\theta &= \sqrt{46}\sqrt{6}\sin 0.8471\\ &\approx 12.45. & &\approx 12.45.\end{aligned}$$

Numerically, they seem equal. Using a right triangle, one can show that

$$\sin\left(\cos^{-1}\left(\frac{11}{\sqrt{46}\sqrt{6}}\right)\right) = \frac{\sqrt{155}}{\sqrt{46}\sqrt{6}},$$

which allows us to verify the theorem exactly.

Right Hand Rule

The anticommutative property of the cross product demonstrates that $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ differ only by a sign – these vectors have the same magnitude but point in the opposite direction. When seeking a vector perpendicular to $\vec{u}$ and $\vec{v}$, we essentially have two directions to choose from, one in the direction of $\vec{u}\times\vec{v}$ and one in the direction of $\vec{v}\times\vec{u}$. Does it matter which we choose? How can we tell which one we will get without graphing, etc.?

Another wonderful property of the cross product, as defined, is that it follows the **right hand rule.** Given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with the same initial point, point the index finger of your right hand in the direction of $\vec{u}$ and let your middle finger point in the direction of $\vec{v}$ (much as we did when establishing the right hand rule for the 3-dimensional coordinate system). Your thumb will naturally extend in the direction of $\vec{u}\times\vec{v}$. One can "practice" this using Figure 10.4.1. If you switch, and point the index finder in the direction of $\vec{v}$ and the middle finger in the direction of $\vec{u}$, your thumb will now point in the opposite direction, allowing you to "visualize" the anticommutative property of the cross product.

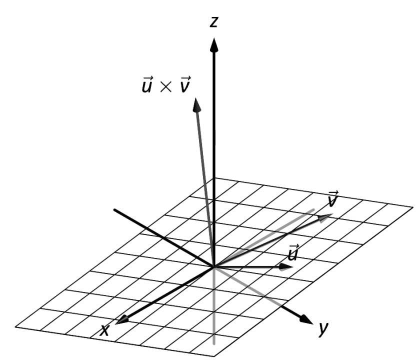


Figure 10.4.1: Illustrating the Right Hand Rule of the cross product.

Applications of the Cross Product

There are a number of ways in which the cross product is useful in mathematics, physics and other areas of science beyond "just" finding a vector perpendicular to two others. We highlight a few here.

Notes:

Area of a Parallelogram

It is a standard geometry fact that the area of a parallelogram is $A = bh$, where b is the length of the base and h is the height of the parallelogram, as illustrated in Figure 10.4.2(a). As shown when defining the Parallelogram Law of vector addition, two vectors $\vec{u}$ and $\vec{v}$ define a parallelogram when drawn from the same initial point, as illustrated in Figure 10.4.2(b). Trigonometry tells us that $h = ||\,\vec{u}\,||\sin\theta$, hence the area of the parallelogram is

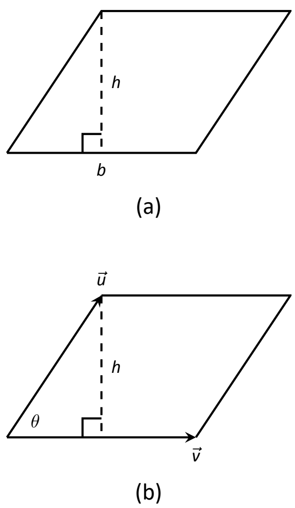


Figure 10.4.2: Using the cross product to find the area of a parallelogram.

$$A = ||\,\vec{u}\,||\,||\,\vec{v}\,||\sin\theta = ||\,\vec{u}\times\vec{v}\,||, \tag{10.4}$$

where the second equality comes from Theorem 10.4.2. We illustrate using Equation (10.4) in the following example.

Example 10.4.4 Finding the area of a parallelogram

1. Find the area of the parallelogram defined by the vectors $\vec{u} = \langle 2, 1\rangle$ and $\vec{v} = \langle 1, 3\rangle$.
2. Verify that the points $A = (1, 1, 1)$, $B = (2, 3, 2)$, $C = (4, 5, 3)$ and $D = (3, 3, 2)$ are the vertices of a parallelogram. Find the area of the parallelogram.

Solution

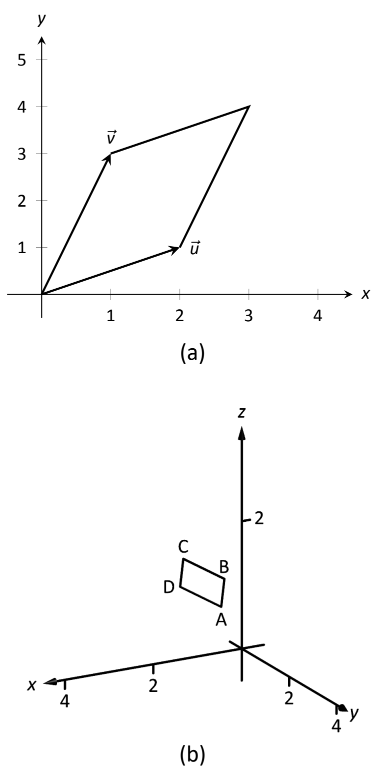


Figure 10.4.3: Sketching the parallelograms in Example 10.4.4.

1. Figure 10.4.3(a) sketches the parallelogram defined by the vectors $\vec{u}$ and $\vec{v}$. We have a slight problem in that our vectors exist in $\mathbb{R}^2$, not $\mathbb{R}^3$, and the cross product is only defined on vectors in $\mathbb{R}^3$. We skirt this issue by viewing $\vec{u}$ and $\vec{v}$ as vectors in the $x-y$ plane of $\mathbb{R}^3$, and rewrite them as $\vec{u} = \langle 2, 1, 0\rangle$ and $\vec{v} = \langle 1, 3, 0\rangle$. We can now compute the cross product. It is easy to show that $\vec{u}\times\vec{v} = \langle 0, 0, 5\rangle$; therefore the area of the parallelogram is $A = ||\,\vec{u}\times\vec{v}\,|| = 5$.
2. To show that the quadrilateral $ABCD$ is a parallelogram (shown in Figure 10.4.3(b)), we need to show that the opposite sides are parallel. We can quickly show that $\overrightarrow{AB} = \overrightarrow{DC} = \langle 1, 2, 1\rangle$ and $\overrightarrow{BC} = \overrightarrow{AD} = \langle 2, 2, 1\rangle$. We find the area by computing the magnitude of the cross product of $\overrightarrow{AB}$ and $\overrightarrow{BC}$:

$$\overrightarrow{AB}\times\overrightarrow{BC} = \langle 0, 1, -2\rangle \quad\Rightarrow\quad ||\,\overrightarrow{AB}\times\overrightarrow{BC}\,|| = \sqrt{5} \approx 2.236.$$

This application is perhaps more useful in finding the area of a triangle (in short, triangles are used more often than parallelograms). We illustrate this in the following example.

Notes:

Example 10.4.5 Area of a triangle

Find the area of the triangle with vertices $A = (1, 2)$, $B = (2, 3)$ and $C = (3, 1)$, as pictured in Figure 10.4.4.

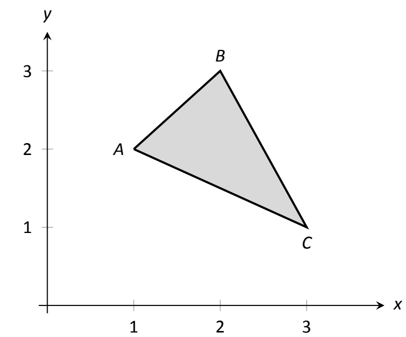


Figure 10.4.4: Finding the area of a triangle in Example 10.4.5.

Solution We found the area of this triangle in Example 7.1.4 to be 1.5 using integration. There we discussed the fact that finding the area of a triangle can be inconvenient using the "$\frac{1}{2}bh$" formula as one has to compute the height, which generally involves finding angles, etc. Using a cross product is much more direct.

We can choose any two sides of the triangle to use to form vectors; we choose $\overrightarrow{AB} = \langle 1, 1\rangle$ and $\overrightarrow{AC} = \langle 2, -1\rangle$. As in the previous example, we will rewrite these vectors with a third component of 0 so that we can apply the cross product. The area of the triangle is

$$\frac{1}{2}||\ \overrightarrow{AB} \times \overrightarrow{AC}\ || = \frac{1}{2}||\ \langle 1, 1, 0\rangle \times \langle 2, -1, 0\rangle\ || = \frac{1}{2}||\ \langle 0, 0, -3\rangle\ || = \frac{3}{2}.$$

We arrive at the same answer as before with less work.

Volume of a Parallelepiped

The three dimensional analogue to the parallelogram is the **parallelepiped**. Each face is parallel to the opposite face, as illustrated in Figure 10.4.5. By crossing $\vec{v}$ and $\vec{w}$, one gets a vector whose magnitude is the area of the base. Dotting this vector with $\vec{u}$ computes the volume of parallelepiped! (Up to a sign; take the absolute value.)

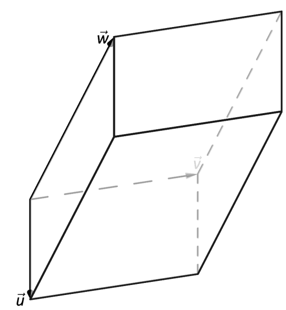


Figure 10.4.5: A parallelepiped is the three dimensional analogue to the parallelogram.

Note: The word "parallelepiped" is pronounced "parallel–eh–pipe–ed."

Thus the volume of a parallelepiped defined by vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ is

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})|. \qquad (10.5)$$

Note how this is the Triple Scalar Product, first seen in Theorem 10.4.1. Applying the identities given in the theorem shows that we can apply the Triple Scalar Product in any "order" we choose to find the volume. That is,

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |\vec{u} \cdot (\vec{w} \times \vec{v})| = |(\vec{u} \times \vec{v}) \cdot \vec{w}|, \quad \text{etc.}$$

Example 10.4.6 Finding the volume of parallelepiped

Find the volume of the parallepiped defined by the vectors $\vec{u} = \langle 1, 1, 0\rangle$, $\vec{v} = \langle -1, 1, 0\rangle$ and $\vec{w} = \langle 0, 1, 1\rangle$.

Solution We apply Equation (10.5). We first find $\vec{v} \times \vec{w} = \langle 1, 1, -1\rangle$. Then

$$|\vec{u} \cdot (\vec{v} \times \vec{w})| = |\ \langle 1, 1, 0\rangle \cdot \langle 1, 1, -1\rangle\ | = 2.$$

So the volume of the parallelepiped is 2 cubic units.

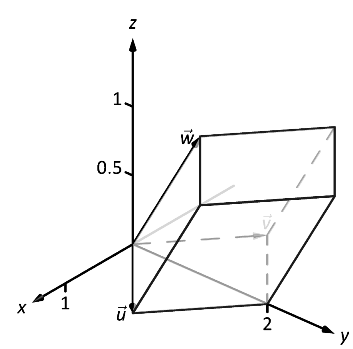


Figure 10.4.6: A parallelepiped in Example 10.4.6.

Notes:

While this application of the Triple Scalar Product is interesting, it is not used all that often: parallelepipeds are not a common shape in physics and engineering. The last application of the cross product is very applicable in engineering.

Torque

Torque is a measure of the turning force applied to an object. A classic scenario involving torque is the application of a wrench to a bolt. When a force is applied to the wrench, the bolt turns. When we represent the force and wrench with vectors $\vec{F}$ and $\vec{\ell}$, we see that the bolt moves (because of the threads) in a direction orthogonal to $\vec{F}$ and $\vec{\ell}$. Torque is usually represented by the Greek letter τ, or tau, and has units of N·m, a Newton–meter, or ft·lb, a foot–pound.

While a full understanding of torque is beyond the purposes of this book, when a force $\vec{F}$ is applied to a lever arm $\vec{\ell}$, the resulting torque is

$$\vec{\tau} = \vec{\ell} \times \vec{F}. \tag{10.6}$$

Example 10.4.7 Computing torque

A lever of length 2ft makes an angle with the horizontal of 45°. Find the resulting torque when a force of 10lb is applied to the end of the level where:

1. the force is perpendicular to the lever, and
2. the force makes an angle of 60° with the lever, as shown in Figure 10.4.7.

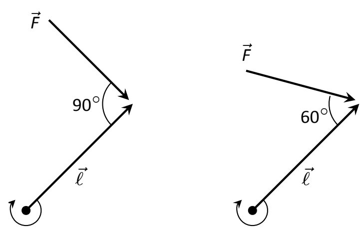


Figure 10.4.7: Showing a force being applied to a lever in Example 10.4.7.

Solution

1. We start by determining vectors for the force and lever arm. Since the lever arm makes a 45° angle with the horizontal and is 2ft long, we can state that $\vec{\ell} = 2\langle \cos 45^\circ, \sin 45^\circ\rangle = \left\langle \sqrt{2}, \sqrt{2}\right\rangle$.

   Since the force vector is perpendicular to the lever arm (as seen in the left hand side of Figure 10.4.7), we can conclude it is making an angle of -45° with the horizontal. As it has a magnitude of 10lb, we can state $\vec{F} = 10\langle \cos(-45^\circ), \sin(-45^\circ)\rangle = \left\langle 5\sqrt{2}, -5\sqrt{2}\right\rangle$.

   Using Equation (10.6) to find the torque requires a cross product. We again let the third component of each vector be 0 and compute the cross product:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F}\\ &= \left\langle \sqrt{2}, \sqrt{2}, 0\right\rangle \times \left\langle 5\sqrt{2}, -5\sqrt{2}, 0\right\rangle\\ &= \langle 0, 0, -20\rangle\end{aligned}$$

Notes:

This clearly has a magnitude of 20 ft-lb.

We can view the force and lever arm vectors as lying "on the page"; our computation of $\vec{\tau}$ shows that the torque goes "into the page." This follows the Right Hand Rule of the cross product, and it also matches well with the example of the wrench turning the bolt. Turning a bolt clockwise moves it in.

2. Our lever arm can still be represented by $\vec{\ell} = \langle \sqrt{2}, \sqrt{2} \rangle$. As our force vector makes a 60° angle with $\vec{\ell}$, we can see (referencing the right hand side of the figure) that $\vec{F}$ makes a -15° angle with the horizontal. Thus

$$\begin{aligned}\vec{F} = 10\,\langle \cos -15^\circ, \sin -15^\circ \rangle &= \left\langle \frac{5(1+\sqrt{3})}{\sqrt{2}}, \frac{5(-1+\sqrt{3})}{\sqrt{2}} \right\rangle \\ &\approx \langle 9.659, -2.588 \rangle\,.\end{aligned}$$

We again make the third component 0 and take the cross product to find the torque:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F} \\ &= \left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle \times \left\langle \frac{5(1+\sqrt{3})}{\sqrt{2}}, \frac{5(-1+\sqrt{3})}{\sqrt{2}}, 0 \right\rangle \\ &= \left\langle 0, 0, -10\sqrt{3} \right\rangle \\ &\approx \langle 0, 0, -17.321 \rangle\,.\end{aligned}$$

As one might expect, when the force and lever arm vectors *are* orthogonal, the magnitude of force is greater than when the vectors *are not* orthogonal.

While the cross product has a variety of applications (as noted in this chapter), its fundamental use is finding a vector perpendicular to two others. Knowing a vector is orthogonal to two others is of incredible importance, as it allows us to find the equations of lines and planes in a variety of contexts. The importance of the cross product, in some sense, relies on the importance of lines and planes, which see widespread use throughout engineering, physics and mathematics. We study lines and planes in the next two sections.

Notes:

Exercises 10.4

Terms and Concepts

1. The cross product of two vectors is a ________, not a scalar.

2. One can visualize the direction of $\vec{u} \times \vec{v}$ using the ________ ________ ________.

3. Give a synonym for "orthogonal."

4. T/F: A fundamental principle of the cross product is that $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

5. ________ is a measure of the turning force applied to an object.

6. T/F: If $\vec{u}$ and $\vec{v}$ are parallel, then $\vec{u} \times \vec{v} = \vec{0}$.

Problems

In Exercises 7 – 16, vectors $\vec{u}$ and $\vec{v}$ are given. Compute $\vec{u} \times \vec{v}$ and show this is orthogonal to both $\vec{u}$ and $\vec{v}$.

7. $\vec{u} = \langle 3, 2, -2 \rangle, \quad \vec{v} = \langle 0, 1, 5 \rangle$

8. $\vec{u} = \langle 5, -4, 3 \rangle, \quad \vec{v} = \langle 2, -5, 1 \rangle$

9. $\vec{u} = \langle 4, -5, -5 \rangle, \quad \vec{v} = \langle 3, 3, 4 \rangle$

10. $\vec{u} = \langle -4, 7, -10 \rangle, \quad \vec{v} = \langle 4, 4, 1 \rangle$

11. $\vec{u} = \langle 1, 0, 1 \rangle, \quad \vec{v} = \langle 5, 0, 7 \rangle$

12. $\vec{u} = \langle 1, 5, -4 \rangle, \quad \vec{v} = \langle -2, -10, 8 \rangle$

13. $\vec{u} = \langle a, b, 0 \rangle, \quad \vec{v} = \langle c, d, 0 \rangle$

14. $\vec{u} = \vec{i}, \quad \vec{v} = \vec{j}$

15. $\vec{u} = \vec{i}, \quad \vec{v} = \vec{k}$

16. $\vec{u} = \vec{j}, \quad \vec{v} = \vec{k}$

17. Pick any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ and show that $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

18. Pick any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ and show that $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$.

In Exercises 19 – 22, the magnitudes of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ are given, along with the angle θ between them. Use this information to find the magnitude of $\vec{u} \times \vec{v}$.

19. $|| \vec{u} || = 2, \quad || \vec{v} || = 5, \quad \theta = 30^\circ$

20. $|| \vec{u} || = 3, \quad || \vec{v} || = 7, \quad \theta = \pi/2$

21. $|| \vec{u} || = 3, \quad || \vec{v} || = 4, \quad \theta = \pi$

22. $|| \vec{u} || = 2, \quad || \vec{v} || = 5, \quad \theta = 5\pi/6$

In Exercises 23 – 26, find the area of the parallelogram defined by the given vectors.

23. $\vec{u} = \langle 1, 1, 2 \rangle, \quad \vec{v} = \langle 2, 0, 3 \rangle$

24. $\vec{u} = \langle -2, 1, 5 \rangle, \quad \vec{v} = \langle -1, 3, 1 \rangle$

25. $\vec{u} = \langle 1, 2 \rangle, \quad \vec{v} = \langle 2, 1 \rangle$

26. $\vec{u} = \langle 2, 0 \rangle, \quad \vec{v} = \langle 0, 3 \rangle$

In Exercises 27 – 30, find the area of the triangle with the given vertices.

27. Vertices: $(0, 0, 0)$, $(1, 3, -1)$ and $(2, 1, 1)$.

28. Vertices: $(5, 2, -1)$, $(3, 6, 2)$ and $(1, 0, 4)$.

29. Vertices: $(1, 1)$, $(1, 3)$ and $(2, 2)$.

30. Vertices: $(3, 1)$, $(1, 2)$ and $(4, 3)$.

In Exercises 31 – 32, find the area of the quadrilateral with the given vertices. (Hint: break the quadrilateral into 2 triangles.)

31. Vertices: $(0, 0)$, $(1, 2)$, $(3, 0)$ and $(4, 3)$.

32. Vertices: $(0, 0, 0)$, $(2, 1, 1)$, $(-1, 2, -8)$ and $(1, -1, 5)$.

In Exercises 33 – 34, find the volume of the parallelepiped defined by the given vectors.

33. $\vec{u} = \langle 1, 1, 1 \rangle, \quad \vec{v} = \langle 1, 2, 3 \rangle, \quad \vec{w} = \langle 1, 0, 1 \rangle$

34. $\vec{u} = \langle -1, 2, 1 \rangle, \quad \vec{v} = \langle 2, 2, 1 \rangle, \quad \vec{w} = \langle 3, 1, 3 \rangle$

In Exercises 35 – 38, find a unit vector orthogonal to both $\vec{u}$ and $\vec{v}$.

35. $\vec{u} = \langle 1, 1, 1 \rangle, \quad \vec{v} = \langle 2, 0, 1 \rangle$

36. $\vec{u} = \langle 1, -2, 1 \rangle, \quad \vec{v} = \langle 3, 2, 1 \rangle$

37. $\vec{u} = \langle 5, 0, 2 \rangle, \quad \vec{v} = \langle -3, 0, 7 \rangle$

38. $\vec{u} = \langle 1, -2, 1 \rangle, \quad \vec{v} = \langle -2, 4, -2 \rangle$

39. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in horizontally from the crankshaft. Find the magnitude of the torque applied to the crankshaft.

40. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in from the crankshaft, making a 30° angle with the horizontal. Find the magnitude of the torque applied to the crankshaft.

41. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench. What is the maximum amount of torque that can be applied to the bolt?

42. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench in a confined space, where the direction of applied force makes a 10° angle with the wrench. How much torque is subsequently applied to the wrench?

43. Show, using the definition of the Cross Product, that $\vec{u}\cdot(\vec{u}\times\vec{v}) = 0$; that is, that $\vec{u}$ is orthogonal to the cross product of $\vec{u}$ and $\vec{v}$.

44. Show, using the definition of the Cross Product, that $\vec{u}\times\vec{u} = \vec{0}$.

10.5 Lines

To find the equation of a line in the *x*-*y* plane, we need two pieces of information: a point and the slope. The slope conveys *direction* information. As vertical lines have an undefined slope, the following statement is more accurate:

> To define a line, one needs a point on the line and the direction of the line.

This holds true for lines in space.

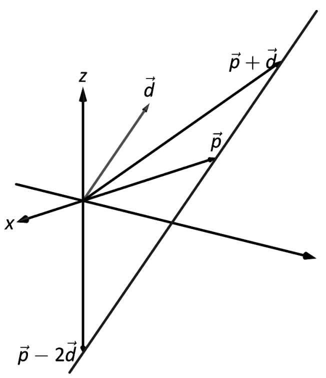


Figure 10.5.2: Defining a line in space.

Let P be a point in space, let $\vec{p}$ be the vector with initial point at the origin and terminal point at P (i.e., $\vec{p}$ "points" to P), and let $\vec{d}$ be a vector. Consider the points on the line through P in the direction of $\vec{d}$.

Clearly one point on the line is P; we can say that the *vector* $\vec{p}$ lies at this point on the line. To find another point on the line, we can start at $\vec{p}$ and move in a direction parallel to $\vec{d}$. For instance, starting at $\vec{p}$ and traveling one length of $\vec{d}$ places one at another point on the line. Consider Figure 10.5.2 where certain points along the line are indicated.

The figure illustrates how every point on the line can be obtained by starting with $\vec{p}$ and moving a certain distance in the direction of $\vec{d}$. That is, we can define the line as a function of t:

$$\vec{\ell}(t) = \vec{p} + t\,\vec{d}. \tag{10.7}$$

In many ways, this is *not* a new concept. Compare Equation (10.7) to the familiar "$y = mx + b$" equation of a line:

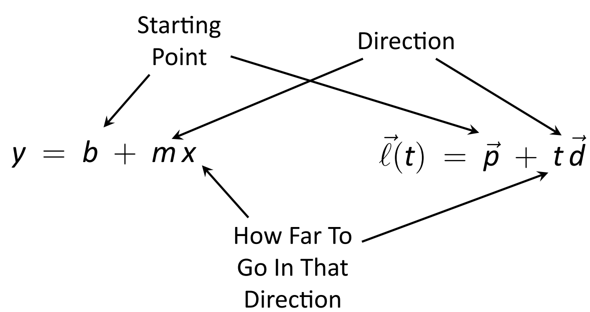


Figure 10.5.1: Understanding the vector equation of a line.

The equations exhibit the same structure: they give a starting point, define a direction, and state how far in that direction to travel.

Equation (10.7) is an example of a **vector–valued function**; the input of the function is a real number and the output is a vector. We will cover vector–valued functions extensively in the next chapter.

Notes:

There are other ways to represent a line. Let $\vec{p} = \langle x_0, y_0, z_0 \rangle$ and let $\vec{d} = \langle a, b, c \rangle$. Then the equation of the line through $\vec{p}$ in the direction of $\vec{d}$ is:

$$\begin{aligned}\vec{\ell}(t) &= \vec{p} + t\vec{d}\\ &= \langle x_0, y_0, z_0 \rangle + t\,\langle a, b, c \rangle\\ &= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle\,.\end{aligned}$$

The last line states that the x values of the line are given by $x = x_0 + at$, the y values are given by $y = y_0 + bt$, and the z values are given by $z = z_0 + ct$. These three equations, taken together, are the **parametric equations of the line** through $\vec{p}$ in the direction of $\vec{d}$.

Finally, each of the equations for x, y and z above contain the variable t. We can solve for t in each equation:

$$\begin{aligned} x = x_0 + at &\quad\Rightarrow\quad t = \frac{x - x_0}{a},\\ y = y_0 + bt &\quad\Rightarrow\quad t = \frac{y - y_0}{b},\\ z = z_0 + ct &\quad\Rightarrow\quad t = \frac{z - z_0}{c},\end{aligned}$$

assuming $a, b, c \neq 0$. Since t is equal to each expression on the right, we can set these equal to each other, forming the **symmetric equations of the line** through $\vec{p}$ in the direction of $\vec{d}$:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Each representation has its own advantages, depending on the context. We summarize these three forms in the following definition, then give examples of their use.

Notes:

Definition 10.5.1 Equations of Lines in Space

Consider the line in space that passes through $\vec{p} = \langle x_0, y_0, z_0\rangle$ in the direction of $\vec{d} = \langle a, b, c\rangle$.

1. The **vector equation** of the line is
$$\vec{\ell}(t) = \vec{p} + t\vec{d}.$$
2. The **parametric equations** of the line are
$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$
3. The **symmetric equations** of the line are
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Example 10.5.1 Finding the equation of a line

Give all three equations, as given in Definition 10.5.1, of the line through $P = (2, 3, 1)$ in the direction of $\vec{d} = \langle -1, 1, 2\rangle$. Does the point $Q = (-1, 6, 6)$ lie on this line?

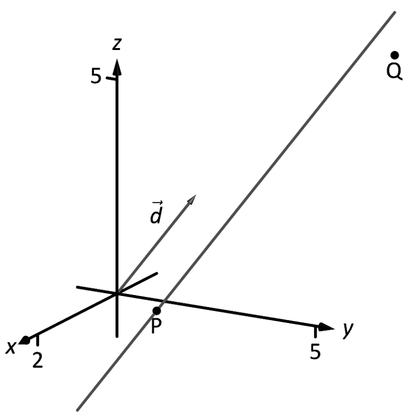


Figure 10.5.3: Graphing a line in Example 10.5.1.

Solution We identify the point $P = (2, 3, 1)$ with the vector $\vec{p} = \langle 2, 3, 1\rangle$. Following the definition, we have

- the vector equation of the line is $\vec{\ell}(t) = \langle 2, 3, 1\rangle + t\langle -1, 1, 2\rangle$;
- the parametric equations of the line are
$$x = 2 - t, \quad y = 3 + t, \quad z = 1 + 2t; \text{ and}$$
- the symmetric equations of the line are
$$\frac{x - 2}{-1} = \frac{y - 3}{1} = \frac{z - 1}{2}.$$

The first two equations of the line are useful when a t value is given: one can immediately find the corresponding point on the line. These forms are good when calculating with a computer; most software programs easily handle equations in these formats. (For instance, the graphics program that made Figure 10.5.3 can be given the input "`(2-t,3+t,1+2*t)`" for $-1 \leq t \leq 3$.).

Does the point $Q = (-1, 6, 6)$ lie on the line? The graph in Figure 10.5.3 makes it clear that it does not. We can answer this question without the graph

Notes:

using any of the three equation forms. Of the three, the symmetric equations are probably best suited for this task. Simply plug in the values of x, y and z and see if equality is maintained:

$$\frac{-1-2}{-1} \stackrel{?}{=} \frac{6-3}{1} \stackrel{?}{=} \frac{6-1}{2} \quad \Rightarrow \quad 3 = 3 \neq 2.5.$$

We see that Q does not lie on the line as it did not satisfy the symmetric equations.

Example 10.5.2 Finding the equation of a line through two points

Find the parametric equations of the line through the points $P = (2, -1, 2)$ and $Q = (1, 3, -1)$.

Solution Recall the statement made at the beginning of this section: to find the equation of a line, we need a point and a direction. We have *two* points; either one will suffice. The direction of the line can be found by the vector with initial point P and terminal point Q: $\overrightarrow{PQ} = \langle -1, 4, -3 \rangle$.

The parametric equations of the line ℓ through P in the direction of $\overrightarrow{PQ}$ are:

$$\ell: \quad x = 2 - t \quad y = -1 + 4t \quad z = 2 - 3t.$$

A graph of the points and line are given in Figure 10.5.4. Note how in the given parametrization of the line, $t = 0$ corresponds to the point P, and $t = 1$ corresponds to the point Q. This relates to the understanding of the vector equation of a line described in Figure 10.5.1. The parametric equations "start" at the point P, and t determines how far in the direction of $\overrightarrow{PQ}$ to travel. When $t = 0$, we travel 0 lengths of $\overrightarrow{PQ}$; when $t = 1$, we travel one length of $\overrightarrow{PQ}$, resulting in the point Q.

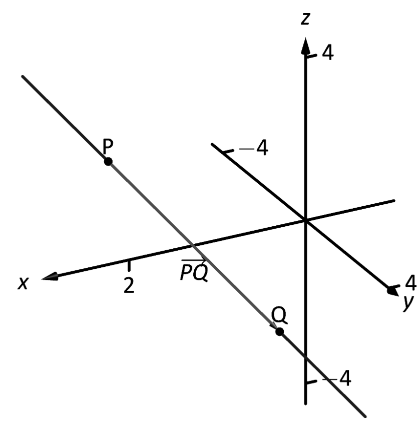


Figure 10.5.4: A graph of the line in Example 10.5.2.

Parallel, Intersecting and Skew Lines

In the plane, two *distinct* lines can either be parallel or they will intersect at exactly one point. In space, given equations of two lines, it can sometimes be difficult to tell whether the lines are distinct or not (i.e., the same line can be represented in different ways). Given lines $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$ and $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$, we have four possibilities: $\vec{\ell}_1$ and $\vec{\ell}_2$ are

the same line	they share all points;
intersecting lines	share only 1 point;
parallel lines	$\vec{d}_1 \parallel \vec{d}_2$, no points in common; or
skew lines	$\vec{d}_1 \nparallel \vec{d}_2$, no points in common.

Notes:

The next two examples investigate these possibilities.

Example 10.5.3 Comparing lines

Consider lines ℓ_1 and ℓ_2, given in parametric equation form:

$$\ell_1: \begin{array}{rcl} x &=& 1+3t \\ y &=& 2-t \\ z &=& t \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& -2+4s \\ y &=& 3+s \\ z &=& 5+2s. \end{array}$$

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

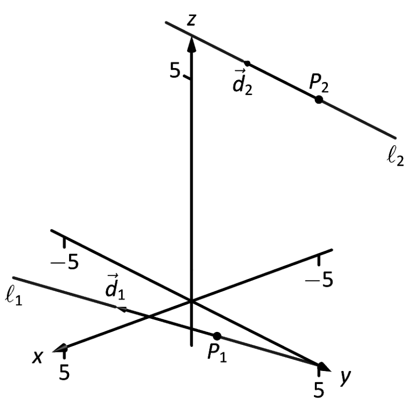


Figure 10.5.5: Sketching the lines from Example 10.5.3.

Solution We start by looking at the directions of each line. Line ℓ_1 has the direction given by $\vec{d}_1 = \langle 3, -1, 1\rangle$ and line ℓ_2 has the direction given by $\vec{d}_2 = \langle 4, 1, 2\rangle$. It should be clear that $\vec{d}_1$ and $\vec{d}_2$ are not parallel, hence ℓ_1 and ℓ_2 are not the same line, nor are they parallel. Figure 10.5.5 verifies this fact (where the points and directions indicated by the equations of each line are identified).

We next check to see if they intersect (if they do not, they are skew lines). To find if they intersect, we look for t and s values such that the respective x, y and z values are the same. That is, we want s and t such that:

$$\begin{array}{rcl} 1+3t &=& -2+4s \\ 2-t &=& 3+s \\ t &=& 5+2s. \end{array}$$

This is a relatively simple system of linear equations. Since the last equation is already solved for t, substitute that value of t into the equation above it:

$$2-(5+2s) = 3+s \quad \Rightarrow \quad s=-2,\ t=1.$$

A key to remember is that we have *three* equations; we need to check if $s = -2,\ t=1$ satisfies the first equation as well:

$$1+3(1) \neq -2+4(-2).$$

It does not. Therefore, we conclude that the lines ℓ_1 and ℓ_2 are skew.

Example 10.5.4 Comparing lines

Consider lines ℓ_1 and ℓ_2, given in parametric equation form:

$$\ell_1: \begin{array}{rcl} x &=& -0.7+1.6t \\ y &=& 4.2+2.72t \\ z &=& 2.3-3.36t \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& 2.8-2.9s \\ y &=& 10.15-4.93s \\ z &=& -5.05+6.09s. \end{array}$$

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

Notes:

Solution It is obviously very difficult to simply look at these equations and discern anything. This is done intentionally. In the "real world," most equations that are used do not have nice, integer coefficients. Rather, there are lots of digits after the decimal and the equations can look "messy."

We again start by deciding whether or not each line has the same direction. The direction of ℓ_1 is given by $\vec{d}_1 = \langle 1.6, 2.72, -3.36\rangle$ and the direction of ℓ_2 is given by $\vec{d}_2 = \langle -2.9, -4.93, 6.09\rangle$. When it is not clear through observation whether two vectors are parallel or not, the standard way of determining this is by comparing their respective unit vectors. Using a calculator, we find:

$$\vec{u}_1 = \frac{\vec{d}_1}{||\,\vec{d}_1\,||} = \langle 0.3471, 0.5901, -0.7289\rangle$$
$$\vec{u}_2 = \frac{\vec{d}_2}{||\,\vec{d}_2\,||} = \langle -0.3471, -0.5901, 0.7289\rangle\,.$$

The two vectors seem to be parallel (at least, their components are equal to 4 decimal places). In most situations, it would suffice to conclude that the lines are at least parallel, if not the same. One way to be sure is to rewrite $\vec{d}_1$ and $\vec{d}_2$ in terms of fractions, not decimals. We have

$$\vec{d}_1 = \left\langle \frac{16}{10}, \frac{272}{100}, -\frac{336}{100}\right\rangle \qquad \vec{d}_2 = \left\langle -\frac{29}{10}, -\frac{493}{100}, \frac{609}{100}\right\rangle.$$

One can then find the magnitudes of each vector in terms of fractions, then compute the unit vectors likewise. After a lot of manual arithmetic (or after briefly using a computer algebra system), one finds that

$$\vec{u}_1 = \left\langle \sqrt{\frac{10}{83}}, \frac{17}{\sqrt{830}}, -\frac{21}{\sqrt{830}}\right\rangle \qquad \vec{u}_2 = \left\langle -\sqrt{\frac{10}{83}}, -\frac{17}{\sqrt{830}}, \frac{21}{\sqrt{830}}\right\rangle.$$

We can now say without equivocation that these lines are parallel.

Are they the same line? The parametric equations for a line describe one point that lies on the line, so we know that the point $P_1 = (-0.7, 4.2, 2.3)$ lies on ℓ_1. To determine if this point also lies on ℓ_2, plug in the *x*, *y* and *z* values of P_1 into the symmetric equations for ℓ_2:

$$\frac{(-0.7)-2.8}{-2.9} \stackrel{?}{=} \frac{(4.2)-10.15}{-4.93} \stackrel{?}{=} \frac{(2.3)-(-5.05)}{6.09} \quad\Rightarrow\quad 1.2069 = 1.2069 = 1.2069.$$

The point P_1 lies on both lines, so we conclude they are the same line, just parametrized differently. Figure 10.5.6 graphs this line along with the points and vectors described by the parametric equations. Note how $\vec{d}_1$ and $\vec{d}_2$ are parallel, though point in opposite directions (as indicated by their unit vectors above).

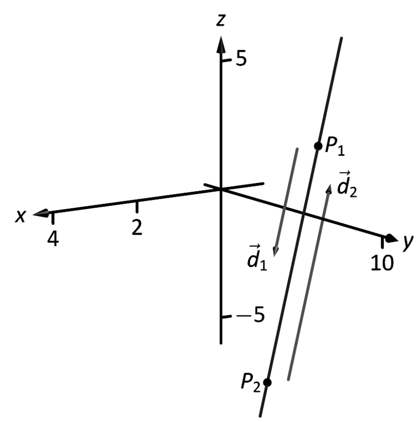


Figure 10.5.6: Graphing the lines in Example 10.5.4.

Notes:

Distances

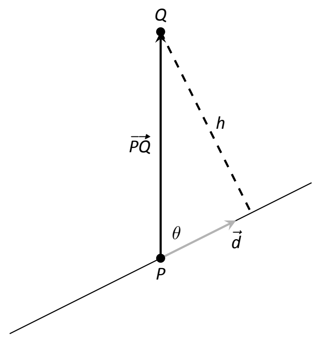


Figure 10.5.7: Establishing the distance from a point to a line.

Given a point Q and a line $\vec{\ell}(t) = \vec{p} + t\vec{d}$ in space, it is often useful to know the distance from the point to the line. (Here we use the standard definition of "distance," i.e., the length of the shortest line segment from the point to the line.) Identifying $\vec{p}$ with the point P, Figure 10.5.7 will help establish a general method of computing this distance h.

From trigonometry, we know $h = ||\,\vec{PQ}\,||\sin\theta$. We have a similar identity involving the cross product: $||\,\vec{PQ}\times\vec{d}\,|| = ||\,\vec{PQ}\,||\,||\,\vec{d}\,||\sin\theta$. Divide both sides of this latter equation by $||\,\vec{d}\,||$ to obtain h:

$$h = \frac{||\,\vec{PQ}\times\vec{d}\,||}{||\,\vec{d}\,||}. \tag{10.8}$$

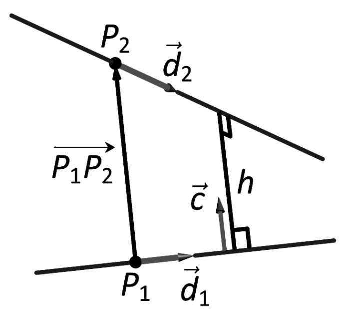


Figure 10.5.8: Establishing the distance between lines.

It is also useful to determine the distance between lines, which we define as the length of the shortest line segment that connects the two lines (an argument from geometry shows that this line segments is perpendicular to both lines). Let lines $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$ and $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$ be given, as shown in Figure 10.5.8. To find the direction orthogonal to both $\vec{d}_1$ and $\vec{d}_2$, we take the cross product: $\vec{c} = \vec{d}_1 \times \vec{d}_2$. The magnitude of the orthogonal projection of $\overrightarrow{P_1P_2}$ onto $\vec{c}$ is the distance h we seek:

$$\begin{aligned} h &= \left|\left|\,\text{proj}_{\,\vec{c}}\,\overrightarrow{P_1P_2}\,\right|\right| \\ &= \left|\left|\,\frac{\overrightarrow{P_1P_2}\cdot\vec{c}}{\vec{c}\cdot\vec{c}}\vec{c}\,\right|\right| \\ &= \frac{|\overrightarrow{P_1P_2}\cdot\vec{c}|}{||\,\vec{c}\,||^2}||\,\vec{c}\,|| \\ &= \frac{|\overrightarrow{P_1P_2}\cdot\vec{c}|}{||\,\vec{c}\,||}. \end{aligned}$$

A problem in the Exercise section is to show that this distance is 0 when the lines intersect. Note the use of the Triple Scalar Product: $\overrightarrow{P_1P_2}\cdot\vec{c} = \overrightarrow{P_1P_2}\cdot(\vec{d}_1\times\vec{d}_2)$.

The following Key Idea restates these two distance formulas.

Notes:

Key Idea 10.5.1 Distances to Lines

1. Let P be a point on a line ℓ that is parallel to $\vec{d}$. The distance h from a point Q to the line ℓ is:

$$h = \frac{||\,\overrightarrow{PQ} \times \vec{d}\,||}{||\,\vec{d}\,||}.$$

2. Let P_1 be a point on line ℓ_1 that is parallel to $\vec{d_1}$, and let P_2 be a point on line ℓ_2 parallel to $\vec{d_2}$, and let $\vec{c} = \vec{d_1} \times \vec{d_2}$, where lines ℓ_1 and ℓ_2 are not parallel. The distance h between the two lines is:

$$h = \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{||\,\vec{c}\,||}.$$

Example 10.5.5 Finding the distance from a point to a line

Find the distance from the point $Q = (1, 1, 3)$ to the line $\vec{\ell}(t) = \langle 1, -1, 1\rangle + t\,\langle 2, 3, 1\rangle$.

Solution The equation of the line gives us the point $P = (1, -1, 1)$ that lies on the line, hence $\overrightarrow{PQ} = \langle 0, 2, 2\rangle$. The equation also gives $\vec{d} = \langle 2, 3, 1\rangle$. Following Key Idea 10.5.1, we have the distance as

$$\begin{aligned} h &= \frac{||\,\overrightarrow{PQ} \times \vec{d}\,||}{||\,\vec{d}\,||} \\ &= \frac{||\ \langle -4, 4, -4\rangle\ ||}{\sqrt{14}} \\ &= \frac{4\sqrt{3}}{\sqrt{14}} \approx 1.852. \end{aligned}$$

The point Q is approximately 1.852 units from the line $\vec{\ell}(t)$.

Example 10.5.6 Finding the distance between lines

Find the distance between the lines

$$\ell_1: \begin{array}{rcl} x &=& 1+3t \\ y &=& 2-t \\ z &=& t \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& -2+4s \\ y &=& 3+s \\ z &=& 5+2s. \end{array}$$

Solution These are the sames lines as given in Example 10.5.3, where

Notes:

we showed them to be skew. The equations allow us to identify the following points and vectors:

$$P_1 = (1,2,0) \quad P_2 = (-2,3,5) \quad \Rightarrow \quad \overrightarrow{P_1P_2} = \langle -3,1,5 \rangle .$$

$$\vec{d}_1 = \langle 3,-1,1 \rangle \quad \vec{d}_2 = \langle 4,1,2 \rangle \quad \Rightarrow \quad \vec{c} = \vec{d}_1 \times \vec{d}_2 = \langle -3,-2,7 \rangle .$$

From Key Idea 10.5.1 we have the distance h between the two lines is

$$\begin{aligned} h &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{||\,\vec{c}\,||} \\ &= \frac{42}{\sqrt{62}} \approx 5.334. \end{aligned}$$

The lines are approximately 5.334 units apart.

One of the key points to understand from this section is this: to describe a line, we need a point and a direction. Whenever a problem is posed concerning a line, one needs to take whatever information is offered and glean point and direction information. Many questions can be asked (and *are* asked in the Exercise section) whose answer immediately follows from this understanding.

Lines are one of two fundamental objects of study in space. The other fundamental object is the *plane*, which we study in detail in the next section. Many complex three dimensional objects are studied by approximating their surfaces with lines and planes.

Notes:

Exercises 10.5

Terms and Concepts

1. To find an equation of a line, what two pieces of information are needed?

2. Two distinct lines in the plane can intersect or be ________.

3. Two distinct lines in space can intersect, be ________ or be ________.

4. Use your own words to describe what it means for two lines in space to be skew.

Problems

In Exercises 5 – 14, write the vector, parametric and symmetric equations of the lines described.

5. Passes through $P = (2, -4, 1)$, parallel to $\vec{d} = \langle 9, 2, 5\rangle$.

6. Passes through $P = (6, 1, 7)$, parallel to $\vec{d} = \langle -3, 2, 5\rangle$.

7. Passes through $P = (2, 1, 5)$ and $Q = (7, -2, 4)$.

8. Passes through $P = (1, -2, 3)$ and $Q = (5, 5, 5)$.

9. Passes through $P = (0, 1, 2)$ and orthogonal to both $\vec{d}_1 = \langle 2, -1, 7\rangle$ and $\vec{d}_2 = \langle 7, 1, 3\rangle$.

10. Passes through $P = (5, 1, 9)$ and orthogonal to both $\vec{d}_1 = \langle 1, 0, 1\rangle$ and $\vec{d}_2 = \langle 2, 0, 3\rangle$.

11. Passes through the point of intersection of $\vec{\ell}_1(t)$ and $\vec{\ell}_2(t)$ and orthogonal to both lines, where $\vec{\ell}_1(t) = \langle 2, 1, 1\rangle + t\,\langle 5, 1, -2\rangle$ and $\vec{\ell}_2(t) = \langle -2, -1, 2\rangle + t\,\langle 3, 1, -1\rangle$.

12. Passes through the point of intersection of $\ell_1(t)$ and $\ell_2(t)$ and orthogonal to both lines, where
$$\ell_1 = \begin{cases} x = t \\ y = -2 + 2t \\ z = 1 + t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 2 + t \\ y = 2 - t \\ z = 3 + 2t \end{cases}.$$

13. Passes through $P = (1, 1)$, parallel to $\vec{d} = \langle 2, 3\rangle$.

14. Passes through $P = (-2, 5)$, parallel to $\vec{d} = \langle 0, 1\rangle$.

In Exercises 15 – 22, determine if the described lines are the same line, parallel lines, intersecting or skew lines. If intersecting, give the point of intersection.

15. $\vec{\ell}_1(t) = \langle 1, 2, 1\rangle + t\,\langle 2, -1, 1\rangle$,
$\vec{\ell}_2(t) = \langle 3, 3, 3\rangle + t\,\langle -4, 2, -2\rangle$.

16. $\vec{\ell}_1(t) = \langle 2, 1, 1\rangle + t\,\langle 5, 1, 3\rangle$,
$\vec{\ell}_2(t) = \langle 14, 5, 9\rangle + t\,\langle 1, 1, 1\rangle$.

17. $\vec{\ell}_1(t) = \langle 3, 4, 1\rangle + t\,\langle 2, -3, 4\rangle$,
$\vec{\ell}_2(t) = \langle -3, 3, -3\rangle + t\,\langle 3, -2, 4\rangle$.

18. $\vec{\ell}_1(t) = \langle 1, 1, 1\rangle + t\,\langle 3, 1, 3\rangle$,
$\vec{\ell}_2(t) = \langle 7, 3, 7\rangle + t\,\langle 6, 2, 6\rangle$.

19. $$\ell_1 = \begin{cases} x = 1 + 2t \\ y = 3 - 2t \\ z = t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 3 - t \\ y = 3 + 5t \\ z = 2 + 7t \end{cases}$$

20. $$\ell_1 = \begin{cases} x = 1.1 + 0.6t \\ y = 3.77 + 0.9t \\ z = -2.3 + 1.5t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 3.11 + 3.4t \\ y = 2 + 5.1t \\ z = 2.5 + 8.5t \end{cases}$$

21. $$\ell_1 = \begin{cases} x = 0.2 + 0.6t \\ y = 1.33 - 0.45t \\ z = -4.2 + 1.05t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 0.86 + 9.2t \\ y = 0.835 - 6.9t \\ z = -3.045 + 16.1t \end{cases}$$

22. $$\ell_1 = \begin{cases} x = 0.1 + 1.1t \\ y = 2.9 - 1.5t \\ z = 3.2 + 1.6t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 4 - 2.1t \\ y = 1.8 + 7.2t \\ z = 3.1 + 1.1t \end{cases}$$

In Exercises 23 – 26, find the distance from the point to the line.

23. $Q = (1, 1, 1)$, $\quad \vec{\ell}(t) = \langle 2, 1, 3\rangle + t\,\langle 2, 1, -2\rangle$

24. $Q = (2, 5, 6)$, $\quad \vec{\ell}(t) = \langle -1, 1, 1\rangle + t\,\langle 1, 0, 1\rangle$

25. $Q = (0, 3)$, $\quad \vec{\ell}(t) = \langle 2, 0\rangle + t\,\langle 1, 1\rangle$

26. $Q = (1, 1)$, $\quad \vec{\ell}(t) = \langle 4, 5\rangle + t\,\langle -4, 3\rangle$

In Exercises 27 – 28, find the distance between the two lines.

27. $\vec{\ell}_1(t) = \langle 1, 2, 1\rangle + t\,\langle 2, -1, 1\rangle$,
$\vec{\ell}_2(t) = \langle 3, 3, 3\rangle + t\,\langle 4, 2, -2\rangle$.

28. $\vec{\ell}_1(t) = \langle 0, 0, 1\rangle + t\,\langle 1, 0, 0\rangle$,
$\vec{\ell}_2(t) = \langle 0, 0, 3\rangle + t\,\langle 0, 1, 0\rangle$.

Exercises 29 – 31 explore special cases of the distance formulas found in Key Idea 10.5.1.

29. Let Q be a point on the line $\vec{\ell}(t)$. Show why the distance formula correctly gives the distance from the point to the line as 0.

30. Let lines $\vec{\ell}_1(t)$ and $\vec{\ell}_2(t)$ be intersecting lines. Show why the distance formula correctly gives the distance between these lines as 0.

31. Let lines $\vec{\ell}_1(t)$ and $\vec{\ell}_2(t)$ be parallel.

    (a) Show why the distance formula for distance between lines cannot be used as stated to find the distance between the lines.

    (b) Show why letting $\vec{c} = (\overrightarrow{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$ allows one to use the formula.

    (c) Show how one can use the formula for the distance between a point and a line to find the distance between parallel lines.

10.6 Planes

Any flat surface, such as a wall, table top or stiff piece of cardboard can be thought of as representing part of a plane. Consider a piece of cardboard with a point P marked on it. One can take a nail and stick it into the cardboard at P such that the nail is perpendicular to the cardboard; see Figure 10.6.1.

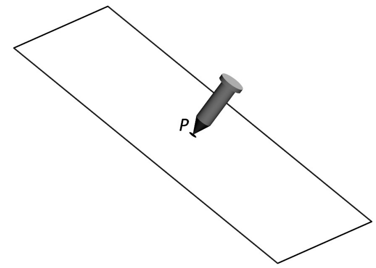


Figure 10.6.1: Illustrating defining a plane with a sheet of cardboard and a nail.

This nail provides a "handle" for the cardboard. Moving the cardboard around moves P to different locations in space. Tilting the nail (but keeping P fixed) tilts the cardboard. Both moving and tilting the cardboard defines a different plane in space. In fact, we can define a plane by: 1) the location of P in space, and 2) the direction of the nail.

The previous section showed that one can define a line given a point on the line and the direction of the line (usually given by a vector). One can make a similar statement about planes: we can define a plane in space given a point on the plane and the direction the plane "faces" (using the description above, the direction of the nail). Once again, the direction information will be supplied by a vector, called a **normal vector**, that is orthogonal to the plane.

What exactly does "orthogonal to the plane" mean? Choose any two points P and Q in the plane, and consider the vector $\overrightarrow{PQ}$. We say a vector $\vec{n}$ is orthogonal to the plane if $\vec{n}$ is perpendicular to $\overrightarrow{PQ}$ for all choices of P and Q; that is, if $\vec{n}\cdot\overrightarrow{PQ} = 0$ for all P and Q.

This gives us way of writing an equation describing the plane. Let $P = (x_0, y_0, z_0)$ be a point in the plane and let $\vec{n} = \langle a, b, c\rangle$ be a normal vector to the plane. A point $Q = (x, y, z)$ lies in the plane defined by P and $\vec{n}$ if, and only if, $\overrightarrow{PQ}$ is orthogonal to $\vec{n}$. Knowing $\overrightarrow{PQ} = \langle x - x_0, y - y_0, z - z_0\rangle$, consider:

$$\begin{aligned} \overrightarrow{PQ}\cdot\vec{n} &= 0\\ \langle x - x_0, y - y_0, z - z_0\rangle\cdot\langle a, b, c\rangle &= 0\\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \end{aligned} \tag{10.9}$$

Equation (10.9) defines an *implicit* function describing the plane. More algebra produces:

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

The right hand side is just a number, so we replace it with d:

$$ax + by + cz = d. \tag{10.10}$$

As long as $c \neq 0$, we can solve for z:

$$z = \frac{1}{c}(d - ax - by). \tag{10.11}$$

Notes:

Equation (10.11) is especially useful as many computer programs can graph functions in this form. Equations (10.9) and (10.10) have specific names, given next.

Definition 10.6.1 Equations of a Plane in Standard and General Forms

The plane passing through the point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$ can be described by an equation with **standard form**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0;$$

the equation's **general form** is

$$ax + by + cz = d.$$

A key to remember throughout this section is this: to find the equation of a plane, we need a point and a normal vector. We will give several examples of finding the equation of a plane, and in each one different types of information are given. In each case, we need to use the given information to find a point on the plane and a normal vector.

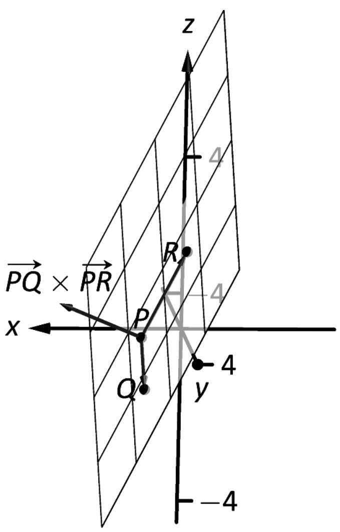


Figure 10.6.2: Sketching the plane in Example 10.6.1.

Example 10.6.1 Finding the equation of a plane.

Write the equation of the plane that passes through the points $P = (1, 1, 0)$, $Q = (1, 2, -1)$ and $R = (0, 1, 2)$ in standard form.

Solution We need a vector $\vec{n}$ that is orthogonal to the plane. Since P, Q and R are in the plane, so are the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$; $\overrightarrow{PQ} \times \overrightarrow{PR}$ is orthogonal to $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and hence the plane itself.

It is straightforward to compute $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2, 1, 1\rangle$. We can use any point we wish in the plane (any of P, Q or R will do) and we arbitrarily choose P. Following Definition 10.6.1, the equation of the plane in standard form is

$$2(x - 1) + (y - 1) + z = 0.$$

The plane is sketched in Figure 10.6.2.

We have just demonstrated the fact that any three non-collinear points define a plane. (This is why a three-legged stool does not "rock;" it's three feet always lie in a plane. A four-legged stool will rock unless all four feet lie in the same plane.)

Example 10.6.2 Finding the equation of a plane.

Verify that lines ℓ_1 and ℓ_2, whose parametric equations are given below, inter-

Notes:

sect, then give the equation of the plane that contains these two lines in general form.

$$\ell_1 : \begin{array}{rcl} x &=& -5+2s \\ y &=& 1+s \\ z &=& -4+2s \end{array} \qquad \ell_2 : \begin{array}{rcl} x &=& 2+3t \\ y &=& 1-2t \\ z &=& 1+t \end{array}$$

Solution The lines clearly are not parallel. If they do not intersect, they are skew, meaning there is not a plane that contains them both. If they do intersect, there is such a plane.

To find their point of intersection, we set the x, y and z equations equal to each other and solve for s and t:

$$\begin{array}{rcl} -5+2s &=& 2+3t \\ 1+s &=& 1-2t \\ -4+2s &=& 1+t \end{array} \quad \Rightarrow \quad s=2, \quad t=-1.$$

When $s=2$ and $t=-1$, the lines intersect at the point $P=(-1,3,0)$.

Let $\vec{d}_1 = \langle 2,1,2\rangle$ and $\vec{d}_2 = \langle 3,-2,1\rangle$ be the directions of lines ℓ_1 and ℓ_2, respectively. A normal vector to the plane containing these the two lines will also be orthogonal to $\vec{d}_1$ and $\vec{d}_2$. Thus we find a normal vector $\vec{n}$ by computing $\vec{n} = \vec{d}_1 \times \vec{d}_2 = \langle 5, 4-7\rangle$.

We can pick any point in the plane with which to write our equation; each line gives us infinite choices of points. We choose P, the point of intersection. We follow Definition 10.6.1 to write the plane's equation in general form:

$$\begin{aligned} 5(x+1)+4(y-3)-7z &= 0 \\ 5x+5+4y-12-7z &= 0 \\ 5x+4y-7z &= 7. \end{aligned}$$

The plane's equation in general form is $5x+4y-7z=7$; it is sketched in Figure 10.6.3.

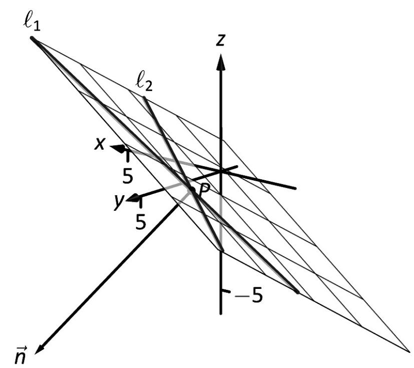


Figure 10.6.3: Sketching the plane in Example 10.6.2.

Example 10.6.3 Finding the equation of a plane

Give the equation, in standard form, of the plane that passes through the point $P=(-1,0,1)$ and is orthogonal to the line with vector equation $\vec{\ell}(t) = \langle -1,0,1\rangle + t\langle 1,2,2\rangle$.

Solution As the plane is to be orthogonal to the line, the plane must be orthogonal to the direction of the line given by $\vec{d} = \langle 1,2,2\rangle$. We use this as our normal vector. Thus the plane's equation, in standard form, is

$$(x+1)+2y+2(z-1)=0.$$

The line and plane are sketched in Figure 10.6.4.

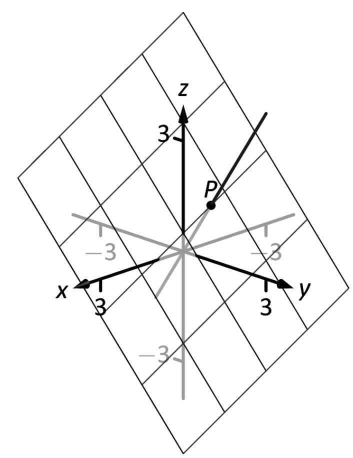


Figure 10.6.4: The line and plane in Example 10.6.3.

Notes:

Example 10.6.4 Finding the intersection of two planes

Give the parametric equations of the line that is the intersection of the planes p_1 and p_2, where:

$$p_1 : x - (y-2) + (z-1) = 0$$
$$p_2 : -2(x-2) + (y+1) + (z-3) = 0$$

Solution To find an equation of a line, we need a point on the line and the direction of the line.

We can find a point on the line by solving each equation of the planes for z:

$$p_1 : z = -x + y - 1$$
$$p_2 : z = 2x - y - 2$$

We can now set these two equations equal to each other (i.e., we are finding values of x and y where the planes have the same z value):

$$-x + y - 1 = 2x - y - 2$$
$$2y = 3x - 1$$
$$y = \frac{1}{2}(3x - 1)$$

We can choose any value for x; we choose $x = 1$. This determines that $y = 1$. We can now use the equations of either plane to find z: when $x = 1$ and $y = 1$, $z = -1$ on both planes. We have found a point P on the line: $P = (1, 1, -1)$.

We now need the direction of the line. Since the line lies in each plane, its direction is orthogonal to a normal vector for each plane. Considering the equations for p_1 and p_2, we can quickly determine their normal vectors. For p_1, $\vec{n}_1 = \langle 1, -1, 1 \rangle$ and for p_2, $\vec{n}_2 = \langle -2, 1, 1 \rangle$. A direction orthogonal to both of these directions is their cross product: $\vec{d} = \vec{n}_1 \times \vec{n}_2 = \langle -2, -3, -1 \rangle$.

The parametric equations of the line through $P = (1, 1, -1)$ in the direction of $d = \langle -2, -3, -1 \rangle$ is:

$$\ell : \quad x = -2t + 1 \quad y = -3t + 1 \quad z = -t - 1.$$

The planes and line are graphed in Figure 10.6.5.

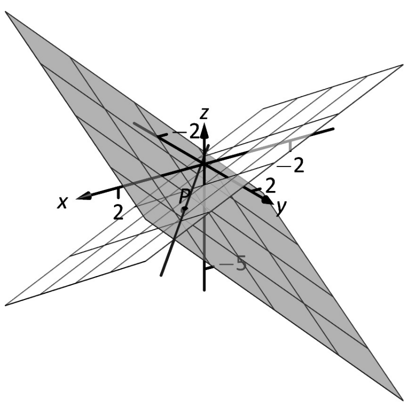


Figure 10.6.5: Graphing the planes and their line of intersection in Example 10.6.4.

Example 10.6.5 Finding the intersection of a plane and a line

Find the point of intersection, if any, of the line $\ell(t) = \langle 3, -3, -1 \rangle + t \langle -1, 2, 1 \rangle$ and the plane with equation in general form $2x + y + z = 4$.

Solution The equation of the plane shows that the vector $\vec{n} = \langle 2, 1, 1 \rangle$ is a normal vector to the plane, and the equation of the line shows that the line

Notes:

moves parallel to $\vec{d} = \langle -1, 2, 1\rangle$. Since these are not orthogonal, we know there is a point of intersection. (If there were orthogonal, it would mean that the plane and line were parallel to each other, either never intersecting or the line was in the plane itself.)

To find the point of intersection, we need to find a t value such that $\ell(t)$ satisfies the equation of the plane. Rewriting the equation of the line with parametric equations will help:

$$\ell(t) = \begin{cases} x = 3 - t \\ y = -3 + 2t \\ z = -1 + t \end{cases}.$$

Replacing x, y and z in the equation of the plane with the expressions containing t found in the equation of the line allows us to determine a t value that indicates the point of intersection:

$$\begin{aligned} 2x + y + z &= 4 \\ 2(3 - t) + (-3 + 2t) + (-1 + t) &= 4 \\ t &= 2. \end{aligned}$$

When $t = 2$, the point on the line satisfies the equation of the plane; that point is $\ell(2) = \langle 1, 1, 1\rangle$. Thus the point $(1, 1, 1)$ is the point of intersection between the plane and the line, illustrated in Figure 10.6.6.

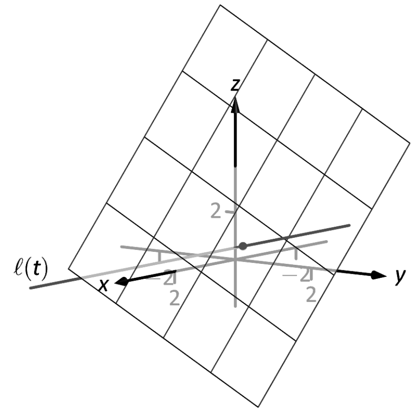


Figure 10.6.6: Illustrating the intersection of a line and a plane in Example 10.6.5.

Distances

Just as it was useful to find distances between points and lines in the previous section, it is also often necessary to find the distance from a point to a plane.

Consider Figure 10.6.7, where a plane with normal vector $\vec{n}$ is sketched containing a point P and a point Q, not on the plane, is given. We measure the distance from Q to the plane by measuring the length of the projection of $\overrightarrow{PQ}$ onto $\vec{n}$. That is, we want:

$$|| \operatorname{proj}_{\vec{n}} \overrightarrow{PQ} \,|| = \left|\left| \frac{\vec{n} \cdot \overrightarrow{PQ}}{|| \vec{n} ||^2} \vec{n} \right|\right| = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{|| \vec{n} ||} \qquad (10.12)$$

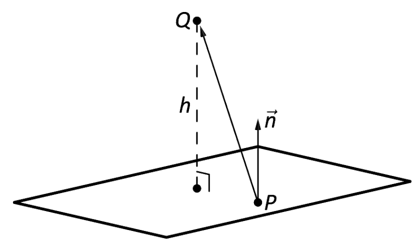


Figure 10.6.7: Illustrating finding the distance from a point to a plane.

Equation (10.12) is important as it does more than just give the distance between a point and a plane. We will see how it allows us to find several other distances as well: the distance between parallel planes and the distance from a line and a plane. Because Equation (10.12) is important, we restate it as a Key Idea.

Notes:

Key Idea 10.6.1 Distance from a Point to a Plane

Let a plane with normal vector $\vec{n}$ be given, and let Q be a point. The distance h from Q to the plane is

$$h = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{||\,\vec{n}\,||},$$

where P is any point in the plane.

Example 10.6.6 Distance between a point and a plane

Find the distance between the point $Q = (2, 1, 4)$ and the plane with equation $2x - 5y + 6z = 9$.

Solution Using the equation of the plane, we find the normal vector $\vec{n} = \langle 2, -5, 6\rangle$. To find a point on the plane, we can let x and y be anything we choose, then let z be whatever satisfies the equation. Letting x and y be 0 seems simple; this makes $z = 1.5$. Thus we let $P = \langle 0, 0, 1.5\rangle$, and $\overrightarrow{PQ} = \langle 2, 1, 2.5\rangle$.

The distance h from Q to the plane is given by Key Idea 10.6.1:

$$\begin{aligned} h &= \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{||\,\vec{n}\,||} \\ &= \frac{|\,\langle 2, -5, 6\rangle \cdot \langle 2, 1, 2.5\rangle\,|}{||\ \langle 2, -5, 6\rangle\ ||} \\ &= \frac{|14|}{\sqrt{65}} \\ &\approx 1.74. \end{aligned}$$

We can use Key Idea 10.6.1 to find other distances. Given two parallel planes, we can find the distance between these planes by letting P be a point on one plane and Q a point on the other. If ℓ is a line parallel to a plane, we can use the Key Idea to find the distance between them as well: again, let P be a point in the plane and let Q be any point on the line. (One can also use Key Idea 10.5.1.) The Exercise section contains problems of these types.

These past two sections have not explored lines and planes in space as an exercise of mathematical curiosity. However, there are many, many applications of these fundamental concepts. Complex shapes can be modeled (or, *approximated*) using planes. For instance, part of the exterior of an aircraft may have a complex, yet smooth, shape, and engineers will want to know how air flows across this piece as well as how heat might build up due to air friction. Many equations that help determine air flow and heat dissipation are difficult to apply to arbitrary surfaces, but simple to apply to planes. By approximating a surface with millions of small planes one can more readily model the needed behavior.

Notes:

Exercises 10.6

Terms and Concepts

1. In order to find the equation of a plane, what two pieces of information must one have?

2. What is the relationship between a plane and one of its normal vectors?

Problems

In Exercises 3 – 6, give any two points in the given plane.

3. $2x - 4y + 7z = 2$

4. $3(x+2) + 5(y-9) - 4z = 0$

5. $x = 2$

6. $4(y+2) - (z-6) = 0$

In Exercises 7 – 20, give the equation of the described plane in standard and general forms.

7. Passes through $(2, 3, 4)$ and has normal vector $\vec{n} = \langle 3, -1, 7\rangle$.

8. Passes through $(1, 3, 5)$ and has normal vector $\vec{n} = \langle 0, 2, 4\rangle$.

9. Passes through the points $(1, 2, 3)$, $(3, -1, 4)$ and $(1, 0, 1)$.

10. Passes through the points $(5, 3, 8)$, $(6, 4, 9)$ and $(3, 3, 3)$.

11. Contains the intersecting lines
$\vec{\ell_1}(t) = \langle 2, 1, 2\rangle + t\langle 1, 2, 3\rangle$ and
$\vec{\ell_2}(t) = \langle 2, 1, 2\rangle + t\langle 2, 5, 4\rangle$.

12. Contains the intersecting lines
$\vec{\ell_1}(t) = \langle 5, 0, 3\rangle + t\langle -1, 1, 1\rangle$ and
$\vec{\ell_2}(t) = \langle 1, 4, 7\rangle + t\langle 3, 0, -3\rangle$.

13. Contains the parallel lines
$\vec{\ell_1}(t) = \langle 1, 1, 1\rangle + t\langle 1, 2, 3\rangle$ and
$\vec{\ell_2}(t) = \langle 1, 1, 2\rangle + t\langle 1, 2, 3\rangle$.

14. Contains the parallel lines
$\vec{\ell_1}(t) = \langle 1, 1, 1\rangle + t\langle 4, 1, 3\rangle$ and
$\vec{\ell_2}(t) = \langle 4, 4, 4\rangle + t\langle 4, 1, 3\rangle$.

15. Contains the point $(2, -6, 1)$ and the line
$$\ell(t) = \begin{cases} x = 2 + 5t \\ y = 2 + 2t \\ z = -1 + 2t \end{cases}$$

16. Contains the point $(5, 7, 3)$ and the line
$$\ell(t) = \begin{cases} x = t \\ y = t \\ z = t \end{cases}$$

17. Contains the point $(5, 7, 3)$ and is orthogonal to the line $\vec{\ell}(t) = \langle 4, 5, 6\rangle + t\langle 1, 1, 1\rangle$.

18. Contains the point $(4, 1, 1)$ and is orthogonal to the line
$$\ell(t) = \begin{cases} x = 4 + 4t \\ y = 1 + 1t \\ z = 1 + 1t \end{cases}$$

19. Contains the point $(-4, 7, 2)$ and is parallel to the plane $3(x-2) + 8(y+1) - 10z = 0$.

20. Contains the point $(1, 2, 3)$ and is parallel to the plane $x = 5$.

In Exercises 21 – 22, give the equation of the line that is the intersection of the given planes.

21. $p1 :\ 3(x-2) + (y-1) + 4z = 0$, and
$p2 :\ 2(x-1) - 2(y+3) + 6(z-1) = 0$.

22. $p1 :\ 5(x-5) + 2(y+2) + 4(z-1) = 0$, and
$p2 :\ 3x - 4(y-1) + 2(z-1) = 0$.

In Exercises 23 – 26, find the point of intersection between the line and the plane.

23. line: $\langle 5, 1, -1\rangle + t\langle 2, 2, 1\rangle$,
plane: $5x - y - z = -3$

24. line: $\langle 4, 1, 0\rangle + t\langle 1, 0, -1\rangle$,
plane: $3x + y - 2z = 8$

25. line: $\langle 1, 2, 3\rangle + t\langle 3, 5, -1\rangle$,
plane: $3x - 2y - z = 4$

26. line: $\langle 1, 2, 3\rangle + t\langle 3, 5, -1\rangle$,
plane: $3x - 2y - z = -4$

In Exercises 27 – 30, find the given distances.

27. The distance from the point $(1, 2, 3)$ to the plane $3(x-1) + (y-2) + 5(z-2) = 0$.

28. The distance from the point $(2, 6, 2)$ to the plane $2(x-1) - y + 4(z+1) = 0$.

29. The distance between the parallel planes
$x + y + z = 0$ and
$(x-2) + (y-3) + (z+4) = 0$

30. The distance between the parallel planes $2(x-1)+2(y+1)+(z-2)=0$ and $2(x-3)+2(y-1)+(z-3)=0$

31. Show why if the point Q lies in a plane, then the distance formula correctly gives the distance from the point to the plane as 0.

32. How is Exercise 30 in Section 10.5 easier to answer once we have an understanding of planes?