11: VECTOR VALUED FUNCTIONS

In the previous chapter, we learned about vectors and were introduced to the power of vectors within mathematics. In this chapter, we'll build on this foundation to define functions whose input is a real number and whose output is a vector. We'll see how to graph these functions and apply calculus techniques to analyze their behavior. Most importantly, we'll see *why* we are interested in doing this: we'll see beautiful applications to the study of moving objects.

11.1 Vector–Valued Functions

We are very familiar with **real valued functions**, that is, functions whose output is a real number. This section introduces **vector–valued functions** – functions whose output is a vector.

Definition 11.1.1 Vector–Valued Functions

A **vector–valued function** is a function of the form

$$\vec{r}(t) = \langle f(t), g(t) \rangle \quad \text{or} \quad \vec{r}(t) = \langle f(t), g(t), h(t) \rangle,$$

where f, g and h are real valued functions.

The **domain** of $\vec{r}$ is the set of all values of t for which $\vec{r}(t)$ is defined. The **range** of $\vec{r}$ is the set of all possible output vectors $\vec{r}(t)$.

Evaluating and Graphing Vector–Valued Functions

Evaluating a vector–valued function at a specific value of t is straightforward; simply evaluate each component function at that value of t. For instance, if $\vec{r}(t) = \langle t^2, t^2 + t - 1\rangle$, then $\vec{r}(-2) = \langle 4, 1\rangle$. We can sketch this vector, as is done in Figure 11.1.1(a). Plotting lots of vectors is cumbersome, though, so generally we do not sketch the whole vector but just the terminal point. The **graph** of a vector–valued function is the set of all terminal points of $\vec{r}(t)$, where the initial point of each vector is always the origin. In Figure 11.1.1(b) we sketch the graph of $\vec{r}$; we can indicate individual points on the graph with their respective vector, as shown.

Vector–valued functions are closely related to parametric equations of graphs. While in both methods we plot points $\big(x(t), y(t)\big)$ or $\big(x(t), y(t), z(t)\big)$ to produce a graph, in the context of vector–valued functions each such point represents a vector. The implications of this will be more fully realized in the next section as we apply calculus ideas to these functions.

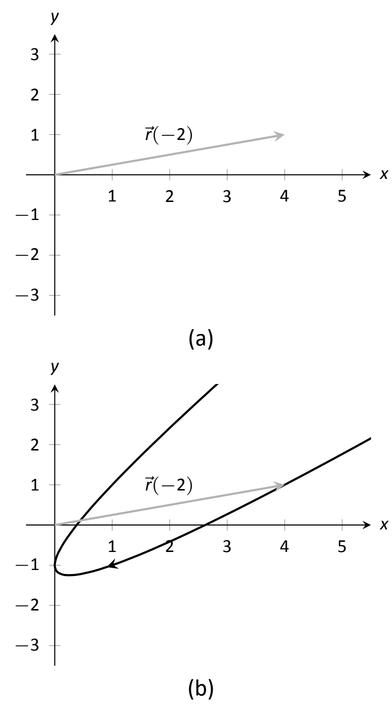
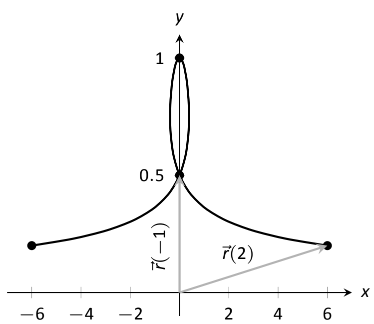


Figure 11.1.1: Sketching the graph of a vector–valued function.

t	$t^3 - t$	$\dfrac{1}{t^2+1}$
-2	-6	1/5
-1	0	1/2
0	0	1
1	0	1/2
2	6	1/5

(a)



(b)

Figure 11.1.2: Sketching the vector–valued function of Example 11.1.1.

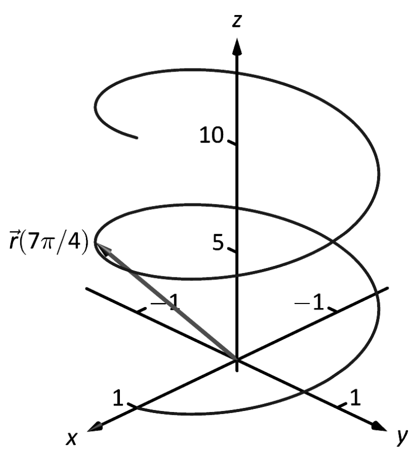


Figure 11.1.3: The graph of $\vec{r}(t)$ in Example 11.1.2.

Example 11.1.1 Graphing vector–valued functions

Graph $\vec{r}(t) = \left\langle t^3 - t, \dfrac{1}{t^2+1} \right\rangle$, for $-2 \leq t \leq 2$. Sketch $\vec{r}(-1)$ and $\vec{r}(2)$.

Solution We start by making a table of t, x and y values as shown in Figure 11.1.2(a). Plotting these points gives an indication of what the graph looks like. In Figure 11.1.2(b), we indicate these points and sketch the full graph. We also highlight $\vec{r}(-1)$ and $\vec{r}(2)$ on the graph.

Example 11.1.2 Graphing vector–valued functions.

Graph $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \leq t \leq 4\pi$.

Solution We can again plot points, but careful consideration of this function is very revealing. Momentarily ignoring the third component, we see the x and y components trace out a circle of radius 1 centered at the origin. Noticing that the z component is t, we see that as the graph winds around the z-axis, it is also increasing at a constant rate in the positive z direction, forming a spiral. This is graphed in Figure 11.1.3. In the graph $\vec{r}(7\pi/4) \approx (0.707, -0.707, 5.498)$ is highlighted to help us understand the graph.

Algebra of Vector–Valued Functions

Definition 11.1.2 Operations on Vector–Valued Functions

Let $\vec{r}_1(t) = \langle f_1(t), g_1(t) \rangle$ and $\vec{r}_2(t) = \langle f_2(t), g_2(t) \rangle$ be vector–valued functions in $\mathbb{R}^2$ and let c be a scalar. Then:

1. $\vec{r}_1(t) \pm \vec{r}_2(t) = \langle\, f_1(t) \pm f_2(t), g_1(t) \pm g_2(t) \,\rangle$.
2. $c\vec{r}_1(t) = \langle\, cf_1(t), cg_1(t) \,\rangle$.

A similar definition holds for vector–valued functions in $\mathbb{R}^3$.

This definition states that we add, subtract and scale vector-valued functions component–wise. Combining vector–valued functions in this way can be very useful (as well as create interesting graphs).

Example 11.1.3 Adding and scaling vector–valued functions.

Let $\vec{r}_1(t) = \langle\, 0.2t, 0.3t \,\rangle$, $\vec{r}_2(t) = \langle\, \cos t, \sin t \,\rangle$ and $\vec{r}(t) = \vec{r}_1(t) + \vec{r}_2(t)$. Graph $\vec{r}_1(t)$, $\vec{r}_2(t)$, $\vec{r}(t)$ and $5\vec{r}(t)$ on $-10 \leq t \leq 10$.

Notes:

Solution We can graph $\vec{r}_1$ and $\vec{r}_2$ easily by plotting points (or just using technology). Let's think about each for a moment to better understand how vector–valued functions work.

We can rewrite $\vec{r}_1(t) = \langle 0.2t, 0.3t \rangle$ as $\vec{r}_1(t) = t\langle 0.2, 0.3\rangle$. That is, the function $\vec{r}_1$ scales the vector $\langle 0.2, 0.3\rangle$ by t. This scaling of a vector produces a line in the direction of $\langle 0.2, 0.3\rangle$.

We are familiar with $\vec{r}_2(t) = \langle \cos t, \sin t \rangle$; it traces out a circle, centered at the origin, of radius 1. Figure 11.1.4(a) graphs $\vec{r}_1(t)$ and $\vec{r}_2(t)$.

Adding $\vec{r}_1(t)$ to $\vec{r}_2(t)$ produces $\vec{r}(t) = \langle \cos t + 0.2t, \sin t + 0.3t \rangle$, graphed in Figure 11.1.4(b). The linear movement of the line combines with the circle to create loops that move in the direction of $\langle 0.2, 0.3\rangle$. (We encourage the reader to experiment by changing $\vec{r}_1(t)$ to $\langle 2t, 3t\rangle$, etc., and observe the effects on the loops.)

Multiplying $\vec{r}(t)$ by 5 scales the function by 5, producing $5\vec{r}(t) = \langle 5\cos t + 1, 5\sin t + 1.5\rangle$, which is graphed in Figure 11.1.4(c) along with $\vec{r}(t)$. The new function is "5 times bigger" than $\vec{r}(t)$. Note how the graph of $5\vec{r}(t)$ in (c) looks identical to the graph of $\vec{r}(t)$ in (b). This is due to the fact that the x and y bounds of the plot in (c) are exactly 5 times larger than the bounds in (b).

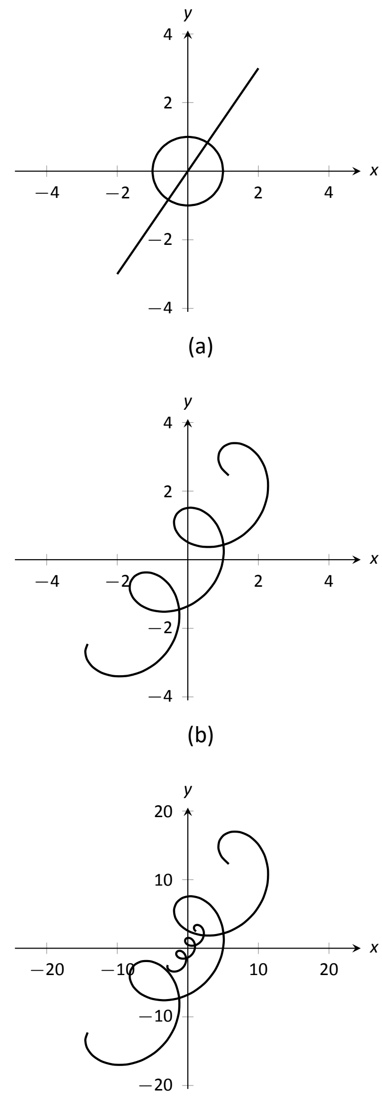


Figure 11.1.4: Graphing the functions in Example 11.1.3.

Example 11.1.4 Adding and scaling vector–valued functions.

A **cycloid** is a graph traced by a point p on a rolling circle, as shown in Figure 11.1.5. Find an equation describing the cycloid, where the circle has radius 1.

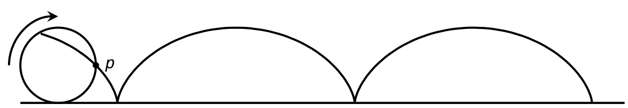


Figure 11.1.5: Tracing a cycloid.

Solution This problem is not very difficult if we approach it in a clever way. We start by letting $\vec{p}(t)$ describe the position of the point p on the circle, where the circle is centered at the origin and only rotates clockwise (i.e., it does not roll). This is relatively simple given our previous experiences with parametric equations; $\vec{p}(t) = \langle \cos t, -\sin t\rangle$.

We now want the circle to roll. We represent this by letting $\vec{c}(t)$ represent the location of the center of the circle. It should be clear that the y component of $\vec{c}(t)$ should be 1; the center of the circle is always going to be 1 if it rolls on a horizontal surface.

The x component of $\vec{c}(t)$ is a linear function of t: $f(t) = mt$ for some scalar m. When $t = 0$, $f(t) = 0$ (the circle starts centered on the y-axis). When $t = 2\pi$, the circle has made one complete revolution, traveling a distance equal to its

Notes:

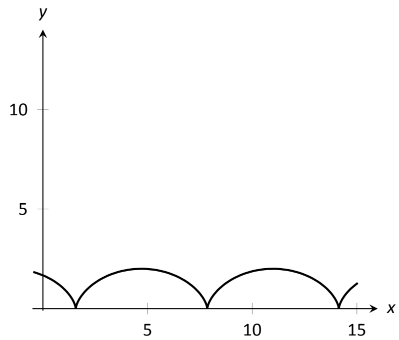


Figure 11.1.6: The cycloid in Example 11.1.4.

circumference, which is also 2π. This gives us a point on our line $f(t) = mt$, the point $(2\pi, 2\pi)$. It should be clear that $m = 1$ and $f(t) = t$. So $\vec{c}(t) = \langle t, 1 \rangle$.

We now combine $\vec{p}$ and $\vec{c}$ together to form the equation of the cycloid: $\vec{r}(t) = \vec{p}(t) + \vec{c}(t) = \langle \cos t + t, -\sin t + 1 \rangle$, which is graphed in Figure 11.1.6.

Displacement

A vector–valued function $\vec{r}(t)$ is often used to describe the position of a moving object at time t. At $t = t_0$, the object is at $\vec{r}(t_0)$; at $t = t_1$, the object is at $\vec{r}(t_1)$. Knowing the locations $\vec{r}(t_0)$ and $\vec{r}(t_1)$ give no indication of the path taken between them, but often we only care about the difference of the locations, $\vec{r}(t_1) - \vec{r}(t_0)$, the **displacement**.

Definition 11.1.3 Displacement

Let $\vec{r}(t)$ be a vector–valued function and let $t_0 < t_1$ be values in the domain. The **displacement** $\vec{d}$ of $\vec{r}$, from $t = t_0$ to $t = t_1$, is

$$\vec{d} = \vec{r}(t_1) - \vec{r}(t_0).$$

When the displacement vector is drawn with initial point at $\vec{r}(t_0)$, its terminal point is $\vec{r}(t_1)$. We think of it as the vector which points from a starting position to an ending position.

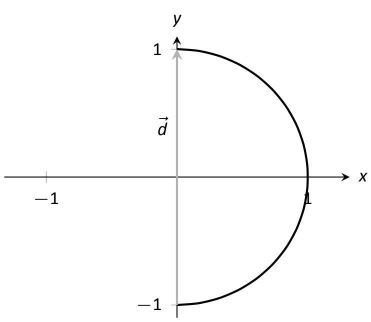


Figure 11.1.7: Graphing the displacement of a position function in Example 11.1.5.

Example 11.1.5 Finding and graphing displacement vectors

Let $\vec{r}(t) = \left\langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t) \right\rangle$. Graph $\vec{r}(t)$ on $-1 \leq t \leq 1$, and find the displacement of $\vec{r}(t)$ on this interval.

Solution The function $\vec{r}(t)$ traces out the unit circle, though at a different rate than the "usual" $\langle \cos t, \sin t \rangle$ parametrization. At $t_0 = -1$, we have $\vec{r}(t_0) = \langle 0, -1 \rangle$; at $t_1 = 1$, we have $\vec{r}(t_1) = \langle 0, 1 \rangle$. The displacement of $\vec{r}(t)$ on $[-1, 1]$ is thus $\vec{d} = \langle 0, 1 \rangle - \langle 0, -1 \rangle = \langle 0, 2 \rangle$.

A graph of $\vec{r}(t)$ on $[-1, 1]$ is given in Figure 11.1.7, along with the displacement vector $\vec{d}$ on this interval.

Measuring displacement makes us contemplate related, yet very different, concepts. Considering the semi–circular path the object in Example 11.1.5 took, we can quickly verify that the object ended up a distance of 2 units from its initial location. That is, we can compute $|| \vec{d} || = 2$. However, measuring *distance from the starting point* is different from measuring *distance traveled*. Being a semi–

Notes:

circle, we can measure the distance traveled by this object as $\pi \approx 3.14$ units. Knowing *distance from the starting point* allows us to compute **average rate of change.**

Definition 11.1.4 Average Rate of Change

Let $\vec{r}(t)$ be a vector–valued function, where each of its component functions is continuous on its domain, and let $t_0 < t_1$. The **average rate of change** of $\vec{r}(t)$ on $[t_0, t_1]$ is

$$\text{average rate of change} = \frac{\vec{r}(t_1) - \vec{r}(t_0)}{t_1 - t_0}.$$

Example 11.1.6 Average rate of change

Let $\vec{r}(t) = \left\langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t)\right\rangle$ as in Example 11.1.5. Find the average rate of change of $\vec{r}(t)$ on $[-1, 1]$ and on $[-1, 5]$.

SOLUTION We computed in Example 11.1.5 that the displacement of $\vec{r}(t)$ on $[-1, 1]$ was $\vec{d} = \langle 0, 2\rangle$. Thus the average rate of change of $\vec{r}(t)$ on $[-1, 1]$ is:

$$\frac{\vec{r}(1) - \vec{r}(-1)}{1 - (-1)} = \frac{\langle 0, 2\rangle}{2} = \langle 0, 1\rangle .$$

We interpret this as follows: the object followed a semi–circular path, meaning it moved towards the right then moved back to the left, while climbing slowly, then quickly, then slowly again. *On average*, however, it progressed straight up at a constant rate of $\langle 0, 1\rangle$ per unit of time.

We can quickly see that the displacement on $[-1, 5]$ is the same as on $[-1, 1]$, so $\vec{d} = \langle 0, 2\rangle$. The average rate of change is different, though:

$$\frac{\vec{r}(5) - \vec{r}(-1)}{5 - (-1)} = \frac{\langle 0, 2\rangle}{6} = \langle 0, 1/3\rangle .$$

As it took "3 times as long" to arrive at the same place, this average rate of change on $[-1, 5]$ is $1/3$ the average rate of change on $[-1, 1]$.

We considered average rates of change in Sections 1.1 and 2.1 as we studied limits and derivatives. The same is true here; in the following section we apply calculus concepts to vector–valued functions as we find limits, derivatives, and integrals. Understanding the average rate of change will give us an understanding of the derivative; displacement gives us one application of integration.

Notes:

Exercises 11.1

Terms and Concepts

1. Vector–valued functions are closely related to ________ ________ of graphs.

2. When sketching vector–valued functions, technically one isn't graphing points, but rather ________.

3. It can be useful to think of ________ as a vector that points from a starting position to an ending position.

4. In the context of vector–valued functions, average rate of change is ________ divided by time.

Problems

In Exercises 5 – 12, sketch the vector–valued function on the given interval.

5. $\vec{r}(t) = \left\langle t^2, t^2 - 1 \right\rangle$, for $-2 \le t \le 2$.

6. $\vec{r}(t) = \left\langle t^2, t^3 \right\rangle$, for $-2 \le t \le 2$.

7. $\vec{r}(t) = \left\langle 1/t, 1/t^2 \right\rangle$, for $-2 \le t \le 2$.

8. $\vec{r}(t) = \left\langle \frac{1}{10}t^2, \sin t \right\rangle$, for $-2\pi \le t \le 2\pi$.

9. $\vec{r}(t) = \left\langle \frac{1}{10}t^2, \sin t \right\rangle$, for $-2\pi \le t \le 2\pi$.

10. $\vec{r}(t) = \langle 3\sin(\pi t), 2\cos(\pi t) \rangle$, on $[0, 2]$.

11. $\vec{r}(t) = \langle 3\cos t, 2\sin(2t) \rangle$, on $[0, 2\pi]$.

12. $\vec{r}(t) = \langle 2\sec t, \tan t \rangle$, on $[-\pi, \pi]$.

In Exercises 13 – 16, sketch the vector–valued function on the given interval in $\mathbb{R}^3$. Technology may be useful in creating the sketch.

13. $\vec{r}(t) = \langle 2\cos t, t, 2\sin t \rangle$, on $[0, 2\pi]$.

14. $\vec{r}(t) = \langle 3\cos t, \sin t, t/\pi \rangle$ on $[0, 2\pi]$.

15. $\vec{r}(t) = \langle \cos t, \sin t, \sin t \rangle$ on $[0, 2\pi]$.

16. $\vec{r}(t) = \langle \cos t, \sin t, \sin(2t) \rangle$ on $[0, 2\pi]$.

In Exercises 17 – 20, find $||\ \vec{r}(t)\ ||$.

17. $\vec{r}(t) = \left\langle t, t^2 \right\rangle$.

18. $\vec{r}(t) = \langle 5\cos t, 3\sin t \rangle$.

19. $\vec{r}(t) = \langle 2\cos t, 2\sin t, t \rangle$.

20. $\vec{r}(t) = \left\langle \cos t, t, t^2 \right\rangle$.

In Exercises 21 – 30, create a vector–valued function whose graph matches the given description.

21. A circle of radius 2, centered at $(1, 2)$, traced counter–clockwise once on $[0, 2\pi]$.

22. A circle of radius 3, centered at $(5, 5)$, traced clockwise once on $[0, 2\pi]$.

23. An ellipse, centered at $(0, 0)$ with vertical major axis of length 10 and minor axis of length 3, traced once counter–clockwise on $[0, 2\pi]$.

24. An ellipse, centered at $(3, -2)$ with horizontal major axis of length 6 and minor axis of length 4, traced once clockwise on $[0, 2\pi]$.

25. A line through $(2, 3)$ with a slope of 5.

26. A line through $(1, 5)$ with a slope of $-1/2$.

27. The line through points $(1, 2, 3)$ and $(4, 5, 6)$, where $\vec{r}(0) = \langle 1, 2, 3 \rangle$ and $\vec{r}(1) = \langle 4, 5, 6 \rangle$.

28. The line through points $(1, 2)$ and $(4, 4)$, where $\vec{r}(0) = \langle 1, 2 \rangle$ and $\vec{r}(1) = \langle 4, 4 \rangle$.

29. A vertically oriented helix with radius of 2 that starts at $(2, 0, 0)$ and ends at $(2, 0, 4\pi)$ after 1 revolution on $[0, 2\pi]$.

30. A vertically oriented helix with radius of 3 that starts at $(3, 0, 0)$ and ends at $(3, 0, 3)$ after 2 revolutions on $[0, 1]$.

In Exercises 31 – 34, find the average rate of change of $\vec{r}(t)$ on the given interval.

31. $\vec{r}(t) = \left\langle t, t^2 \right\rangle$ on $[-2, 2]$.

32. $\vec{r}(t) = \langle t, t + \sin t \rangle$ on $[0, 2\pi]$.

33. $\vec{r}(t) = \langle 3\cos t, 2\sin t, t \rangle$ on $[0, 2\pi]$.

34. $\vec{r}(t) = \left\langle t, t^2, t^3 \right\rangle$ on $[-1, 3]$.

11.2 Calculus and Vector–Valued Functions

The previous section introduced us to a new mathematical object, the vector–valued function. We now apply calculus concepts to these functions. We start with the limit, then work our way through derivatives to integrals.

Limits of Vector–Valued Functions

The initial definition of the limit of a vector–valued function is a bit intimidating, as was the definition of the limit in Definition 1.2.1. The theorem following the definition shows that in practice, taking limits of vector–valued functions is no more difficult than taking limits of real–valued functions.

Definition 11.2.1 Limits of Vector–Valued Functions

Let I be an open interval containing c, and let $\vec{r}(t)$ be a vector–valued function defined on I, except possibly at c. The **limit of $\vec{r}(t)$, as t approaches c, is $\vec{L}$**, expressed as

$$\lim_{t\to c}\vec{r}(t) = \vec{L},$$

means that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $t \neq c$, if $|t - c| < \delta$, we have $||\,\vec{r}(t) - \vec{L}\,|| < \varepsilon$.

Note: we can define one-sided limits in a manner very similar to Definition 11.2.1.

Note how the measurement of distance between real numbers is the absolute value of their difference; the measure of distance between vectors is the vector norm, or magnitude, of their difference.

Theorem 11.2.1 states that we can compute limits of vector–valued functions component–wise.

Theorem 11.2.1 Limits of Vector–Valued Functions

1. Let $\vec{r}(t) = \langle\, f(t), g(t)\,\rangle$ be a vector–valued function in $\mathbb{R}^2$ defined on an open interval I containing c, except possibly at c. Then

$$\lim_{t\to c}\vec{r}(t) = \left\langle \lim_{t\to c} f(t)\,,\ \lim_{t\to c} g(t)\right\rangle.$$

2. Let $\vec{r}(t) = \langle\, f(t), g(t), h(t)\,\rangle$ be a vector–valued function in $\mathbb{R}^3$ defined on an open interval I containing c, except possibly at c. Then

$$\lim_{t\to c}\vec{r}(t) = \left\langle \lim_{t\to c} f(t)\,,\ \lim_{t\to c} g(t)\,,\ \lim_{t\to c} h(t)\right\rangle$$

Notes:

Example 11.2.1 Finding limits of vector–valued functions

Let $\vec{r}(t) = \left\langle \frac{\sin t}{t},\ t^2 - 3t + 3,\ \cos t \right\rangle$. Find $\lim_{t\to 0} \vec{r}(t)$.

Solution We apply the theorem and compute limits component–wise.

$$\begin{aligned}\lim_{t\to 0} \vec{r}(t) &= \left\langle \lim_{t\to 0} \frac{\sin t}{t}\ ,\ \lim_{t\to 0} t^2 - 3t + 3\ ,\ \lim_{t\to 0} \cos t \right\rangle \\ &= \langle 1, 3, 1 \rangle\ .\end{aligned}$$

Continuity

Definition 11.2.2 Continuity of Vector–Valued Functions

Let $\vec{r}(t)$ be a vector–valued function defined on an open interval I containing c.

1. $\vec{r}(t)$ is **continuous at** c if $\lim_{t\to c} \vec{r}(t) = \vec{r}(c)$.

2. If $\vec{r}(t)$ is continuous at all c in I, then $\vec{r}(t)$ is **continuous on I.**

Note: Using one-sided limits, we can also define continuity on closed intervals as done before.

We again have a theorem that lets us evaluate continuity component–wise.

Theorem 11.2.2 Continuity of Vector–Valued Functions

Let $\vec{r}(t)$ be a vector–valued function defined on an open interval I containing c. Then $\vec{r}(t)$ is continuous at c if, and only if, each of its component functions is continuous at c.

Example 11.2.2 Evaluating continuity of vector–valued functions

Let $\vec{r}(t) = \left\langle \frac{\sin t}{t},\ t^2 - 3t + 3,\ \cos t \right\rangle$. Determine whether $\vec{r}$ is continuous at $t = 0$ and $t = 1$.

Solution While the second and third components of $\vec{r}(t)$ are defined at $t = 0$, the first component, $(\sin t)/t$, is not. Since the first component is not even defined at $t = 0$, $\vec{r}(t)$ is not defined at $t = 0$, and hence it is not continuous at $t = 0$.

At $t = 1$ each of the component functions is continuous. Therefore $\vec{r}(t)$ is continuous at $t = 1$.

Notes:

Derivatives

Consider a vector–valued function $\vec{r}$ defined on an open interval I containing t_0 and t_1. We can compute the displacement of $\vec{r}$ on $[t_0, t_1]$, as shown in Figure 11.2.1(a). Recall that dividing the displacement vector by $t_1 - t_0$ gives the average rate of change on $[t_0, t_1]$, as shown in (b).

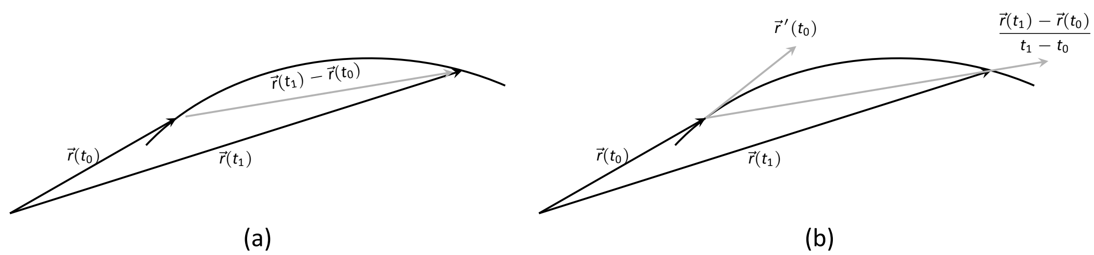


Figure 11.2.1: Illustrating displacement, leading to an understanding of the derivative of vector–valued functions.

The **derivative** of a vector–valued function is a measure of the *instantaneous* rate of change, measured by taking the limit as the length of $[t_0, t_1]$ goes to 0. Instead of thinking of an interval as $[t_0, t_1]$, we think of it as $[c, c+h]$ for some value of h (hence the interval has length h). The *average* rate of change is

$$\frac{\vec{r}(c+h) - \vec{r}(c)}{h}$$

for any value of $h \neq 0$. We take the limit as $h \to 0$ to measure the instantaneous rate of change; this is the derivative of $\vec{r}$.

Definition 11.2.3 Derivative of a Vector–Valued Function

Let $\vec{r}(t)$ be continuous on an open interval I containing c.

1. The **derivative of $\vec{r}$ at** $t = c$ is
$$\vec{r}\,'(c) = \lim_{h\to 0} \frac{\vec{r}(c+h) - \vec{r}(c)}{h}.$$

2. The **derivative of $\vec{r}$** is
$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

Alternate notations for the derivative of $\vec{r}$ include:

$$\vec{r}\,'(t) = \frac{d}{dt}\big(\vec{r}(t)\big) = \frac{d\vec{r}}{dt}.$$

Notes:

If a vector–valued function has a derivative for all c in an open interval I, we say that $\vec{r}(t)$ is **differentiable** on I.

Note: again, using one-sided limits, we can define differentiability on closed intervals. We'll make use of this a few times in this chapter.

Once again we might view this definition as intimidating, but recall that we can evaluate limits component–wise. The following theorem verifies that this means we can compute derivatives component–wise as well, making the task not too difficult.

Theorem 11.2.3 Derivatives of Vector–Valued Functions

1. Let $\vec{r}(t) = \langle f(t), g(t) \rangle$. Then
$$\vec{r}\,'(t) = \langle f'(t), g'(t) \rangle .$$
2. Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. Then
$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle .$$

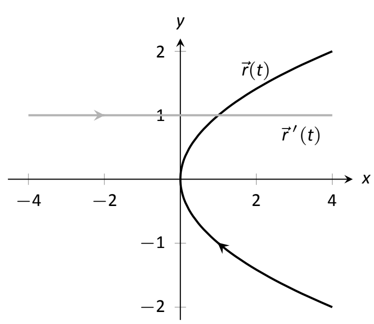
Example 11.2.3 Derivatives of vector–valued functions

Let $\vec{r}(t) = \left\langle t^2, t \right\rangle$.

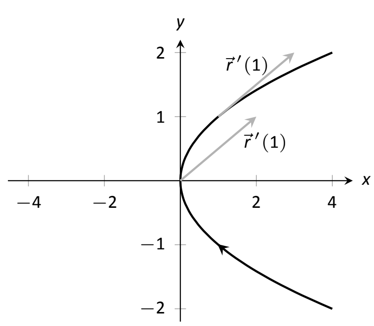
1. Sketch $\vec{r}(t)$ and $\vec{r}\,'(t)$ on the same axes.
2. Compute $\vec{r}\,'(1)$ and sketch this vector with its initial point at the origin and at $\vec{r}(1)$.

Solution

1. Theorem 11.2.3 allows us to compute derivatives component–wise, so
$$\vec{r}\,'(t) = \langle 2t, 1 \rangle .$$
$\vec{r}(t)$ and $\vec{r}\,'(t)$ are graphed together in Figure 11.2.2(a). Note how plotting the two of these together, in this way, is not very illuminating. When dealing with real–valued functions, plotting $f(x)$ with $f'(x)$ gave us useful information as we were able to compare f and f' at the same x-values. When dealing with vector–valued functions, it is hard to tell which points on the graph of $\vec{r}\,'$ correspond to which points on the graph of $\vec{r}$.
2. We easily compute $\vec{r}\,'(1) = \langle 2, 1 \rangle$, which is drawn in Figure 11.2.2 with its initial point at the origin, as well as at $\vec{r}(1) = \langle 1, 1 \rangle$. These are sketched in Figure 11.2.2(b).



(a)



(b)

Figure 11.2.2: Graphing the derivative of a vector–valued function in Example 11.2.3.

Notes:

Example 11.2.4 Derivatives of vector–valued functions

Let $\vec{r}(t) = \langle \cos t, \sin t, t\rangle$. Compute $\vec{r}\,'(t)$ and $\vec{r}\,'(\pi/2)$. Sketch $\vec{r}\,'(\pi/2)$ with its initial point at the origin and at $\vec{r}(\pi/2)$.

Solution We compute $\vec{r}\,'$ as $\vec{r}\,'(t) = \langle -\sin t, \cos t, 1\rangle$. At $t = \pi/2$, we have $\vec{r}\,'(\pi/2) = \langle -1, 0, 1\rangle$. Figure 11.2.3 shows two graphs of $\vec{r}(t)$, from different perspectives, with $\vec{r}\,'(\pi/2)$ plotted with its initial point at the origin and at $\vec{r}(\pi/2)$.

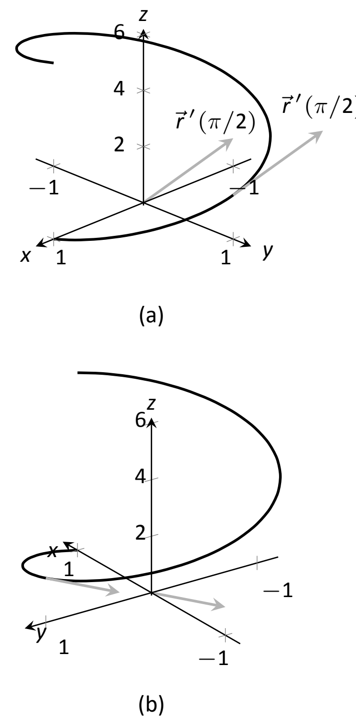


Figure 11.2.3: Viewing a vector–valued function, and its derivative at one point, from two different perspectives.

In Examples 11.2.3 and 11.2.4, sketching a particular derivative with its initial point at the origin did not seem to reveal anything significant. However, when we sketched the vector with its initial point on the corresponding point on the graph, we did see something significant: the vector appeared to be *tangent* to the graph. We have not yet defined what "tangent" means in terms of curves in space; in fact, we use the derivative to define this term.

Definition 11.2.4 Tangent Vector, Tangent Line

Let $\vec{r}(t)$ be a differentiable vector–valued function on an open interval I containing c, where $\vec{r}\,'(c) \neq \vec{0}$.

1. A vector $\vec{v}$ is **tangent to the graph of** $\vec{r}(t)$ **at** $t = c$ if $\vec{v}$ is parallel to $\vec{r}\,'(c)$.
2. The **tangent line** to the graph of $\vec{r}(t)$ at $t = c$ is the line through $\vec{r}(c)$ with direction parallel to $\vec{r}\,'(c)$. An equation of the tangent line is
$$\vec{\ell}(t) = \vec{r}(c) + t\,\vec{r}\,'(c).$$

Example 11.2.5 Finding tangent lines to curves in space

Let $\vec{r}(t) = \left\langle t, t^2, t^3\right\rangle$ on $[-1.5, 1.5]$. Find the vector equation of the line tangent to the graph of $\vec{r}$ at $t = -1$.

Solution To find the equation of a line, we need a point on the line and the line's direction. The point is given by $\vec{r}(-1) = \langle -1, 1, -1\rangle$. (To be clear, $\langle -1, 1, -1\rangle$ is a *vector*, not a point, but we use the point "pointed to" by this vector.)

The direction comes from $\vec{r}\,'(-1)$. We compute, component–wise, $\vec{r}\,'(t) = \left\langle 1, 2t, 3t^2\right\rangle$. Thus $\vec{r}\,'(-1) = \langle 1, -2, 3\rangle$.

The vector equation of the line is $\ell(t) = \langle -1, 1, -1\rangle + t\,\langle 1, -2, 3\rangle$. This line and $\vec{r}(t)$ are sketched, from two perspectives, in Figure 11.2.4 (a) and (b).

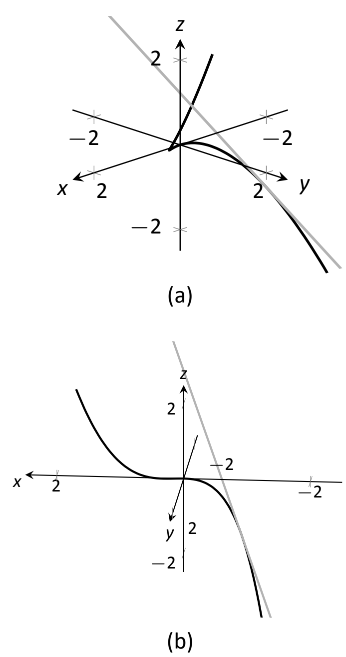


Figure 11.2.4: Graphing a curve in space with its tangent line.

Notes:

Example 11.2.6 Finding tangent lines to curves
Find the equations of the lines tangent to $\vec{r}(t) = \left\langle t^3, t^2 \right\rangle$ at $t = -1$ and $t = 0$.

SOLUTION We find that $\vec{r}\,'(t) = \left\langle 3t^2, 2t \right\rangle$. At $t = -1$, we have

$$\vec{r}(-1) = \langle -1, 1 \rangle \quad \text{and} \quad \vec{r}\,'(-1) = \langle 3, -2 \rangle ,$$

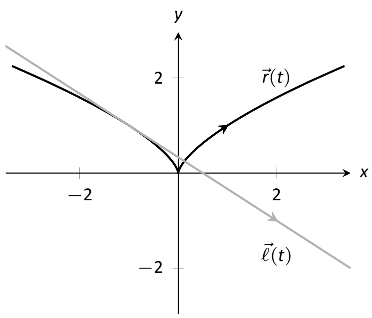


Figure 11.2.5: Graphing $\vec{r}(t)$ and its tangent line in Example 11.2.6.

so the equation of the line tangent to the graph of $\vec{r}(t)$ at $t = -1$ is

$$\ell(t) = \langle -1, 1 \rangle + t \langle 3, -2 \rangle .$$

This line is graphed with $\vec{r}(t)$ in Figure 11.2.5.

At $t = 0$, we have $\vec{r}\,'(0) = \langle 0, 0 \rangle = \vec{0}$! This implies that the tangent line "has no direction." We cannot apply Definition 11.2.4, hence cannot find the equation of the tangent line.

We were unable to compute the equation of the tangent line to $\vec{r}(t) = \left\langle t^3, t^2 \right\rangle$ at $t = 0$ because $\vec{r}\,'(0) = \vec{0}$. The graph in Figure 11.2.5 shows that there is a cusp at this point. This leads us to another definition of **smooth**, previously defined by Definition 9.2.2 in Section 9.2.

Definition 11.2.5 Smooth Vector–Valued Functions

Let $\vec{r}(t)$ be a differentiable vector–valued function on an open interval I where $\vec{r}\,'(t)$ is continuous on I. $\vec{r}(t)$ is **smooth** on I if $\vec{r}\,'(t) \neq \vec{0}$ on I.

Having established derivatives of vector–valued functions, we now explore the relationships between the derivative and other vector operations. The following theorem states how the derivative interacts with vector addition and the various vector products.

Notes:

Theorem 11.2.4 Properties of Derivatives of Vector–Valued Functions

Let $\vec{r}$ and $\vec{s}$ be differentiable vector–valued functions, let f be a differentiable real–valued function, and let c be a real number.

1. $\dfrac{d}{dt}\Big(\vec{r}(t) \pm \vec{s}(t)\Big) = \vec{r}\,'(t) \pm \vec{s}\,'(t)$
2. $\dfrac{d}{dt}\Big(c\vec{r}(t)\Big) = c\vec{r}\,'(t)$
3. $\dfrac{d}{dt}\Big(f(t)\vec{r}(t)\Big) = f\,'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$ **Product Rule**
4. $\dfrac{d}{dt}\Big(\vec{r}(t) \cdot \vec{s}(t)\Big) = \vec{r}\,'(t) \cdot \vec{s}(t) + \vec{r}(t) \cdot \vec{s}\,'(t)$ **Product Rule**
5. $\dfrac{d}{dt}\Big(\vec{r}(t) \times \vec{s}(t)\Big) = \vec{r}\,'(t) \times \vec{s}(t) + \vec{r}(t) \times \vec{s}\,'(t)$ **Product Rule**
6. $\dfrac{d}{dt}\Big(\vec{r}\big(f(t)\big)\Big) = \vec{r}\,'\big(f(t)\big)f\,'(t)$ **Chain Rule**

Example 11.2.7 Using derivative properties of vector–valued functions

Let $\vec{r}(t) = \langle t, t^2 - 1\rangle$ and let $\vec{u}(t)$ be the unit vector that points in the direction of $\vec{r}(t)$.

1. Graph $\vec{r}(t)$ and $\vec{u}(t)$ on the same axes, on $[-2, 2]$.
2. Find $\vec{u}\,'(t)$ and sketch $\vec{u}\,'(-2)$, $\vec{u}\,'(-1)$ and $\vec{u}\,'(0)$. Sketch each with initial point the corresponding point on the graph of $\vec{u}$.

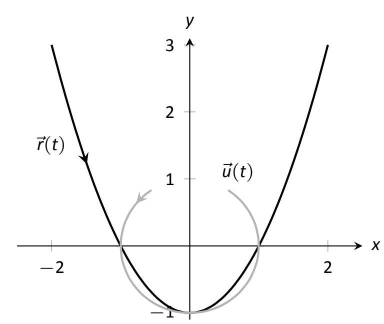


Figure 11.2.6: Graphing $\vec{r}(t)$ and $\vec{u}(t)$ in Example 11.2.7.

Solution

1. To form the unit vector that points in the direction of $\vec{r}$, we need to divide $\vec{r}(t)$ by its magnitude.

$$|| \,\vec{r}(t)\, || = \sqrt{t^2 + (t^2-1)^2} \quad \Rightarrow \quad \vec{u}(t) = \frac{1}{\sqrt{t^2+(t^2-1)^2}}\left\langle t, t^2-1\right\rangle.$$

$\vec{r}(t)$ and $\vec{u}(t)$ are graphed in Figure 11.2.6. Note how the graph of $\vec{u}(t)$ forms part of a circle; this must be the case, as the length of $\vec{u}(t)$ is 1 for all t.

Notes:

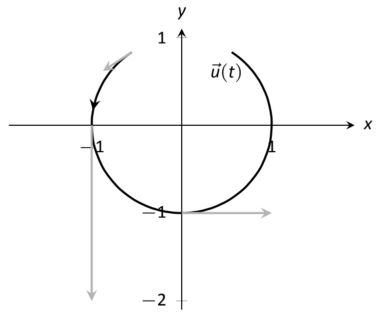


Figure 11.2.7: Graphing some of the derivatives of $\vec u(t)$ in Example 11.2.7.

2. To compute $\vec u\,'(t)$, we use Theorem 11.2.4, writing

$$\vec u(t) = f(t)\vec r(t), \quad \text{where} \quad f(t) = \frac{1}{\sqrt{t^2+(t^2-1)^2}} = \big(t^2+(t^2-1)^2\big)^{-1/2}.$$

(We *could* write

$$\vec u(t) = \left\langle \frac{t}{\sqrt{t^2+(t^2-1)^2}}, \frac{t^2-1}{\sqrt{t^2+(t^2-1)^2}} \right\rangle$$

and then take the derivative. It is a matter of preference; this latter method requires two applications of the Quotient Rule where our method uses the Product and Chain Rules.)

We find $f\,'(t)$ using the Chain Rule:

$$\begin{aligned} f\,'(t) &= -\frac{1}{2}\big(t^2+(t^2-1)^2\big)^{-3/2}\big(2t+2(t^2-1)(2t)\big) \\ &= -\frac{2t(2t^2-1)}{2\big(\sqrt{t^2+(t^2-1)^2}\,\big)^3} \end{aligned}$$

We now find $\vec u\,'(t)$ using part 3 of Theorem 11.2.4:

$$\begin{aligned} \vec u\,'(t) &= f\,'(t)\vec u(t) + f(t)\vec u\,'(t) \\ &= -\frac{2t(2t^2-1)}{2\big(\sqrt{t^2+(t^2-1)^2}\,\big)^3}\left\langle t, t^2-1\right\rangle + \frac{1}{\sqrt{t^2+(t^2-1)^2}}\left\langle 1, 2t\right\rangle. \end{aligned}$$

This is admittedly very "messy;" such is usually the case when we deal with unit vectors. We can use this formula to compute $\vec u\,'(-2)$, $\vec u\,'(-1)$ and $\vec u\,'(0)$:

$$\begin{aligned} \vec u\,'(-2) &= \left\langle -\frac{15}{13\sqrt{13}}, -\frac{10}{13\sqrt{13}} \right\rangle \approx \langle -0.320, -0.213\rangle \\ \vec u\,'(-1) &= \langle 0, -2\rangle \\ \vec u\,'(0) &= \langle 1, 0\rangle \end{aligned}$$

Each of these is sketched in Figure 11.2.7. Note how the length of the vector gives an indication of how quickly the circle is being traced at that point. When $t = -2$, the circle is being drawn relatively slow; when $t = -1$, the circle is being traced much more quickly.

It is a basic geometric fact that a line tangent to a circle at a point P is perpendicular to the line passing through the center of the circle and P. This is

Notes:

illustrated in Figure 11.2.7; each tangent vector is perpendicular to the line that passes through its initial point and the center of the circle. Since the center of the circle is the origin, we can state this another way: $\vec{u}\,'(t)$ is orthogonal to $\vec{u}(t)$.

Recall that the dot product serves as a test for orthogonality: if $\vec{u} \cdot \vec{v} = 0$, then $\vec{u}$ is orthogonal to $\vec{v}$. Thus in the above example, $\vec{u}(t) \cdot \vec{u}\,'(t) = 0$.

This is true of any vector–valued function that has a constant length, that is, that traces out part of a circle. It has important implications later on, so we state it as a theorem (and leave its formal proof as an Exercise.)

Theorem 11.2.5 Vector–Valued Functions of Constant Length

Let $\vec{r}(t)$ be a vector–valued function of constant length that is differentiable on an open interval I. That is, $||\,\vec{r}(t)\,|| = c$ for all t in I (equivalently, $\vec{r}(t) \cdot \vec{r}(t) = c^2$ for all t in I). Then $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$ for all t in I.

Integration

Before formally defining integrals of vector–valued functions, consider the following equation that our calculus experience tells us *should* be true:

$$\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(b) - \vec{r}(a).$$

That is, the integral of a rate of change function should give total change. In the context of vector–valued functions, this total change is displacement. The above equation *is* true; we now develop the theory to show why.

We can define antiderivatives and the indefinite integral of vector–valued functions in the same manner we defined indefinite integrals in Definition 5.1.1. However, we cannot define the definite integral of a vector–valued function as we did in Definition 5.2.1. That definition was based on the signed area between a function $y = f(x)$ and the x-axis. An area–based definition will not be useful in the context of vector–valued functions. Instead, we define the definite integral of a vector–valued function in a manner similar to that of Theorem 5.3.2, utilizing Riemann sums.

Notes:

Definition 11.2.6 Antiderivatives, Indefinite and Definite Integrals of Vector–Valued Functions

Let $\vec{r}(t)$ be a continuous vector–valued function on $[a,b]$. An **antiderivative** of $\vec{r}(t)$ is a function $\vec{R}(t)$ such that $\vec{R}'(t) = \vec{r}(t)$.

The set of all antiderivatives of $\vec{r}(t)$ is the **indefinite integral of** $\vec{r}(t)$, denoted by

$$\int \vec{r}(t)\ dt.$$

The definite integral of $\vec{r}(t)$ on $[a,b]$ is

$$\int_a^b \vec{r}(t)\ dt = \lim_{||\Delta t||\to 0} \sum_{i=1}^n \vec{r}(c_i)\Delta t_i,$$

where Δt_i is the length of the i^{th} subinterval of a partition of $[a,b]$, $||\Delta t||$ is the length of the largest subinterval in the partition, and c_i is any value in the i^{th} subinterval of the partition.

It is probably difficult to infer meaning from the definition of the definite integral. The important thing to realize from the definition is that it is built upon limits, which we can evaluate component–wise.

The following theorem simplifies the computation of definite integrals; the rest of this section and the following section will give meaning and application to these integrals.

Theorem 11.2.6 Indefinite and Definite Integrals of Vector–Valued Functions

Let $\vec{r}(t) = \langle f(t), g(t)\rangle$ be a vector–valued function in $\mathbb{R}^2$ that are continuous on $[a,b]$.

1. $\displaystyle \int \vec{r}(t)\ dt = \left\langle \int f(t)\ dt, \int g(t)\ dt \right\rangle$
2. $\displaystyle \int_a^b \vec{r}(t)\ dt = \left\langle \int_a^b f(t)\ dt, \int_a^b g(t)\ dt \right\rangle$

A similar statement holds for vector–valued functions in $\mathbb{R}^3$.

Notes:

Example 11.2.8 Evaluating a definite integral of a vector–valued function

Let $\vec{r}(t) = \left\langle e^{2t}, \sin t \right\rangle$. Evaluate $\displaystyle\int_0^1 \vec{r}(t)\, dt$.

Solution We follow Theorem 11.2.6.

$$\begin{aligned}
\int_0^1 \vec{r}(t)\, dt &= \int_0^1 \left\langle e^{2t}, \sin t \right\rangle\, dt \\
&= \left\langle \int_0^1 e^{2t}\, dt\ , \int_0^1 \sin t\, dt \right\rangle \\
&= \left\langle \frac{1}{2}e^{2t}\Big|_0^1\ , -\cos t\Big|_0^1 \right\rangle \\
&= \left\langle \frac{1}{2}(e^2 - 1)\ , -\cos(1) + 1 \right\rangle \\
&\approx \langle 3.19, 0.460 \rangle\ .
\end{aligned}$$

Example 11.2.9 Solving an initial value problem

Let $\vec{r}\,''(t) = \langle 2, \cos t, 12t \rangle$. Find $\vec{r}(t)$, where $\vec{r}(0) = \langle -7, -1, 2 \rangle$ and $\vec{r}\,'(0) = \langle 5, 3, 0 \rangle$.

Solution Knowing $\vec{r}\,''(t) = \langle 2, \cos t, 12t \rangle$, we find $\vec{r}\,'(t)$ by evaluating the indefinite integral.

$$\begin{aligned}
\int \vec{r}\,''(t)\, dt &= \left\langle \int 2\, dt\ , \int \cos t\, dt\ , \int 12t\, dt \right\rangle \\
&= \left\langle 2t + C_1, \sin t + C_2, 6t^2 + C_3 \right\rangle \\
&= \left\langle 2t, \sin t, 6t^2 \right\rangle + \langle C_1, C_2, C_3 \rangle \\
&= \left\langle 2t, \sin t, 6t^2 \right\rangle + \vec{C}.
\end{aligned}$$

Note how each indefinite integral creates its own constant which we collect as one constant vector $\vec{C}$. Knowing $\vec{r}\,'(0) = \langle 5, 3, 0 \rangle$ allows us to solve for $\vec{C}$:

$$\begin{aligned}
\vec{r}\,'(t) &= \left\langle 2t, \sin t, 6t^2 \right\rangle + \vec{C} \\
\vec{r}\,'(0) &= \langle 0, 0, 0 \rangle + \vec{C} \\
\langle 5, 3, 0 \rangle &= \vec{C}.
\end{aligned}$$

So $\vec{r}\,'(t) = \left\langle 2t, \sin t, 6t^2 \right\rangle + \langle 5, 3, 0 \rangle = \left\langle 2t + 5, \sin t + 3, 6t^2 \right\rangle$. To find $\vec{r}(t)$, we integrate once more.

$$\begin{aligned}
\int \vec{r}\,'(t)\, dt &= \left\langle \int 2t + 5\, dt, \int \sin t + 3\, dt, \int 6t^2\, dt \right\rangle \\
&= \left\langle t^2 + 5t, -\cos t + 3t, 2t^3 \right\rangle + \vec{C}.
\end{aligned}$$

Notes:

With $\vec{r}(0) = \langle -7, -1, 2 \rangle$, we solve for $\vec{C}$:

$$\begin{aligned} \vec{r}(t) &= \left\langle t^2 + 5t, -\cos t + 3t, 2t^3 \right\rangle + \vec{C} \\ \vec{r}(0) &= \langle 0, -1, 0 \rangle + \vec{C} \\ \langle -7, -1, 2 \rangle &= \langle 0, -1, 0 \rangle + \vec{C} \\ \langle -7, 0, 2 \rangle &= \vec{C}. \end{aligned}$$

So $\vec{r}(t) = \left\langle t^2 + 5t, -\cos t + 3t, 2t^3 \right\rangle + \langle -7, 0, 2 \rangle = \left\langle t^2 + 5t - 7, -\cos t + 3t, 2t^3 + 2 \right\rangle$.

What does the integration of a vector–valued function *mean*? There are many applications, but none as direct as "the area under the curve" that we used in understanding the integral of a real–valued function.

A key understanding for us comes from considering the integral of a derivative:

$$\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(t)\Big|_a^b = \vec{r}(b) - \vec{r}(a).$$

Integrating a *rate of change* function gives *displacement*.

Noting that vector–valued functions are closely related to parametric equations, we can describe the arc length of the graph of a vector–valued function as an integral. Given parametric equations $x = f(t)$, $y = g(t)$, the arc length on $[a, b]$ of the graph is

$$\text{Arc Length} = \int_a^b \sqrt{f\,'(t)^2 + g\,'(t)^2}\,dt,$$

as stated in Theorem 9.3.1. If $\vec{r}(t) = \langle f(t), g(t) \rangle$, note that $\sqrt{f\,'(t)^2 + g\,'(t)^2} = \|\, \vec{r}\,'(t) \,\|$. Therefore we can express the arc length of the graph of a vector–valued function as an integral of the magnitude of its derivative.

Theorem 11.2.7 Arc Length of a Vector–Valued Function

Let $\vec{r}(t)$ be a vector–valued function where $\vec{r}\,'(t)$ is continuous on $[a, b]$. The arc length L of the graph of $\vec{r}(t)$ is

$$L = \int_a^b \|\, \vec{r}\,'(t) \,\|\, dt.$$

Note that we are actually integrating a scalar–function here, not a vector–valued function.

The next section takes what we have established thus far and applies it to objects in motion. We will let $\vec{r}(t)$ describe the path of an object in the plane or in space and will discover the information provided by $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$.

Notes:

Exercises 11.2

Terms and Concepts

1. Limits, derivatives and integrals of vector–valued functions are all evaluated ________–wise.

2. The definite integral of a rate of change function gives ________.

3. Why is it generally not useful to graph both $\vec{r}(t)$ and $\vec{r}\,'(t)$ on the same axes?

4. Theorem 11.2.4 contains three product rules. What are the three different types of products used in these rules?

Problems

In Exercises 5 – 8, evaluate the given limit.

5. $\lim\limits_{t\to 5} \langle 2t+1, 3t^2-1, \sin t\rangle$

6. $\lim\limits_{t\to 3} \left\langle e^t, \dfrac{t^2-9}{t+3}\right\rangle$

7. $\lim\limits_{t\to 0} \left\langle \dfrac{t}{\sin t}, (1+t)^{\frac{1}{t}}\right\rangle$

8. $\lim\limits_{h\to 0} \dfrac{\vec{r}(t+h)-\vec{r}(t)}{h}$, where $\vec{r}(t) = \langle t^2, t, 1\rangle$.

In Exercises 9 – 10, identify the interval(s) on which $\vec{r}(t)$ is continuous.

9. $\vec{r}(t) = \langle t^2, 1/t\rangle$

10. $\vec{r}(t) = \langle \cos t, e^t, \ln t\rangle$

In Exercises 11 – 16, find the derivative of the given function.

11. $\vec{r}(t) = \langle \cos t, e^t, \ln t\rangle$

12. $\vec{r}(t) = \left\langle \dfrac{1}{t}, \dfrac{2t-1}{3t+1}, \tan t\right\rangle$

13. $\vec{r}(t) = (t^2)\,\langle \sin t, 2t+5\rangle$

14. $r(t) = \langle t^2+1, t-1\rangle \cdot \langle \sin t, 2t+5\rangle$

15. $\vec{r}(t) = \langle t^2+1, t-1, 1\rangle \times \langle \sin t, 2t+5, 1\rangle$

16. $\vec{r}(t) = \langle \cosh t, \sinh t\rangle$

In Exercises 17 – 20, find $\vec{r}\,'(t)$. Sketch $\vec{r}(t)$ and $\vec{r}\,'(1)$, with the initial point of $\vec{r}\,'(1)$ at $\vec{r}(1)$.

17. $\vec{r}(t) = \langle t^2+t, t^2-t\rangle$

18. $\vec{r}(t) = \langle t^2-2t+2, t^3-3t^2+2t\rangle$

19. $\vec{r}(t) = \langle t^2+1, t^3-t\rangle$

20. $\vec{r}(t) = \langle t^2-4t+5, t^3-6t^2+11t-6\rangle$

In Exercises 21 – 24, give the equation of the line tangent to the graph of $\vec{r}(t)$ at the given t value.

21. $\vec{r}(t) = \langle t^2+t, t^2-t\rangle$ at $t=1$.

22. $\vec{r}(t) = \langle 3\cos t, \sin t\rangle$ at $t=\pi/4$.

23. $\vec{r}(t) = \langle 3\cos t, 3\sin t, t\rangle$ at $t=\pi$.

24. $\vec{r}(t) = \langle e^t, \tan t, t\rangle$ at $t=0$.

In Exercises 25 – 28, find the value(s) of t for which $\vec{r}(t)$ is not smooth.

25. $\vec{r}(t) = \langle \cos t, \sin t - t\rangle$

26. $\vec{r}(t) = \langle t^2-2t+1, t^3+t^2-5t+3\rangle$

27. $\vec{r}(t) = \langle \cos t - \sin t, \sin t - \cos t, \cos(4t)\rangle$

28. $\vec{r}(t) = \langle t^3-3t+2, -\cos(\pi t), \sin^2(\pi t)\rangle$

Exercises 29 – 32 ask you to verify parts of Theorem 11.2.4. In each let $f(t) = t^3$, $\vec{r}(t) = \langle t^2, t-1, 1\rangle$ and $\vec{s}(t) = \langle \sin t, e^t, t\rangle$. Compute the various derivatives as indicated.

29. Simplify $f(t)\vec{r}(t)$, then find its derivative; show this is the same as $f\,'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$.

30. Simplify $\vec{r}(t)\cdot\vec{s}(t)$, then find its derivative; show this is the same as $\vec{r}\,'(t)\cdot\vec{s}(t) + \vec{r}(t)\cdot\vec{s}\,'(t)$.

31. Simplify $\vec{r}(t)\times\vec{s}(t)$, then find its derivative; show this is the same as $\vec{r}\,'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}\,'(t)$.

32. Simplify $\vec{r}\big(f(t)\big)$, then find its derivative; show this is the same as $\vec{r}\,'\big(f(t)\big)f\,'(t)$.

In Exercises 33 – 36, evaluate the given definite or indefinite integral.

33. $\displaystyle\int \langle t^3, \cos t, te^t\rangle\ dt$

34. $\displaystyle\int \left\langle \frac{1}{1+t^2}, \sec^2 t\right\rangle\ dt$

35. $\displaystyle\int_0^{\pi} \langle -\sin t, \cos t\rangle\ dt$

36. $\displaystyle\int_{-2}^{2} \langle 2t+1, 2t-1\rangle\ dt$

In Exercises 37 – 40, solve the given initial value problems.

37. Find $\vec{r}(t)$, given that $\vec{r}\,'(t) = \langle t, \sin t\rangle$ and $\vec{r}(0) = \langle 2, 2\rangle$.

38. Find $\vec{r}(t)$, given that $\vec{r}\,'(t) = \langle 1/(t+1), \tan t\rangle$ and $\vec{r}(0) = \langle 1, 2\rangle$.

39. Find $\vec{r}(t)$, given that $\vec{r}\,''(t) = \left\langle t^2, t, 1\right\rangle$, $\vec{r}\,'(0) = \langle 1, 2, 3\rangle$ and $\vec{r}(0) = \langle 4, 5, 6\rangle$.

40. Find $\vec{r}(t)$, given that $\vec{r}\,''(t) = \left\langle \cos t, \sin t, e^t\right\rangle$, $\vec{r}\,'(0) = \langle 0, 0, 0\rangle$ and $\vec{r}(0) = \langle 0, 0, 0\rangle$.

In Exercises 41 – 44 , find the arc length of $\vec{r}(t)$ on the indicated interval.

41. $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t\rangle$ on $[0, 2\pi]$.

42. $\vec{r}(t) = \langle 5\cos t, 3\sin t, 4\sin t\rangle$ on $[0, 2\pi]$.

43. $\vec{r}(t) = \left\langle t^3, t^2, t^3\right\rangle$ on $[0, 1]$.

44. $\vec{r}(t) = \left\langle e^{-t}\cos t, e^{-t}\sin t\right\rangle$ on $[0, 1]$.

45. Prove Theorem 11.2.5; that is, show if $\vec{r}(t)$ has constant length and is differentiable, then $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$. (Hint: use the Product Rule to compute $\frac{d}{dt}\big(\vec{r}(t)\cdot\vec{r}(t)\big)$.)

11.3 The Calculus of Motion

A common use of vector–valued functions is to describe the motion of an object in the plane or in space. A **position function** $\vec{r}(t)$ gives the position of an object at **time** t. This section explores how derivatives and integrals are used to study the motion described by such a function.

Definition 11.3.1 Velocity, Speed and Acceleration

Let $\vec{r}(t)$ be a position function in $\mathbb{R}^2$ or $\mathbb{R}^3$.

1. **Velocity**, denoted $\vec{v}(t)$, is the instantaneous rate of position change; that is, $\vec{v}(t) = \vec{r}\,'(t)$.
2. **Speed** is the magnitude of velocity, $||\ \vec{v}(t)\ ||$.
3. **Acceleration**, denoted $\vec{a}(t)$, is the instantaneous rate of velocity change; that is, $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

Example 11.3.1 Finding velocity and acceleration

An object is moving with position function $\vec{r}(t) = \left\langle t^2 - t, t^2 + t \right\rangle$, $-3 \leq t \leq 3$, where distances are measured in feet and time is measured in seconds.

1. Find $\vec{v}(t)$ and $\vec{a}(t)$.
2. Sketch $\vec{r}(t)$; plot $\vec{v}(-1)$, $\vec{a}(-1)$, $\vec{v}(1)$ and $\vec{a}(1)$, each with their initial point at their corresponding point on the graph of $\vec{r}(t)$.
3. When is the object's speed minimized?

Solution

1. Taking derivatives, we find

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t - 1, 2t + 1 \rangle \quad \text{and} \quad \vec{a}(t) = \vec{r}\,''(t) = \langle 2, 2 \rangle .$$

   Note that acceleration is constant.

2. $\vec{v}(-1) = \langle -3, -1 \rangle$, $\vec{a}(-1) = \langle 2, 2 \rangle$; $\quad \vec{v}(1) = \langle 1, 3 \rangle$, $\vec{a}(1) = \langle 2, 2 \rangle$. These are plotted with $\vec{r}(t)$ in Figure 11.3.1(a).

   We can think of acceleration as "pulling" the velocity vector in a certain direction. At $t = -1$, the velocity vector points down and to the left; at $t = 1$, the velocity vector has been pulled in the $\langle 2, 2 \rangle$ direction and is

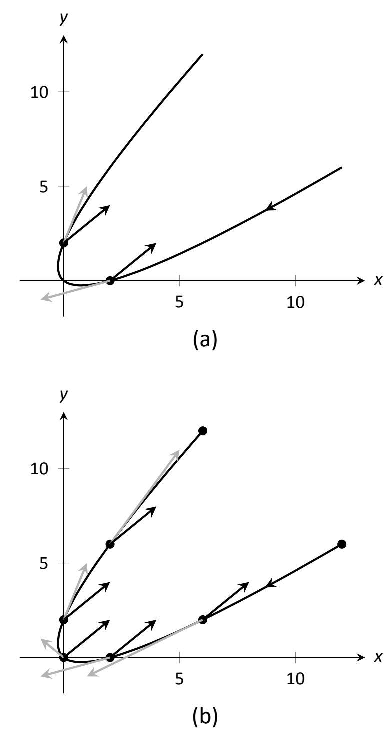


Figure 11.3.1: Graphing the position, velocity and acceleration of an object in Example 11.3.1.

Notes:

now pointing up and to the right. In Figure 11.3.1(b) we plot more velocity/acceleration vectors, making more clear the effect acceleration has on velocity.

Since $\vec{a}(t)$ is constant in this example, as t grows large $\vec{v}(t)$ becomes almost parallel to $\vec{a}(t)$. For instance, when $t = 10$, $\vec{v}(10) = \langle 19, 21\rangle$, which is nearly parallel to $\langle 2, 2\rangle$.

3. The object's speed is given by

$$|| \vec{v}(t) || = \sqrt{(2t-1)^2 + (2t+1)^2} = \sqrt{8t^2+2}.$$

To find the minimal speed, we could apply calculus techniques (such as set the derivative equal to 0 and solve for t, etc.) but we can find it by inspection. Inside the square root we have a quadratic which is minimized when $t = 0$. Thus the speed is minimized at $t = 0$, with a speed of $\sqrt{2}$ ft/s.

The graph in Figure 11.3.1(b) also implies speed is minimized here. The filled dots on the graph are located at integer values of t between -3 and 3. Dots that are far apart imply the object traveled a far distance in 1 second, indicating high speed; dots that are close together imply the object did not travel far in 1 second, indicating a low speed. The dots are closest together near $t = 0$, implying the speed is minimized near that value.

Example 11.3.2 Analyzing Motion

Two objects follow an identical path at different rates on $[-1, 1]$. The position function for Object 1 is $\vec{r}_1(t) = \left\langle t, t^2\right\rangle$; the position function for Object 2 is $\vec{r}_2(t) = \left\langle t^3, t^6\right\rangle$, where distances are measured in feet and time is measured in seconds. Compare the velocity, speed and acceleration of the two objects on the path.

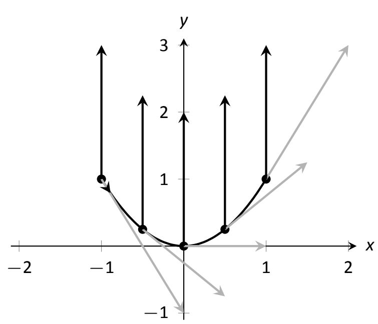


Figure 11.3.2: Plotting velocity and acceleration vectors for Object 1 in Example 11.3.2.

Solution We begin by computing the velocity and acceleration function for each object:

$$\vec{v}_1(t) = \langle 1, 2t\rangle \qquad \vec{v}_2(t) = \left\langle 3t^2, 6t^5\right\rangle$$
$$\vec{a}_1(t) = \langle 0, 2\rangle \qquad \vec{a}_2(t) = \left\langle 6t, 30t^4\right\rangle$$

We immediately see that Object 1 has constant acceleration, whereas Object 2 does not.

At $t = -1$, we have $\vec{v}_1(-1) = \langle 1, -2\rangle$ and $\vec{v}_2(-1) = \langle 3, -6\rangle$; the velocity of Object 2 is three times that of Object 1 and so it follows that the speed of Object 2 is three times that of Object 1 ($3\sqrt{5}$ ft/s compared to $\sqrt{5}$ ft/s.)

Notes:

At $t = 0$, the velocity of Object 1 is $\vec{v}(1) = \langle 1, 0\rangle$ and the velocity of Object 2 is $\vec{0}$! This tells us that Object 2 comes to a complete stop at $t = 0$.

In Figure 11.3.2, we see the velocity and acceleration vectors for Object 1 plotted for $t = -1, -1/2, 0, 1/2$ and $t = 1$. Note again how the constant acceleration vector seems to "pull" the velocity vector from pointing down, right to up, right. We could plot the analogous picture for Object 2, but the velocity and acceleration vectors are rather large ($\vec{a}_2(-1) = \langle -6, 30\rangle$!)

Instead, we simply plot the locations of Object 1 and 2 on intervals of $1/5^{\text{th}}$ of a second, shown in Figure 11.3.3(a) and (b). Note how the x-values of Object 1 increase at a steady rate. This is because the x-component of $\vec{a}(t)$ is 0; there is no acceleration in the x-component. The dots are not evenly spaced; the object is moving faster near $t = -1$ and $t = 1$ than near $t = 0$.

In part (b) of the Figure, we see the points plotted for Object 2. Note the large change in position from $t = -1$ to $t = -0.8$; the object starts moving very quickly. However, it slows considerably at it approaches the origin, and comes to a complete stop at $t = 0$. While it looks like there are 3 points near the origin, there are in reality 5 points there.

Since the objects begin and end at the same location, they have the same displacement. Since they begin and end at the same time, with the same displacement, they have the same average rate of change (i.e, they have the same average velocity). Since they follow the same path, they have the same distance traveled. Even though these three measurements are the same, the objects obviously travel the path in very different ways.

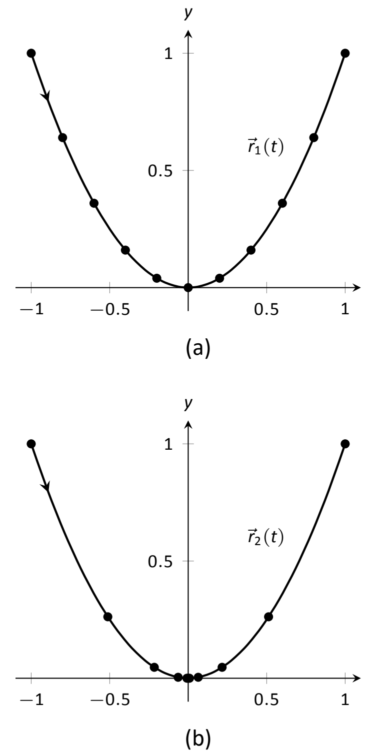


Figure 11.3.3: Comparing the positions of Objects 1 and 2 in Example 11.3.2.

Example 11.3.3 Analyzing the motion of a whirling ball on a string

A young boy whirls a ball, attached to a string, above his head in a counter-clockwise circle. The ball follows a circular path and makes 2 revolutions per second. The string has length 2ft.

1. Find the position function $\vec{r}(t)$ that describes this situation.
2. Find the acceleration of the ball and give a physical interpretation of it.
3. A tree stands 10ft in front of the boy. At what t-values should the boy release the string so that the ball hits the tree?

SOLUTION

1. The ball whirls in a circle. Since the string is 2ft long, the radius of the circle is 2. The position function $\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$ describes a circle with radius 2, centered at the origin, but makes a full revolution every 2π seconds, not two revolutions per second. We modify the period of the

Notes:

trigonometric functions to be 1/2 by multiplying t by 4π. The final position function is thus

$$\vec{r}(t) = \langle 2\cos(4\pi t), 2\sin(4\pi t)\rangle .$$

(Plot this for $0 \leq t \leq 1/2$ to verify that one revolution is made in 1/2 a second.)

2. To find $\vec{a}(t)$, we take the derivative of $\vec{r}(t)$ twice.

$$\begin{aligned}\vec{v}(t) = \vec{r}\,'(t) &= \langle -8\pi\sin(4\pi t), 8\pi\cos(4\pi t)\rangle \\ \vec{a}(t) = \vec{r}\,''(t) &= \left\langle -32\pi^2\cos(4\pi t), -32\pi^2\sin(4\pi t)\right\rangle \\ &= -32\pi^2\left\langle \cos(4\pi t), \sin(4\pi t)\right\rangle .\end{aligned}$$

Note how $\vec{a}(t)$ is parallel to $\vec{r}(t)$, but has a different magnitude and points in the opposite direction. Why is this?

Recall the classic physics equation, "Force = mass × acceleration." A force acting on a mass induces acceleration (i.e., the mass moves); acceleration acting on a mass induces a force (gravity gives our mass a *weight*). Thus force and acceleration are closely related. A moving ball "wants" to travel in a straight line. Why does the ball in our example move in a circle? It is attached to the boy's hand by a string. The string applies a force to the ball, affecting it's motion: the string *accelerates* the ball. This is not acceleration in the sense of "it travels faster;" rather, this acceleration is changing the velocity of the ball. In what direction is this force/acceleration being applied? In the direction of the string, towards the boy's hand.

The magnitude of the acceleration is related to the speed at which the ball is traveling. A ball whirling quickly is rapidly changing direction/velocity. When velocity is changing rapidly, the acceleration must be "large."

3. When the boy releases the string, the string no longer applies a force to the ball, meaning acceleration is $\vec{0}$ and the ball can now move in a straight line in the direction of $\vec{v}(t)$.

Let $t = t_0$ be the time when the boy lets go of the string. The ball will be at $\vec{r}(t_0)$, traveling in the direction of $\vec{v}(t_0)$. We want to find t_0 so that this line contains the point $(0, 10)$ (since the tree is 10ft directly in front of the boy).

There are many ways to find this time value. We choose one that is relatively simple computationally. As shown in Figure 11.3.4, the vector from the release point to the tree is $\langle 0, 10\rangle - \vec{r}(t_0)$. This line segment is tangent to the circle, which means it is also perpendicular to $\vec{r}(t_0)$ itself, so their dot product is 0.

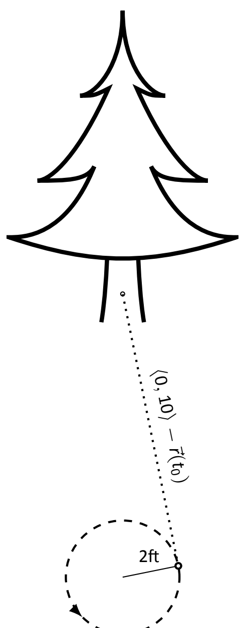


Figure 11.3.4: Modeling the flight of a ball in Example 11.3.3.

Notes:

$$\begin{aligned}
\vec{r}(t_0) \cdot \big(\langle 0, 10 \rangle - \vec{r}(t_0) \big) &= 0 \\
\langle 2\cos(4\pi t_0), 2\sin(4\pi t_0) \rangle \cdot \langle -2\cos(4\pi t_0), 10 - 2\sin(4\pi t_0) \rangle &= 0 \\
-4\cos^2(4\pi t_0) + 20\sin(4\pi t_0) - 4\sin^2(4\pi t_0) &= 0 \\
20\sin(4\pi t_0) - 4 &= 0 \\
\sin(4\pi t_0) &= 1/5 \\
4\pi t_0 &= \sin^{-1}(1/5) \\
4\pi t_0 &\approx 0.2 + 2\pi n,
\end{aligned}$$

where n is an integer. Solving for t_0 we have:

$$t_0 \approx 0.016 + n/2$$

This is a wonderful formula. Every 1/2 second after $t = 0.016$s the boy can release the string (since the ball makes 2 revolutions per second, he has two chances each second to release the ball).

Example 11.3.4 Analyzing motion in space

An object moves in a spiral with position function $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, where distances are measured in meters and time is in minutes. Describe the object's speed and acceleration at time t.

SOLUTION With $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, we have:

$$\begin{aligned}
\vec{v}(t) &= \langle -\sin t, \cos t, 1 \rangle \quad \text{and} \\
\vec{a}(t) &= \langle -\cos t, -\sin t, 0 \rangle .
\end{aligned}$$

The speed of the object is $||\,\vec{v}(t)\,|| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$m/min; it moves at a constant speed. Note that the object does not accelerate in the z-direction, but rather moves up at a constant rate of 1m/min.

The objects in Examples 11.3.3 and 11.3.4 traveled at a constant speed. That is, $||\,\vec{v}(t)\,|| = c$ for some constant c. Recall Theorem 11.2.5, which states that if a vector–valued function $\vec{r}(t)$ has constant length, then $\vec{r}(t)$ is perpendicular to its derivative: $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$. In these examples, the velocity function has constant length, therefore we can conclude that the velocity is perpendicular to the acceleration: $\vec{v}(t) \cdot \vec{a}(t) = 0$. A quick check verifies this.

There is an intuitive understanding of this. If acceleration is parallel to velocity, then it is only affecting the object's speed; it does not change the direction of travel. (For example, consider a dropped stone. Acceleration and velocity are

Notes:

parallel – straight down – and the direction of velocity never changes, though speed does increase.) If acceleration is not perpendicular to velocity, then there is some acceleration in the direction of travel, influencing the speed. If speed is constant, then acceleration must be orthogonal to velocity, as it then only affects direction, and not speed.

Key Idea 11.3.1 Objects With Constant Speed

If an object moves with constant speed, then its velocity and acceleration vectors are orthogonal. That is, $\vec{v}(t)\cdot\vec{a}(t)=0$.

Projectile Motion

An important application of vector–valued position functions is *projectile motion*: the motion of objects under only the influence of gravity. We will measure time in seconds, and distances will either be in meters or feet. We will show that we can completely describe the path of such an object knowing its initial position and initial velocity (i.e., where it *is* and where it *is going.*)

Suppose an object has initial position $\vec{r}(0)=\langle x_0,y_0\rangle$ and initial velocity $\vec{v}(0)=\langle v_x,v_y\rangle$. It is customary to rewrite $\vec{v}(0)$ in terms of its speed v_0 and direction $\vec{u}$, where $\vec{u}$ is a unit vector. Recall all unit vectors in $\mathbb{R}^2$ can be written as $\langle\cos\theta,\sin\theta\rangle$, where θ is an angle measure counter–clockwise from the x-axis. (We refer to θ as the **angle of elevation.**) Thus $\vec{v}(0)=v_0\left\langle\cos\theta,\sin\theta\right\rangle$.

Since the acceleration of the object is known, namely $\vec{a}(t)=\langle 0,-g\rangle$, where g is the gravitational constant, we can find $\vec{r}(t)$ knowing our two initial conditions. We first find $\vec{v}(t)$:

Note: This text uses $g=32\text{ft/s}^2$ when using Imperial units, and $g=9.8\text{m/s}^2$ when using SI units.

$$\begin{aligned}\vec{v}(t)&=\int\vec{a}(t)\,dt\\ \vec{v}(t)&=\int\langle 0,-g\rangle\ dt\\ \vec{v}(t)&=\langle 0,-gt\rangle+\vec{C}.\end{aligned}$$

Knowing $\vec{v}(0)=v_0\left\langle\cos\theta,\sin\theta\right\rangle$, we have $\vec{C}=v_0\left\langle\cos\theta,\sin\theta\right\rangle$ and so

$$\vec{v}(t)=\Big\langle v_0\cos\theta,-gt+v_0\sin\theta\Big\rangle.$$

Notes:

We integrate once more to find $\vec{r}(t)$:

$$\begin{aligned}\vec{r}(t) &= \int \vec{v}(t)\, dt\\ \vec{r}(t) &= \int \big\langle v_0\cos\theta, -gt + v_0\sin\theta\big\rangle\; dt\\ \vec{r}(t) &= \left\langle \big(v_0\cos\theta\big)t, -\frac{1}{2}gt^2 + \big(v_0\sin\theta\big)t\right\rangle + \vec{C}.\end{aligned}$$

Knowing $\vec{r}(0) = \langle x_0, y_0\rangle$, we conclude $\vec{C} = \langle x_0, y_0\rangle$ and

$$\vec{r}(t) = \left\langle \big(v_0\cos\theta\big)t + x_0\;, -\frac{1}{2}gt^2 + \big(v_0\sin\theta\big)t + y_0\;\right\rangle.$$

Key Idea 11.3.2 Projectile Motion

The position function of a projectile propelled from an initial position of $\vec{r}_0 = \langle x_0, y_0\rangle$, with initial speed v_0, with angle of elevation θ and neglecting all accelerations but gravity is

$$\vec{r}(t) = \left\langle \big(v_0\cos\theta\big)t + x_0\;, -\frac{1}{2}gt^2 + \big(v_0\sin\theta\big)t + y_0\;\right\rangle.$$

Letting $\vec{v}_0 = v_0\,\langle\cos\theta, \sin\theta\rangle$, $\vec{r}(t)$ can be written as

$$\vec{r}(t) = \left\langle 0, -\frac{1}{2}gt^2\right\rangle + \vec{v}_0 t + \vec{r}_0.$$

We demonstrate how to use this position function in the next two examples.

Example 11.3.5 Projectile Motion

Sydney shoots her Red Ryder® bb gun across level ground from an elevation of 4ft, where the barrel of the gun makes a 5° angle with the horizontal. Find how far the bb travels before landing, assuming the bb is fired at the advertised rate of 350ft/s and ignoring air resistance.

Solution A direct application of Key Idea 11.3.2 gives

$$\begin{aligned}\vec{r}(t) &= \big\langle(350\cos 5^\circ)t, -16t^2 + (350\sin 5^\circ)t + 4\big\rangle\\ &\approx \big\langle 346.67t, -16t^2 + 30.50t + 4\big\rangle,\end{aligned}$$

Notes:

where we set her initial position to be $\langle 0,4\rangle$. We need to find *when* the bb lands, then we can find *where*. We accomplish this by setting the y-component equal to 0 and solving for t:

$$\begin{aligned}-16t^2+30.50t+4&=0\\ t&=\frac{-30.50\pm\sqrt{30.50^2-4(-16)(4)}}{-32}\\ t&\approx 2.03s.\end{aligned}$$

(We discarded a negative solution that resulted from our quadratic equation.)

We have found that the bb lands 2.03s after firing; with $t=2.03$, we find the x-component of our position function is $346.67(2.03)=703.74$ft. The bb lands about 704 feet away.

Example 11.3.6 Projectile Motion

Alex holds his sister's bb gun at a height of 3ft and wants to shoot a target that is 6ft above the ground, 25ft away. At what angle should he hold the gun to hit his target? (We still assume the muzzle velocity is 350ft/s.)

Solution The position function for the path of Alex's bb is

$$\vec{r}(t)=\left\langle(350\cos\theta)t,-16t^2+(350\sin\theta)t+3\right\rangle.$$

We need to find θ so that $\vec{r}(t)=\langle 25,6\rangle$ for some value of t. That is, we want to find θ and t such that

$$(350\cos\theta)t=25\quad\text{and}\quad -16t^2+(350\sin\theta)t+3=6.$$

This is not trivial (though not "hard"). We start by solving each equation for $\cos\theta$ and $\sin\theta$, respectively.

$$\cos\theta=\frac{25}{350t}\quad\text{and}\quad\sin\theta=\frac{3+16t^2}{350t}.$$

Using the Pythagorean Identity $\cos^2\theta+\sin^2\theta=1$, we have

$$\left(\frac{25}{350t}\right)^2+\left(\frac{3+16t^2}{350t}\right)^2=1$$

Multiply both sides by $(350t)^2$:

$$\begin{aligned}25^2+(3+16t^2)^2&=350^2t^2\\ 256t^4-122,404t^2+634&=0.\end{aligned}$$

Notes:

This is a quadratic *in* t^2. That is, we can apply the quadratic formula to find t^2, then solve for t itself.

$$\begin{aligned} t^2 &= \frac{122,404 \pm \sqrt{122,404^2 - 4(256)(634)}}{512} \\ t^2 &= 0.0052,\ 478.135 \\ t &= \pm 0.072,\ \pm 21.866 \end{aligned}$$

Clearly the negative t values do not fit our context, so we have $t = 0.072$ and $t = 21.866$. Using $\cos\theta = 25/(350t)$, we can solve for θ:

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{25}{350 \cdot 0.072}\right) \quad \text{and} \quad \cos^{-1}\left(\frac{25}{350 \cdot 21.866}\right) \\ \theta &= 7.03^\circ \quad \text{and} \quad 89.8^\circ. \end{aligned}$$

Alex has two choices of angle. He can hold the rifle at an angle of about 7° with the horizontal and hit his target 0.07s after firing, or he can hold his rifle almost straight up, with an angle of 89.8°, where he'll hit his target about 22s later. The first option is clearly the option he should choose.

Distance Traveled

Consider a driver who sets her cruise–control to 60mph, and travels at this speed for an hour. We can ask:

1. How far did the driver travel?
2. How far from her starting position is the driver?

The first is easy to answer: she traveled 60 miles. The second is impossible to answer with the given information. We do not know if she traveled in a straight line, on an oval racetrack, or along a slowly–winding highway.

This highlights an important fact: to compute distance traveled, we need only to know the speed, given by $\|\,\vec{v}(t)\,\|$.

Theorem 11.3.1 Distance Traveled

Let $\vec{v}(t)$ be a velocity function for a moving object. The distance traveled by the object on $[a, b]$ is:

$$\text{distance traveled} = \int_a^b \|\,\vec{v}(t)\,\|\, dt.$$

Note that this is just a restatement of Theorem 11.2.7: arc length is the same as distance traveled, just viewed in a different context.

Notes:

Example 11.3.7 Distance Traveled, Displacement, and Average Speed

A particle moves in space with position function $\vec{r}(t) = \langle t, t^2, \sin(\pi t) \rangle$ on $[-2, 2]$, where t is measured in seconds and distances are in meters. Find:

1. The distance traveled by the particle on $[-2, 2]$.
2. The displacement of the particle on $[-2, 2]$.
3. The particle's average speed.

Solution

1. We use Theorem 11.3.1 to establish the integral:

$$\begin{aligned} \text{distance traveled} &= \int_{-2}^{2} ||\,\vec{v}(t)\,||\, dt \\ &= \int_{-2}^{2} \sqrt{1 + (2t)^2 + \pi^2 \cos^2(\pi t)}\, dt. \end{aligned}$$

   This cannot be solved in terms of elementary functions so we turn to numerical integration, finding the distance to be 12.88m.

2. The displacement is the vector

$$\vec{r}(2) - \vec{r}(-2) = \langle 2, 4, 0 \rangle - \langle -2, 4, 0 \rangle = \langle 4, 0, 0 \rangle\,.$$

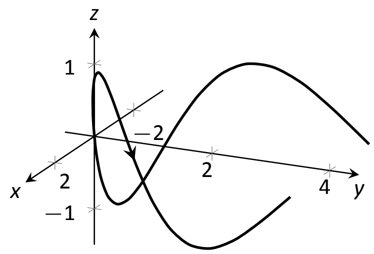
   That is, the particle ends with an x-value increased by 4 and with y- and z-values the same (see Figure 11.3.5).

3. We found above that the particle traveled 12.88m over 4 seconds. We can compute average speed by dividing: 12.88/4 = 3.22m/s.

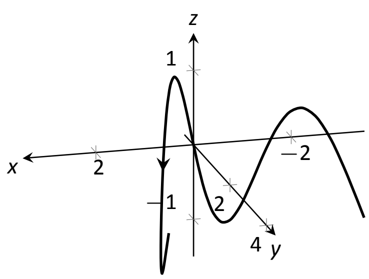
   We should also consider Definition 5.4.1 of Section 5.4, which says that the average value of a function f on $[a, b]$ is $\frac{1}{b-a}\int_a^b f(x)\, dx$. In our context, the average value of the speed is

$$\text{average speed} = \frac{1}{2-(-2)}\int_{-2}^{2} ||\,\vec{v}(t)\,||\, dt \approx \frac{1}{4}12.88 = 3.22\text{m/s}.$$

   Note how the physical context of a particle traveling gives meaning to a more abstract concept learned earlier.



(a)



(b)

Figure 11.3.5: The path of the particle, from two perspectives, in Example 11.3.7.

In Definition 5.4.1 of Chapter 5 we defined the average value of a function $f(x)$ on $[a, b]$ to be

$$\frac{1}{b-a}\int_a^b f(x)\, dx.$$

Notes:

Note how in Example 11.3.7 we computed the average speed as

$$\frac{\text{distance traveled}}{\text{travel time}} = \frac{1}{2-(-2)}\int_{-2}^{2} ||\,\vec{v}(t)\,||\,dt;$$

that is, we just found the average value of $||\,\vec{v}(t)\,||$ on $[-2,2]$.

Likewise, given position function $\vec{r}(t)$, the average velocity on $[a,b]$ is

$$\frac{\text{displacement}}{\text{travel time}} = \frac{1}{b-a}\int_a^b \vec{r}\,'(t)\,dt = \frac{\vec{r}(b)-\vec{r}(a)}{b-a};$$

that is, it is the average value of $\vec{r}\,'(t)$, or $\vec{v}(t)$, on $[a,b]$.

Key Idea 11.3.3 Average Speed, Average Velocity

Let $\vec{r}(t)$ be a differentiable position function on $[a,b]$.

The **average speed** is:

$$\frac{\text{distance traveled}}{\text{travel time}} = \frac{\int_a^b ||\,\vec{v}(t)\,||\,dt}{b-a} = \frac{1}{b-a}\int_a^b ||\,\vec{v}(t)\,||\,dt.$$

The **average velocity** is:

$$\frac{\text{displacement}}{\text{travel time}} = \frac{\int_a^b \vec{r}\,'(t)\,dt}{b-a} = \frac{1}{b-a}\int_a^b \vec{r}\,'(t)\,dt.$$

The next two sections investigate more properties of the graphs of vector–valued functions and we'll apply these new ideas to what we just learned about motion.

Notes:

Exercises 11.3

Terms and Concepts

1. How is *velocity* different from *speed*?

2. What is the difference between *displacement* and *distance traveled*?

3. What is the difference between *average velocity* and *average speed*?

4. *Distance traveled* is the same as ________ ________, just viewed in a different context.

5. Describe a scenario where an object's average speed is a large number, but the magnitude of the average velocity is not a large number.

6. Explain why it is not possible to have an average velocity with a large magnitude but a small average speed.

Problems

In Exercises 7 – 10 , a position function $\vec{r}(t)$ is given. Find $\vec{v}(t)$ and $\vec{a}(t)$.

7. $\vec{r}(t) = \langle 2t+1, 5t-2, 7\rangle$

8. $\vec{r}(t) = \left\langle 3t^2-2t+1, -t^2+t+14\right\rangle$

9. $\vec{r}(t) = \langle \cos t, \sin t\rangle$

10. $\vec{r}(t) = \langle t/10, -\cos t, \sin t\rangle$

In Exercises 11 – 14 , a position function $\vec{r}(t)$ is given. Sketch $\vec{r}(t)$ on the indicated interval. Find $\vec{v}(t)$ and $\vec{a}(t)$, then add $\vec{v}(t_0)$ and $\vec{a}(t_0)$ to your sketch, with their initial points at $\vec{r}(t_0)$, for the given value of t_0.

11. $\vec{r}(t) = \langle t, \sin t\rangle$ on $[0, \pi/2]$; $t_0 = \pi/4$

12. $\vec{r}(t) = \left\langle t^2, \sin t^2\right\rangle$ on $[0, \pi/2]$; $t_0 = \sqrt{\pi/4}$

13. $\vec{r}(t) = \left\langle t^2+t, -t^2+2t\right\rangle$ on $[-2, 2]$; $t_0 = 1$

14. $\vec{r}(t) = \left\langle \dfrac{2t+3}{t^2+1}, t^2\right\rangle$ on $[-1, 1]$; $t_0 = 0$

In Exercises 15 – 24 , a position function $\vec{r}(t)$ of an object is given. Find the speed of the object in terms of t, and find where the speed is minimized/maximized on the indicated interval.

15. $\vec{r}(t) = \left\langle t^2, t\right\rangle$ on $[-1, 1]$

16. $\vec{r}(t) = \left\langle t^2, t^2-t^3\right\rangle$ on $[-1, 1]$

17. $\vec{r}(t) = \langle 5\cos t, 5\sin t\rangle$ on $[0, 2\pi]$

18. $\vec{r}(t) = \langle 2\cos t, 5\sin t\rangle$ on $[0, 2\pi]$

19. $\vec{r}(t) = \langle \sec t, \tan t\rangle$ on $[0, \pi/4]$

20. $\vec{r}(t) = \langle t+\cos t, 1-\sin t\rangle$ on $[0, 2\pi]$

21. $\vec{r}(t) = \langle 12t, 5\cos t, 5\sin t\rangle$ on $[0, 4\pi]$

22. $\vec{r}(t) = \left\langle t^2-t, t^2+t, t\right\rangle$ on $[0, 1]$

23. $\vec{r}(t) = \left\langle t, t^2, \sqrt{1-t^2}\right\rangle$ on $[-1, 1]$

24. **Projectile Motion:** $\vec{r}(t) = \left\langle (v_0\cos\theta)t, -\dfrac{1}{2}gt^2 + (v_0\sin\theta)t\right\rangle$ on $\left[0, \dfrac{2v_0\sin\theta}{g}\right]$

In Exercises 25 – 28 , position functions $\vec{r}_1(t)$ and $\vec{r}_2(s)$ for two objects are given that follow the same path on the respective intervals.

(a) Show that the positions are the same at the indicated t_0 and s_0 values; i.e., show $\vec{r}_1(t_0) = \vec{r}_2(s_0)$.

(b) Find the velocity, speed and acceleration of the two objects at t_0 and s_0, respectively.

25. $\vec{r}_1(t) = \left\langle t, t^2\right\rangle$ on $[0, 1]$; $t_0 = 1$
$\vec{r}_2(s) = \left\langle s^2, s^4\right\rangle$ on $[0, 1]$; $s_0 = 1$

26. $\vec{r}_1(t) = \langle 3\cos t, 3\sin t\rangle$ on $[0, 2\pi]$; $t_0 = \pi/2$
$\vec{r}_2(s) = \langle 3\cos(4s), 3\sin(4s)\rangle$ on $[0, \pi/2]$; $s_0 = \pi/8$

27. $\vec{r}_1(t) = \langle 3t, 2t\rangle$ on $[0, 2]$; $t_0 = 2$
$\vec{r}_2(s) = \langle 6s-6, 4s-4\rangle$ on $[1, 2]$; $s_0 = 2$

28. $\vec{r}_1(t) = \left\langle t, \sqrt{t}\right\rangle$ on $[0, 1]$; $t_0 = 1$
$\vec{r}_2(s) = \left\langle \sin t, \sqrt{\sin t}\right\rangle$ on $[0, \pi/2]$; $s_0 = \pi/2$

In Exercises 29 – 32 , find the position function of an object given its acceleration and initial velocity and position.

29. $\vec{a}(t) = \langle 2, 3\rangle$; $\vec{v}(0) = \langle 1, 2\rangle$, $\vec{r}(0) = \langle 5, -2\rangle$

30. $\vec{a}(t) = \langle 2, 3\rangle$; $\vec{v}(1) = \langle 1, 2\rangle$, $\vec{r}(1) = \langle 5, -2\rangle$

31. $\vec{a}(t) = \langle \cos t, -\sin t\rangle$; $\vec{v}(0) = \langle 0, 1\rangle$, $\vec{r}(0) = \langle 0, 0\rangle$

32. $\vec{a}(t) = \langle 0, -32\rangle$; $\vec{v}(0) = \langle 10, 50\rangle$, $\vec{r}(0) = \langle 0, 0\rangle$

In Exercises 33 – 36 , find the displacement, distance traveled, average velocity and average speed of the described object on the given interval.

33. An object with position function $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t\rangle$, where distances are measured in feet and time is in seconds, on $[0, 2\pi]$.

34. An object with position function $\vec{r}(t) = \langle 5\cos t, -5\sin t\rangle$, where distances are measured in feet and time is in seconds, on $[0, \pi]$.

35. An object with velocity function $\vec{v}(t) = \langle \cos t, \sin t\rangle$, where distances are measured in feet and time is in seconds, on $[0, 2\pi]$.

36. An object with velocity function $\vec{v}(t) = \langle 1, 2, -1\rangle$, where distances are measured in feet and time is in seconds, on $[0, 10]$.

Exercises 37 – 42 ask you to solve a variety of problems based on the principles of projectile motion.

37. A boy whirls a ball, attached to a 3ft string, above his head in a counter–clockwise circle. The ball makes 2 revolutions per second.

    At what t-values should the boy release the string so that the ball heads directly for a tree standing 10ft in front of him?

38. David faces Goliath with only a stone in a 3ft sling, which he whirls above his head at 4 revolutions per second. They stand 20ft apart.

    (a) At what t-values must David release the stone in his sling in order to hit Goliath?

    (b) What is the speed at which the stone is traveling when released?

    (c) Assume David releases the stone from a height of 6ft and Goliath's forehead is 9ft above the ground. What angle of elevation must David apply to the stone to hit Goliath's head?

39. A hunter aims at a deer which is 40 yards away. Her crossbow is at a height of 5ft, and she aims for a spot on the deer 4ft above the ground. The crossbow fires her arrows at 300ft/s.

    (a) At what angle of elevation should she hold the crossbow to hit her target?

    (b) If the deer is moving perpendicularly to her line of sight at a rate of 20mph, by approximately how much should she lead the deer in order to hit it in the desired location?

40. A baseball player hits a ball at 100mph, with an initial height of 3ft and an angle of elevation of 20°, at Boston's Fenway Park. The ball flies towards the famed "Green Monster," a wall 37ft high located 310ft from home plate.

    (a) Show that as hit, the ball hits the wall.

    (b) Show that if the angle of elevation is 21°, the ball clears the Green Monster.

41. A Cessna flies at 1000ft at 150mph and drops a box of supplies to the professor (and his wife) on an island. Ignoring wind resistance, how far horizontally will the supplies travel before they land?

42. A football quarterback throws a pass from a height of 6ft, intending to hit his receiver 20yds away at a height of 5ft.

    (a) If the ball is thrown at a rate of 50mph, what angle of elevation is needed to hit his intended target?

    (b) If the ball is thrown at with an angle of elevation of 8°, what initial ball speed is needed to hit his target?

11.4 Unit Tangent and Normal Vectors

Unit Tangent Vector

Given a smooth vector–valued function $\vec{r}(t)$, we defined in Definition 11.2.4 that any vector parallel to $\vec{r}\,'(t_0)$ is *tangent* to the graph of $\vec{r}(t)$ at $t = t_0$. It is often useful to consider just the *direction* of $\vec{r}\,'(t)$ and not its magnitude. Therefore we are interested in the unit vector in the direction of $\vec{r}\,'(t)$. This leads to a definition.

Definition 11.4.1 Unit Tangent Vector

Let $\vec{r}(t)$ be a smooth function on an open interval I. The unit tangent vector $\vec{T}(t)$ is

$$\vec{T}(t) = \frac{1}{\|\,\vec{r}\,'(t)\,\|}\vec{r}\,'(t).$$

Example 11.4.1 Computing the unit tangent vector

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$. Find $\vec{T}(t)$ and compute $\vec{T}(0)$ and $\vec{T}(1)$.

Solution We apply Definition 11.4.1 to find $\vec{T}(t)$.

$$\begin{aligned}\vec{T}(t) &= \frac{1}{\|\,\vec{r}\,'(t)\,\|}\vec{r}\,'(t)\\ &= \frac{1}{\sqrt{\big(-3\sin t\big)^2 + \big(3\cos t\big)^2 + 4^2}}\langle -3\sin t, 3\cos t, 4\rangle\\ &= \left\langle -\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right\rangle.\end{aligned}$$

We can now easily compute $\vec{T}(0)$ and $\vec{T}(1)$:

$$\vec{T}(0) = \left\langle 0, \frac{3}{5}, \frac{4}{5}\right\rangle; \quad \vec{T}(1) = \left\langle -\frac{3}{5}\sin 1, \frac{3}{5}\cos 1, \frac{4}{5}\right\rangle \approx \langle -0.505, 0.324, 0.8\rangle.$$

These are plotted in Figure 11.4.1 with their initial points at $\vec{r}(0)$ and $\vec{r}(1)$, respectively. (They look rather "short" since they are only length 1.)

The unit tangent vector $\vec{T}(t)$ always has a magnitude of 1, though it is sometimes easy to doubt that is true. We can help solidify this thought in our minds by computing $\|\,\vec{T}(1)\,\|$:

$$\|\,\vec{T}(1)\,\| \approx \sqrt{(-0.505)^2 + 0.324^2 + 0.8^2} = 1.000001.$$

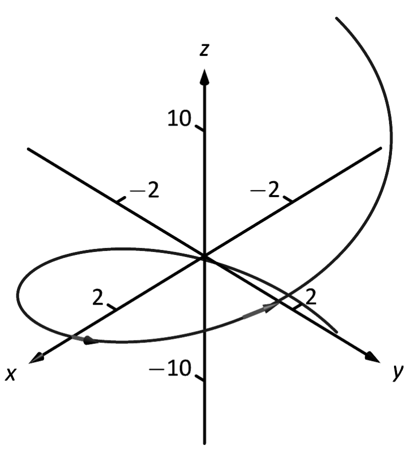


Figure 11.4.1: Plotting unit tangent vectors in Example 11.4.1.

Notes:

We have rounded in our computation of $\vec{T}(1)$, so we don't get 1 exactly. We leave it to the reader to use the exact representation of $\vec{T}(1)$ to verify it has length 1.

In many ways, the previous example was "too nice." It turned out that $\vec{r}\,'(t)$ was always of length 5. In the next example the length of $\vec{r}\,'(t)$ is variable, leaving us with a formula that is not as clean.

Example 11.4.2 Computing the unit tangent vector

Let $\vec{r}(t) = \left\langle t^2 - t, t^2 + t\right\rangle$. Find $\vec{T}(t)$ and compute $\vec{T}(0)$ and $\vec{T}(1)$.

Solution We find $\vec{r}\,'(t) = \langle 2t-1, 2t+1\rangle$, and

$$\|\,\vec{r}\,'(t)\,\| = \sqrt{(2t-1)^2 + (2t+1)^2} = \sqrt{8t^2+2}.$$

Therefore

$$\vec{T}(t) = \frac{1}{\sqrt{8t^2+2}}\langle 2t-1, 2t+1\rangle = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}}\right\rangle.$$

When $t = 0$, we have $\vec{T}(0) = \left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle$; when $t = 1$, we have $\vec{T}(1) = \left\langle 1/\sqrt{10}, 3/\sqrt{10}\right\rangle$. We leave it to the reader to verify each of these is a unit vector. They are plotted in Figure 11.4.2

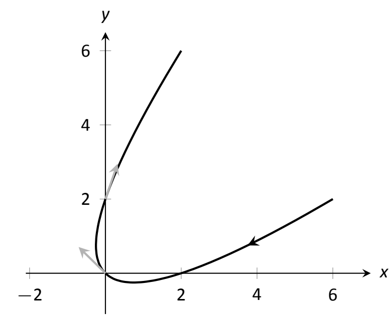


Figure 11.4.2: Plotting unit tangent vectors in Example 11.4.2.

Unit Normal Vector

Just as knowing the direction tangent to a path is important, knowing a direction orthogonal to a path is important. When dealing with real-valued functions, we defined the normal line at a point to the be the line through the point that was perpendicular to the tangent line at that point. We can do a similar thing with vector–valued functions. Given $\vec{r}(t)$ in $\mathbb{R}^2$, we have 2 directions perpendicular to the tangent vector, as shown in Figure 11.4.3. It is good to wonder "Is one of these two directions preferable over the other?"

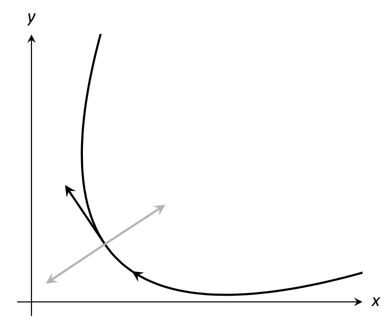


Figure 11.4.3: Given a direction in the plane, there are always two directions orthogonal to it.

Given $\vec{r}(t)$ in $\mathbb{R}^3$, there are infinitely many vectors orthogonal to the tangent vector at a given point. Again, we might wonder "Is one of these infinite choices preferable over the others? Is one of these the 'right' choice?"

The answer in both $\mathbb{R}^2$ and $\mathbb{R}^3$ is "Yes, there is one vector that is not only preferable, it is the 'right' one to choose." Recall Theorem 11.2.5, which states that if $\vec{r}(t)$ has constant length, then $\vec{r}(t)$ is orthogonal to $\vec{r}\,'(t)$ for all t. We know $\vec{T}(t)$, the unit tangent vector, has constant length. Therefore $\vec{T}(t)$ is orthogonal to $\vec{T}\,'(t)$.

We'll see that $\vec{T}\,'(t)$ is more than just a convenient choice of vector that is orthogonal to $\vec{r}\,'(t)$; rather, it is the "right" choice. Since all we care about is the direction, we define this newly found vector to be a unit vector.

Note: $\vec{T}(t)$ is a unit vector, by definition. This *does not* imply that $\vec{T}\,'(t)$ is also a unit vector.

Notes:

Definition 11.4.2 Unit Normal Vector

Let $\vec{r}(t)$ be a vector–valued function where the unit tangent vector, $\vec{T}(t)$, is smooth on an open interval I. The **unit normal vector** $\vec{N}(t)$ is

$$\vec{N}(t) = \frac{1}{||\,\vec{T}\,'(t)\,||}\vec{T}\,'(t).$$

Example 11.4.3 Computing the unit normal vector

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ as in Example 11.4.1. Sketch both $\vec{T}(\pi/2)$ and $\vec{N}(\pi/2)$ with initial points at $\vec{r}(\pi/2)$.

Solution In Example 11.4.1, we found $\vec{T}(t) = \Big\langle (-3/5)\sin t, (3/5)\cos t, 4/5\Big\rangle$. Therefore

$$\vec{T}\,'(t) = \left\langle -\frac{3}{5}\cos t, -\frac{3}{5}\sin t, 0\right\rangle \quad \text{and} \quad ||\,\vec{T}\,'(t)\,|| = \frac{3}{5}.$$

Thus

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{3/5} = \langle -\cos t, -\sin t, 0\rangle\,.$$

We compute $\vec{T}(\pi/2) = \langle -3/5, 0, 4/5\rangle$ and $\vec{N}(\pi/2) = \langle 0, -1, 0\rangle$. These are sketched in Figure 11.4.4.

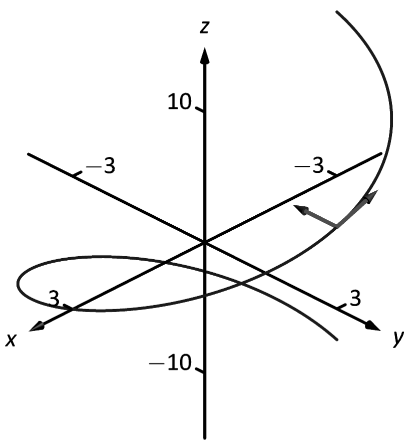


Figure 11.4.4: Plotting unit tangent and normal vectors in Example 11.4.4.

The previous example was once again "too nice." In general, the expression for $\vec{T}(t)$ contains fractions of square–roots, hence the expression of $\vec{T}\,'(t)$ is very messy. We demonstrate this in the next example.

Example 11.4.4 Computing the unit normal vector

Let $\vec{r}(t) = \big\langle t^2 - t, t^2 + t\big\rangle$ as in Example 11.4.2. Find $\vec{N}(t)$ and sketch $\vec{r}(t)$ with the unit tangent and normal vectors at $t = -1, 0$ and 1.

Solution In Example 11.4.2, we found

$$\vec{T}(t) = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}}\right\rangle.$$

Finding $\vec{T}\,'(t)$ requires two applications of the Quotient Rule:

Notes:

$$T'(t) = \left\langle \frac{\sqrt{8t^2+2}(2) - (2t-1)\left(\frac{1}{2}(8t^2+2)^{-1/2}(16t)\right)}{8t^2+2}, \right.$$
$$\left. \frac{\sqrt{8t^2+2}(2) - (2t+1)\left(\frac{1}{2}(8t^2+2)^{-1/2}(16t)\right)}{8t^2+2} \right\rangle$$
$$= \left\langle \frac{4(2t+1)}{(8t^2+2)^{3/2}}, \frac{4(1-2t)}{(8t^2+2)^{3/2}} \right\rangle$$

This is not a unit vector; to find $\vec{N}(t)$, we need to divide $\vec{T}'(t)$ by it's magnitude.

$$\begin{aligned} \|\,\vec{T}'(t)\,\| &= \sqrt{\frac{16(2t+1)^2}{(8t^2+2)^3} + \frac{16(1-2t)^2}{(8t^2+2)^3}} \\ &= \sqrt{\frac{16(8t^2+2)}{(8t^2+2)^3}} \\ &= \frac{4}{8t^2+2}. \end{aligned}$$

Finally,

$$\begin{aligned} \vec{N}(t) &= \frac{1}{4/(8t^2+2)} \left\langle \frac{4(2t+1)}{(8t^2+2)^{3/2}}, \frac{4(1-2t)}{(8t^2+2)^{3/2}} \right\rangle \\ &= \left\langle \frac{2t+1}{\sqrt{8t^2+2}}, -\frac{2t-1}{\sqrt{8t^2+2}} \right\rangle. \end{aligned}$$

Using this formula for $\vec{N}(t)$, we compute the unit tangent and normal vectors for $t = -1, 0$ and 1 and sketch them in Figure 11.4.5.

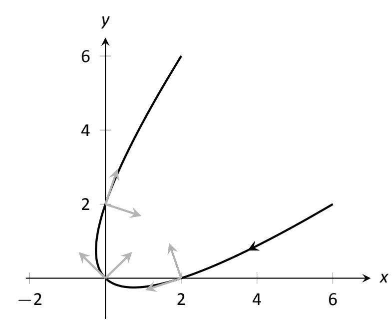


Figure 11.4.5: Plotting unit tangent and normal vectors in Example 11.4.4.

The final result for $\vec{N}(t)$ in Example 11.4.4 is suspiciously similar to $\vec{T}(t)$. There is a clear reason for this. If $\vec{u} = \langle u_1, u_2\rangle$ is a unit vector in $\mathbb{R}^2$, then the *only* unit vectors orthogonal to $\vec{u}$ are $\langle -u_2, u_1\rangle$ and $\langle u_2, -u_1\rangle$. Given $\vec{T}(t)$, we can quickly determine $\vec{N}(t)$ if we know which term to multiply by (-1).

Consider again Figure 11.4.5, where we have plotted some unit tangent and normal vectors. Note how $\vec{N}(t)$ always points "inside" the curve, or to the concave side of the curve. This is not a coincidence; this is true in general. Knowing the direction that $\vec{r}(t)$ "turns" allows us to quickly find $\vec{N}(t)$.

Notes:

Theorem 11.4.1 Unit Normal Vectors in $\mathbb{R}^2$

Let $\vec{r}(t)$ be a vector–valued function in $\mathbb{R}^2$ where $\vec{T}\,'(t)$ is smooth on an open interval *I*. Let t_0 be in *I* and $\vec{T}(t_0) = \langle t_1, t_2 \rangle$ Then $\vec{N}(t_0)$ is either

$$\vec{N}(t_0) = \langle -t_2, t_1 \rangle \quad \text{or} \quad \vec{N}(t_0) = \langle t_2, -t_1 \rangle\,,$$

whichever is the vector that points to the concave side of the graph of $\vec{r}$.

Application to Acceleration

Let $\vec{r}(t)$ be a position function. It is a fact (stated later in Theorem 11.4.2) that acceleration, $\vec{a}(t)$, lies in the plane defined by $\vec{T}$ and $\vec{N}$. That is, there are scalar functions $a_{\text{T}}(t)$ and $a_{\text{N}}(t)$ such that

$$\vec{a}(t) = a_{\text{T}}(t)\vec{T}(t) + a_{\text{N}}(t)\vec{N}(t).$$

We generally drop the "of *t*" part of the notation and just write a_{T} and a_{N}.

The scalar a_{T} measures "how much" acceleration is in the direction of travel, that is, it measures the component of acceleration that affects the speed. The scalar a_{N} measures "how much" acceleration is perpendicular to the direction of travel, that is, it measures the component of acceleration that affects the direction of travel.

We can find a_{T} using the orthogonal projection of $\vec{a}(t)$ onto $\vec{T}(t)$ (review Definition 10.3.3 in Section 10.3 if needed). Recalling that since $\vec{T}(t)$ is a unit vector, $\vec{T}(t)\cdot\vec{T}(t) = 1$, so we have

Note: Keep in mind that both a_{T} and a_{N} are functions of *t*; that is, the scalar changes depending on *t*. It is convention to drop the "(t)" notation from $a_{\text{T}}(t)$ and simply write a_{T}.

$$\text{proj}_{\,\vec{T}(t)}\,\vec{a}(t) = \frac{\vec{a}(t)\cdot\vec{T}(t)}{\vec{T}(t)\cdot\vec{T}(t)}\vec{T}(t) = \underbrace{\big(\vec{a}(t)\cdot\vec{T}(t)\big)}_{a_{\text{T}}}\vec{T}(t).$$

Thus the amount of $\vec{a}(t)$ in the direction of $\vec{T}(t)$ is $a_{\text{T}} = \vec{a}(t)\cdot\vec{T}(t)$. The same logic gives $a_{\text{N}} = \vec{a}(t)\cdot\vec{N}(t)$.

While this is a fine way of computing a_{T}, there are simpler ways of finding a_{N} (as finding $\vec{N}$ itself can be complicated). The following theorem gives alternate formulas for a_{T} and a_{N}.

Notes:

Theorem 11.4.2 Acceleration in the Plane Defined by $\vec{T}$ and $\vec{N}$

Let $\vec{r}(t)$ be a position function with acceleration $\vec{a}(t)$ and unit tangent and normal vectors $\vec{T}(t)$ and $\vec{N}(t)$. Then $\vec{a}(t)$ lies in the plane defined by $\vec{T}(t)$ and $\vec{N}(t)$; that is, there exists scalars a_T and a_N such that

$$\vec{a}(t) = a_\text{T}\vec{T}(t) + a_\text{N}\vec{N}(t).$$

Moreover,

$$a_\text{T} = \vec{a}(t)\cdot\vec{T}(t) = \frac{d}{dt}\Big(||\,\vec{v}(t)\,||\Big)$$
$$a_\text{N} = \vec{a}(t)\cdot\vec{N}(t) = \sqrt{||\,\vec{a}(t)\,||^2 - a_\text{T}^2} = \frac{||\,\vec{a}(t)\times\vec{v}(t)\,||}{||\,\vec{v}(t)\,||} = ||\,\vec{v}(t)\,||\,||\,\vec{T}\,'(t)\,||$$

Note the second formula for a_T: $\frac{d}{dt}\Big(||\,\vec{v}(t)\,||\Big)$. This measures the rate of change of speed, which again is the amount of acceleration in the direction of travel.

Example 11.4.5 Computing a_T and a_N

Let $\vec{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$ as in Examples 11.4.1 and 11.4.3. Find a_T and a_N.

Solution The previous examples give $\vec{a}(t) = \langle -3\cos t, -3\sin t, 0\rangle$ and

$$\vec{T}(t) = \left\langle -\frac{3}{5}\sin t, \frac{3}{5}\cos t, \frac{4}{5}\right\rangle \quad \text{and} \quad \vec{N}(t) = \langle -\cos t, -\sin t, 0\rangle\,.$$

We can find a_T and a_N directly with dot products:

$$a_\text{T} = \vec{a}(t)\cdot\vec{T}(t) = \frac{9}{5}\cos t\sin t - \frac{9}{5}\cos t\sin t + 0 = 0.$$
$$a_\text{N} = \vec{a}(t)\cdot\vec{N}(t) = 3\cos^2 t + 3\sin^2 t + 0 = 3.$$

Thus $\vec{a}(t) = 0\vec{T}(t) + 3\vec{N}(t) = 3\vec{N}(t)$, which is clearly the case.

What is the practical interpretation of these numbers? $a_\text{T} = 0$ means the object is moving at a constant speed, and hence all acceleration comes in the form of direction change.

Example 11.4.6 Computing a_T and a_N

Let $\vec{r}(t) = \langle t^2 - t, t^2 + t\rangle$ as in Examples 11.4.2 and 11.4.4. Find a_T and a_N.

Notes:

Solution The previous examples give $\vec{a}(t) = \langle 2, 2 \rangle$ and

$$\vec{T}(t) = \left\langle \frac{2t-1}{\sqrt{8t^2+2}}, \frac{2t+1}{\sqrt{8t^2+2}} \right\rangle \quad \text{and} \quad \vec{N}(t) = \left\langle \frac{2t+1}{\sqrt{8t^2+2}}, -\frac{2t-1}{\sqrt{8t^2+2}} \right\rangle.$$

While we can compute a_{N} using $\vec{N}(t)$, we instead demonstrate using another formula from Theorem 11.4.2.

$$a_{\text{T}} = \vec{a}(t) \cdot \vec{T}(t) = \frac{4t-2}{\sqrt{8t^2+2}} + \frac{4t+2}{\sqrt{8t^2+2}} = \frac{8t}{\sqrt{8t^2+2}}.$$

$$a_{\text{N}} = \sqrt{\|\,\vec{a}(t)\,\|^2 - a_{\text{T}}^2} = \sqrt{8 - \left(\frac{8t}{\sqrt{8t^2+2}}\right)^2} = \frac{4}{\sqrt{8t^2+2}}.$$

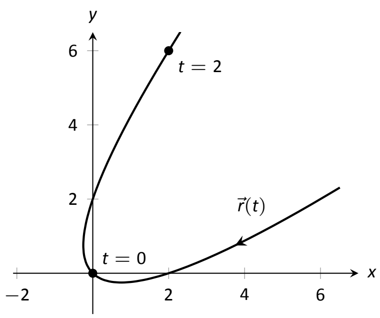


Figure 11.4.6: Graphing $\vec{r}(t)$ in Example 11.4.6.

When $t = 2$, $a_{\text{T}} = \dfrac{16}{\sqrt{34}} \approx 2.74$ and $a_{\text{N}} = \dfrac{4}{\sqrt{34}} \approx 0.69$. We interpret this to mean that at $t = 2$, the particle is accelerating mostly by increasing speed, not by changing direction. As the path near $t = 2$ is relatively straight, this should make intuitive sense. Figure 11.4.6 gives a graph of the path for reference.

Contrast this with $t = 0$, where $a_{\text{T}} = 0$ and $a_{\text{N}} = 4/\sqrt{2} \approx 2.82$. Here the particle's speed is not changing and all acceleration is in the form of direction change.

Example 11.4.7 Analyzing projectile motion

A ball is thrown from a height of 240ft with an initial speed of 64ft/s and an angle of elevation of 30°. Find the position function $\vec{r}(t)$ of the ball and analyze a_{T} and a_{N}.

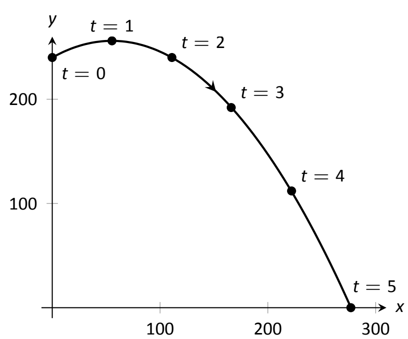


Figure 11.4.7: Plotting the position of a thrown ball, with 1s increments shown.

Solution Using Key Idea 11.3.2 of Section 11.3 we form the position function of the ball:

$$\vec{r}(t) = \left\langle \big(64\cos 30^\circ\big)t, -16t^2 + \big(64\sin 30^\circ\big)t + 240 \right\rangle,$$

which we plot in Figure 11.4.7.

From this we find $\vec{v}(t) = \langle 64\cos 30^\circ, -32t + 64\sin 30^\circ \rangle$ and $\vec{a}(t) = \langle 0, -32 \rangle$. Computing $\vec{T}(t)$ is not difficult, and with some simplification we find

$$\vec{T}(t) = \left\langle \frac{\sqrt{3}}{\sqrt{t^2-2t+4}}, \frac{1-t}{\sqrt{t^2-2t+4}} \right\rangle.$$

With $\vec{a}(t)$ as simple as it is, finding a_{T} is also simple:

$$a_{\text{T}} = \vec{a}(t) \cdot \vec{T}(t) = \frac{32t-32}{\sqrt{t^2-2t+4}}.$$

Notes:

We choose to not find $\vec{N}(t)$ and find a_N through the formula $a_N = \sqrt{||\,\vec{a}(t)\,||^2 - a_T^2}$:

$$a_N = \sqrt{32^2 - \left(\frac{32t - 32}{\sqrt{t^2 - 2t + 4}}\right)^2} = \frac{32\sqrt{3}}{\sqrt{t^2 - 2t + 4}}.$$

Figure 11.4.8 gives a table of values of a_T and a_N. When $t = 0$, we see the ball's speed is decreasing; when $t = 1$ the speed of the ball is unchanged. This corresponds to the fact that at $t = 1$ the ball reaches its highest point.

After $t = 1$ we see that a_N is decreasing in value. This is because as the ball falls, it's path becomes straighter and most of the acceleration is in the form of speeding up the ball, and not in changing its direction.

t	a_T	a_N
0	-16	27.7
1	0	32
2	16	27.7
3	24.2	20.9
4	27.7	16
5	29.4	12.7

Figure 11.4.8: A table of values of a_T and a_N in Example 11.4.7.

Our understanding of the unit tangent and normal vectors is aiding our understanding of motion. The work in Example 11.4.7 gave quantitative analysis of what we intuitively knew.

The next section provides two more important steps towards this analysis. We currently describe position only in terms of time. In everyday life, though, we often describe position in terms of distance ("The gas station is about 2 miles ahead, on the left."). The *arc length parameter* allows us to reference position in terms of distance traveled.

We also intuitively know that some paths are straighter than others – and some are curvier than others, but we lack a measurement of "curviness." The arc length parameter provides a way for us to compute *curvature*, a quantitative measurement of how curvy a curve is.

Notes:

Exercises 11.4

Terms and Concepts

1. If $\vec{T}(t)$ is a unit tangent vector, what is $\|\ \vec{T}(t)\ \|$?

2. If $\vec{N}(t)$ is a unit normal vector, what is $\vec{N}(t)\cdot\vec{r}\,'(t)$?

3. The acceleration vector $\vec{a}(t)$ lies in the plane defined by what two vectors?

4. a_{T} measures how much the acceleration is affecting the ________ of an object.

Problems

In Exercises 5 – 8 , given $\vec{r}(t)$, find $\vec{T}(t)$ and evaluate it at the indicated value of t.

5. $\vec{r}(t)=\left\langle 2t^2, t^2-t\right\rangle, \quad t=1$

6. $\vec{r}(t)=\langle t,\cos t\rangle, \quad t=\pi/4$

7. $\vec{r}(t)=\left\langle \cos^3 t,\sin^3 t\right\rangle, \quad t=\pi/4$

8. $\vec{r}(t)=\langle \cos t,\sin t\rangle, \quad t=\pi$

In Exercises 9 – 12 , find the equation of the line tangent to the curve at the indicated t-value using the unit tangent vector. Note: these are the same problems as in Exercises 5 – 8.

9. $\vec{r}(t)=\left\langle 2t^2, t^2-t\right\rangle, \quad t=1$

10. $\vec{r}(t)=\langle t,\cos t\rangle, \quad t=\pi/4$

11. $\vec{r}(t)=\left\langle \cos^3 t,\sin^3 t\right\rangle, \quad t=\pi/4$

12. $\vec{r}(t)=\langle \cos t,\sin t\rangle, \quad t=\pi$

In Exercises 13 – 16 , find $\vec{N}(t)$ using Definition 11.4.2. Confirm the result using Theorem 11.4.1.

13. $\vec{r}(t)=\langle 3\cos t, 3\sin t\rangle$

14. $\vec{r}(t)=\left\langle t, t^2\right\rangle$

15. $\vec{r}(t)=\langle \cos t, 2\sin t\rangle$

16. $\vec{r}(t)=\left\langle e^t, e^{-t}\right\rangle$

In Exercises 17 – 20 , a position function $\vec{r}(t)$ is given along with its unit tangent vector $\vec{T}(t)$ evaluated at $t=a$, for some value of a.

(a) Confirm that $\vec{T}(a)$ is as stated.

(b) Using a graph of $\vec{r}(t)$ and Theorem 11.4.1, find $\vec{N}(a)$.

17. $\vec{r}(t)=\langle 3\cos t, 5\sin t\rangle; \quad \vec{T}(\pi/4)=\left\langle -\frac{3}{\sqrt{34}},\frac{5}{\sqrt{34}}\right\rangle.$

18. $\vec{r}(t)=\left\langle t,\frac{1}{t^2+1}\right\rangle; \quad \vec{T}(1)=\left\langle \frac{2}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right\rangle.$

19. $\vec{r}(t)=(1+2\sin t)\langle \cos t,\sin t\rangle; \quad \vec{T}(0)=\left\langle \frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}}\right\rangle.$

20. $\vec{r}(t)=\left\langle \cos^3 t,\sin^3 t\right\rangle; \quad \vec{T}(\pi/4)=\left\langle -\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\rangle.$

In Exercises 21 – 24 , find $\vec{N}(t)$.

21. $\vec{r}(t)=\langle 4t, 2\sin t, 2\cos t\rangle$

22. $\vec{r}(t)=\langle 5\cos t, 3\sin t, 4\sin t\rangle$

23. $\vec{r}(t)=\langle a\cos t, a\sin t, bt\rangle; \quad a>0$

24. $\vec{r}(t)=\langle \cos(at),\sin(at),t\rangle$

In Exercises 25 – 30 , find a_{T} and a_{N} given $\vec{r}(t)$. Sketch $\vec{r}(t)$ on the indicated interval, and comment on the relative sizes of a_{T} and a_{N} at the indicated t values.

25. $\vec{r}(t)=\left\langle t,t^2\right\rangle$ on $[-1,1]$; consider $t=0$ and $t=1$.

26. $\vec{r}(t)=\langle t,1/t\rangle$ on $(0,4]$; consider $t=1$ and $t=2$.

27. $\vec{r}(t)=\langle 2\cos t, 2\sin t\rangle$ on $[0,2\pi]$; consider $t=0$ and $t=\pi/2$.

28. $\vec{r}(t)=\left\langle \cos(t^2),\sin(t^2)\right\rangle$ on $(0,2\pi]$; consider $t=\sqrt{\pi/2}$ and $t=\sqrt{\pi}$.

29. $\vec{r}(t)=\langle a\cos t, a\sin t, bt\rangle$ on $[0,2\pi]$, where $a,b>0$; consider $t=0$ and $t=\pi/2$.

30. $\vec{r}(t)=\langle 5\cos t, 4\sin t, 3\sin t\rangle$ on $[0,2\pi]$; consider $t=0$ and $t=\pi/2$.

11.5 The Arc Length Parameter and Curvature

In normal conversation we describe position in terms of both *time* and *distance*. For instance, imagine driving to visit a friend. If she calls and asks where you are, you might answer "I am 20 minutes from your house," or you might say "I am 10 miles from your house." Both answers provide your friend with a general idea of where you are.

Currently, our vector–valued functions have defined points with a parameter t, which we often take to represent time. Consider Figure 11.5.1(a), where $\vec{r}(t) = \left\langle t^2 - t, t^2 + t\right\rangle$ is graphed and the points corresponding to $t = 0,\ 1$ and 2 are shown. Note how the arc length between $t = 0$ and $t = 1$ is smaller than the arc length between $t = 1$ and $t = 2$; if the parameter t is time and $\vec{r}$ is position, we can say that the particle traveled faster on $[1, 2]$ than on $[0, 1]$.

Now consider Figure 11.5.1(b), where the same graph is parametrized by a different variable s. Points corresponding to $s = 0$ through $s = 6$ are plotted. The arc length of the graph between each adjacent pair of points is 1. We can view this parameter s as distance; that is, the arc length of the graph from $s = 0$ to $s = 3$ is 3, the arc length from $s = 2$ to $s = 6$ is 4, etc. If one wants to find the point 2.5 units from an initial location (i.e., $s = 0$), one would compute $\vec{r}(2.5)$. This parameter s is very useful, and is called the **arc length parameter**.

How do we find the arc length parameter?

Start with any parametrization of $\vec{r}$. We can compute the arc length of the graph of $\vec{r}$ on the interval $[0, t]$ with

$$\text{arc length } = \int_0^t ||\,\vec{r}\,'(u)\,||\,du.$$

We can turn this into a function: as t varies, we find the arc length s from 0 to t. This function is

$$s(t) = \int_0^t ||\,\vec{r}\,'(u)\,||\,du. \tag{11.1}$$

This establishes a relationship between s and t. Knowing this relationship explicitly, we can rewrite $\vec{r}(t)$ as a function of s: $\vec{r}(s)$. We demonstrate this in an example.

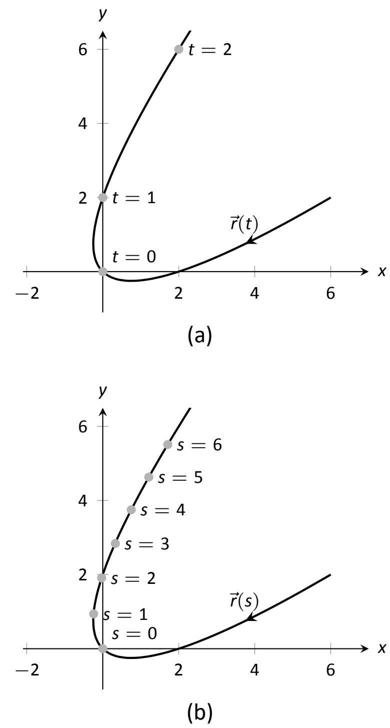


Figure 11.5.1: Introducing the arc length parameter.

Example 11.5.1 Finding the arc length parameter

Let $\vec{r}(t) = \langle 3t - 1, 4t + 2\rangle$. Parametrize $\vec{r}$ with the arc length parameter s.

Solution Using Equation (11.1), we write

$$s(t) = \int_0^t ||\,\vec{r}\,'(u)\,||\,du.$$

Notes:

We can integrate this, explicitly finding a relationship between s and t:

$$\begin{aligned} s(t) &= \int_0^t ||\,\vec{r}\,'(u)\,||\,du \\ &= \int_0^t \sqrt{3^2+4^2}\,du \\ &= \int_0^t 5\,du \\ &= 5t. \end{aligned}$$

Since $s = 5t$, we can write $t = s/5$ and replace t in $\vec{r}(t)$ with $s/5$:

$$\vec{r}(s) = \langle 3(s/5)-1, 4(s/5)+2\rangle = \left\langle \frac{3}{5}s-1, \frac{4}{5}s+2\right\rangle.$$

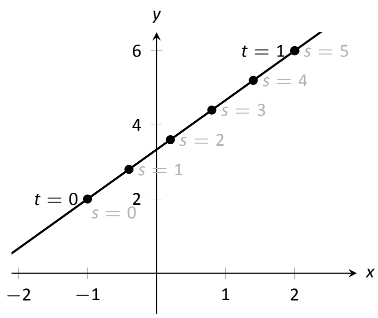


Figure 11.5.2: Graphing $\vec{r}$ in Example 11.5.1 with parameters t and s.

Clearly, as shown in Figure 11.5.2, the graph of $\vec{r}$ is a line, where $t = 0$ corresponds to the point $(-1, 2)$. What point on the line is 2 units away from this initial point? We find it with $\vec{r}(2) = \langle 1/5, 18/5\rangle$.

Is the point $(1/5, 18/5)$ really 2 units away from $(-1, 2)$? We use the Distance Formula to check:

$$d = \sqrt{\left(\frac{1}{5}-(-1)\right)^2 + \left(\frac{18}{5}-2\right)^2} = \sqrt{\frac{36}{25}+\frac{64}{25}} = \sqrt{4} = 2.$$

Yes, $\vec{r}(2)$ is indeed 2 units away, in the direction of travel, from the initial point.

Things worked out very nicely in Example 11.5.1; we were able to establish directly that $s = 5t$. Usually, the arc length parameter is much more difficult to describe in terms of t, a result of integrating a square–root. There are a number of things that we can learn about the arc length parameter from Equation (11.1), though, that are incredibly useful.

First, take the derivative of s with respect to t. The Fundamental Theorem of Calculus (see Theorem 5.4.1) states that

$$\frac{ds}{dt} = s\,'(t) = ||\,\vec{r}\,'(t)\,||. \tag{11.2}$$

Letting t represent time and $\vec{r}(t)$ represent position, we see that the rate of change of s with respect to t is speed; that is, the rate of change of "distance traveled" is speed, which should match our intuition.

The Chain Rule states that

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \frac{d\vec{r}}{ds}\cdot\frac{ds}{dt} \\ \vec{r}\,'(t) &= \vec{r}\,'(s)\cdot||\,\vec{r}\,'(t)\,||. \end{aligned}$$

Notes:

Solving for $\vec{r}\,'(s)$, we have

$$\vec{r}\,'(s) = \frac{\vec{r}\,'(t)}{||\,\vec{r}\,'(t)\,||} = \vec{T}(t), \tag{11.3}$$

where $\vec{T}(t)$ is the unit tangent vector. Equation 11.3 is often misinterpreted, as one is tempted to think it states $\vec{r}\,'(t) = \vec{T}(t)$, but there is a big difference between $\vec{r}\,'(s)$ and $\vec{r}\,'(t)$. The key to take from it is that $\vec{r}\,'(s)$ is a unit vector. In fact, the following theorem states that this characterizes the arc length parameter.

Theorem 11.5.1 Arc Length Parameter

Let $\vec{r}(s)$ be a vector–valued function. The parameter s is the arc length parameter if, and only if, $||\,\vec{r}\,'(s)\,|| = 1$.

Curvature

Consider points A and B on the curve graphed in Figure 11.5.3(a). One can readily argue that the curve curves more sharply at A than at B. It is useful to use a number to describe how sharply the curve bends; that number is the **curvature** of the curve.

We derive this number in the following way. Consider Figure 11.5.3(b), where unit tangent vectors are graphed around points A and B. Notice how the direction of the unit tangent vector changes quite a bit near A, whereas it does not change as much around B. This leads to an important concept: measuring the rate of change of the unit tangent vector with respect to arc length gives us a measurement of curvature.

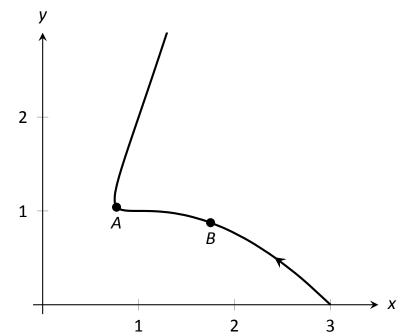
Definition 11.5.1 Curvature

Let $\vec{r}(s)$ be a vector–valued function where s is the arc length parameter. The curvature κ of the graph of $\vec{r}(s)$ is

$$\kappa = \left|\left|\,\frac{d\vec{T}}{ds}\,\right|\right| = ||\,\vec{T}\,'(s)\,||\,.$$

If $\vec{r}(s)$ is parametrized by the arc length parameter, then

$$\vec{T}(s) = \frac{\vec{r}\,'(s)}{||\,\vec{r}\,'(s)\,||} \quad \text{and} \quad \vec{N}(s) = \frac{\vec{T}\,'(s)}{||\,\vec{T}\,'(s)\,||}.$$



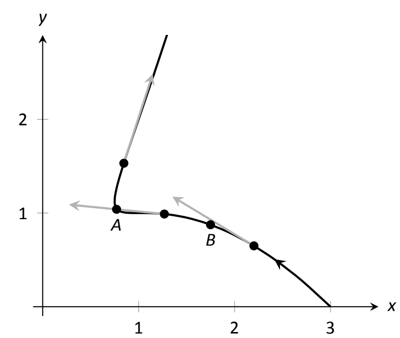


Figure 11.5.3: Establishing the concept of curvature.

Notes:

Having defined $||\,\vec{T}\,'(s)\,|| = \kappa$, we can rewrite the second equation as

$$\vec{T}\,'(s) = \kappa\vec{N}(s). \tag{11.4}$$

We already knew that $\vec{T}\,'(s)$ is in the same direction as $\vec{N}(s)$; that is, we can think of $\vec{T}(s)$ as being "pulled" in the direction of $\vec{N}(s)$. How "hard" is it being pulled? By a factor of κ. When the curvature is large, $\vec{T}(s)$ is being "pulled hard" and the direction of $\vec{T}(s)$ changes rapidly. When κ is small, $T(s)$ is not being pulled hard and hence its direction is not changing rapidly.

We use Definition 11.5.1 to find the curvature of the line in Example 11.5.1.

Example 11.5.2 Finding the curvature of a line

Use Definition 11.5.1 to find the curvature of $\vec{r}(t) = \langle 3t-1, 4t+2\rangle$.

Solution In Example 11.5.1, we found that the arc length parameter was defined by $s = 5t$, so $\vec{r}(s) = \langle 3s/5 - 1, 4s/5 + 2\rangle$ parametrized $\vec{r}$ with the arc length parameter. To find κ, we need to find $\vec{T}\,'(s)$.

$$\begin{aligned}\vec{T}(s) &= \vec{r}\,'(s) \quad \text{(recall this is a unit vector)}\\ &= \langle 3/5, 4/5\rangle\,.\end{aligned}$$

Therefore

$$\vec{T}\,'(s) = \langle 0, 0\rangle$$

and

$$\kappa = \big|\big|\,\vec{T}\,'(s)\,\big|\big| = 0.$$

It probably comes as no surprise that the curvature of a line is 0. (How "curvy" is a line? It is not curvy at all.)

While the definition of curvature is a beautiful mathematical concept, it is nearly impossible to use most of the time; writing $\vec{r}$ in terms of the arc length parameter is generally very hard. Fortunately, there are other methods of calculating this value that are much easier. There is a tradeoff: the definition is "easy" to understand though hard to compute, whereas these other formulas are easy to compute though it may be hard to understand why they work.

Notes:

Theorem 11.5.2 Formulas for Curvature

Let C be a smooth curve in the plane or in space.

1. If C is defined by $y = f(x)$, then
$$\kappa = \frac{|f''(x)|}{\Big(1 + \big(f'(x)\big)^2\Big)^{3/2}}.$$

2. If C is defined as a vector–valued function in the plane, $\vec{r}(t) = \langle x(t), y(t)\rangle$, then
$$\kappa = \frac{|x'y'' - x''y'|}{\big((x')^2 + (y')^2\big)^{3/2}}.$$

3. If C is defined in space by a vector–valued function $\vec{r}(t)$, then
$$\kappa = \frac{||\,\vec{T}\,'(t)\,||}{||\,\vec{r}\,'(t)\,||} = \frac{||\,\vec{r}\,'(t)\times\vec{r}\,''(t)\,||}{||\,\vec{r}\,'(t)\,||^3} = \frac{\vec{a}(t)\cdot\vec{N}(t)}{||\,\vec{v}(t)\,||^2}.$$

We practice using these formulas.

Example 11.5.3 Finding the curvature of a circle

Find the curvature of a circle with radius r, defined by $\vec{c}(t) = \langle r\cos t, r\sin t\rangle$.

Solution Before we start, we should expect the curvature of a circle to be constant, and not dependent on t. (Why?)

We compute κ using the second part of Theorem 11.5.2.

$$\begin{aligned}\kappa &= \frac{|(-r\sin t)(-r\sin t) - (-r\cos t)(r\cos t)|}{\big((-r\sin t)^2 + (r\cos t)^2\big)^{3/2}}\\ &= \frac{r^2(\sin^2 t + \cos^2 t)}{\big(r^2(\sin^2 t + \cos^2 t)\big)^{3/2}}\\ &= \frac{r^2}{r^3} = \frac{1}{r}.\end{aligned}$$

We have found that a circle with radius r has curvature $\kappa = 1/r$.

Example 11.5.3 gives a great result. Before this example, if we were told

Notes:

"The curve has a curvature of 5 at point *A*," we would have no idea what this really meant. Is 5 "big" – does is correspond to a really sharp turn, or a not-so-sharp turn? Now we can think of 5 in terms of a circle with radius 1/5. Knowing the units (inches vs. miles, for instance) allows us to determine how sharply the curve is curving.

Let a point *P* on a smooth curve *C* be given, and let κ be the curvature of the curve at *P*. A circle that:

- passes through *P*,
- lies on the concave side of *C*,
- has a common tangent line as *C* at *P* and
- has radius $r = 1/\kappa$ (hence has curvature κ)

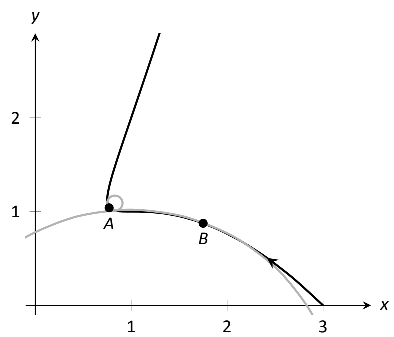


Figure 11.5.4: Illustrating the osculating circles for the curve seen in Figure 11.5.3.

is the **osculating circle**, or **circle of curvature**, to *C* at *P*, and *r* is the **radius of curvature**. Figure 11.5.4 shows the graph of the curve seen earlier in Figure 11.5.3 and its osculating circles at *A* and *B*. A sharp turn corresponds to a circle with a small radius; a gradual turn corresponds to a circle with a large radius. Being able to think of curvature in terms of the radius of a circle is very useful. (The word "osculating" comes from a Latin word related to kissing; an osculating circle "kisses" the graph at a particular point. Many beautiful ideas in mathematics have come from studying the osculating circles to a curve.)

Example 11.5.4 Finding curvature

Find the curvature of the parabola defined by $y = x^2$ at the vertex and at $x = 1$.

SOLUTION We use the first formula found in Theorem 11.5.2.

$$\begin{aligned}\kappa(x) &= \frac{|2|}{\big(1+(2x)^2\big)^{3/2}}\\ &= \frac{2}{\big(1+4x^2\big)^{3/2}}.\end{aligned}$$

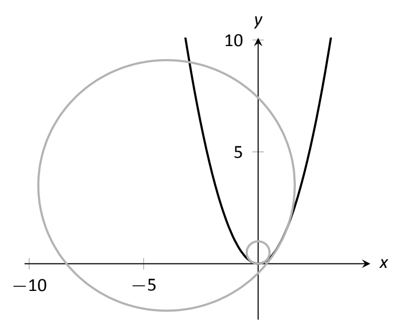


Figure 11.5.5: Examining the curvature of $y = x^2$.

At the vertex ($x = 0$), the curvature is $\kappa = 2$. At $x = 1$, the curvature is $\kappa = 2/(5)^{3/2} \approx 0.179$. So at $x = 0$, the curvature of $y = x^2$ is that of a circle of radius $1/2$; at $x = 1$, the curvature is that of a circle with radius $\approx 1/0.179 \approx 5.59$. This is illustrated in Figure 11.5.5. At $x = 3$, the curvature is 0.009; the graph is nearly straight as the curvature is very close to 0.

Example 11.5.5 Finding curvature

Find where the curvature of $\vec{r}(t) = \left\langle t, t^2, 2t^3\right\rangle$ is maximized.

Notes:

Solution We use the third formula in Theorem 11.5.2 as $\vec{r}(t)$ is defined in space. We leave it to the reader to verify that

$$\vec{r}\,'(t) = \left\langle 1, 2t, 6t^2 \right\rangle, \quad \vec{r}\,''(t) = \langle 0, 2, 12t \rangle, \quad \text{and} \quad \vec{r}\,'(t)\times\vec{r}\,''(t) = \left\langle 12t^2, -12t, 2 \right\rangle.$$

Thus

$$\begin{aligned} \kappa(t) &= \frac{\|\,\vec{r}\,'(t)\times\vec{r}\,''(t)\,\|}{\|\,\vec{r}\,'(t)\,\|^3} \\ &= \frac{\|\,\left\langle 12t^2, -12t, 2\right\rangle\,\|}{\|\,\langle 1, 2t, 6t^2\rangle\,\|^3} \\ &= \frac{\sqrt{144t^4+144t^2+4}}{\left(\sqrt{1+4t^2+36t^4}\right)^3} \end{aligned}$$

While this is not a particularly "nice" formula, it does explicitly tell us what the curvature is at a given t value. To maximize $\kappa(t)$, we should solve $\kappa'(t) = 0$ for t. This is doable, but *very* time consuming. Instead, consider the graph of $\kappa(t)$ as given in Figure 11.5.6(a). We see that κ is maximized at two t values; using a numerical solver, we find these values are $t \approx \pm 0.189$. In part (b) of the figure we graph $\vec{r}(t)$ and indicate the points where curvature is maximized.

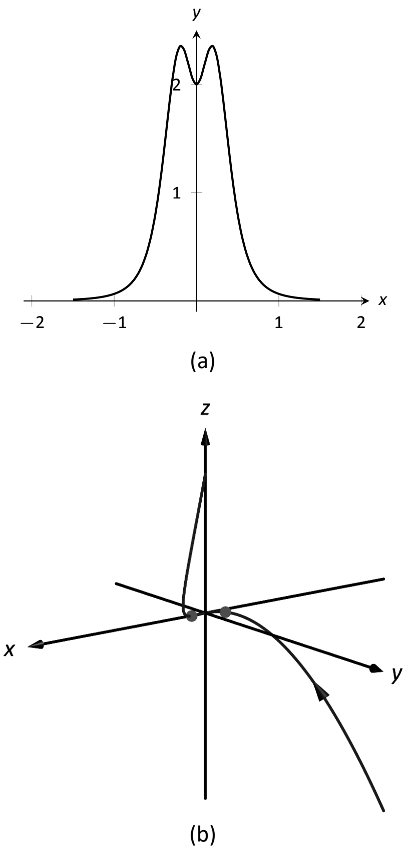


Figure 11.5.6: Understanding the curvature of a curve in space.

Curvature and Motion

Let $\vec{r}(t)$ be a position function of an object, with velocity $\vec{v}(t) = \vec{r}\,'(t)$ and acceleration $\vec{a}(t) = \vec{r}\,''(t)$. In Section 11.4 we established that acceleration is in the plane formed by $\vec{T}(t)$ and $\vec{N}(t)$, and that we can find scalars a_{T} and a_{N} such that

$$\vec{a}(t) = a_{\text{T}}\vec{T}(t) + a_{\text{N}}\vec{N}(t).$$

Theorem 11.4.2 gives formulas for a_{T} and a_{N}:

$$a_{\text{T}} = \frac{d}{dt}\Big(\|\,\vec{v}(t)\,\|\Big) \quad \text{and} \quad a_{\text{N}} = \frac{\|\,\vec{v}(t)\times\vec{a}(t)\,\|}{\|\,\vec{v}(t)\,\|}.$$

We understood that the amount of acceleration in the direction of $\vec{T}$ relates only to how the speed of the object is changing, and that the amount of acceleration in the direction of $\vec{N}$ relates to how the direction of travel of the object is changing. (That is, if the object travels at constant speed, $a_{\text{T}} = 0$; if the object travels in a constant direction, $a_{\text{N}} = 0$.)

In Equation (11.2) at the beginning of this section, we found $s\,'(t) = \|\,\vec{v}(t)\,\|$. We can combine this fact with the above formula for a_{T} to write

$$a_{\text{T}} = \frac{d}{dt}\Big(\|\,\vec{v}(t)\,\|\Big) = \frac{d}{dt}\big(s\,'(t)\big) = s\,''(t).$$

Notes:

Since $s\,'(t)$ is speed, $s\,''(t)$ is the rate at which speed is changing with respect to time. We see once more that the component of acceleration in the direction of travel relates only to speed, not to a change in direction.

Now compare the formula for a_{N} above to the formula for curvature in Theorem 11.5.2:

$$a_{\text{N}} = \frac{||\,\vec v(t) \times \vec a(t)\,||}{||\,\vec v(t)\,||} \quad \text{and} \quad \kappa = \frac{||\,\vec r\,'(t) \times \vec r\,''(t)\,||}{||\,\vec r\,'(t)\,||^3} = \frac{||\,\vec v(t) \times \vec a(t)\,||}{||\,\vec v(t)\,||^3}.$$

Thus

$$\begin{aligned} a_{\text{N}} &= \kappa||\,\vec v(t)\,||^2 \qquad (11.5)\\ &= \kappa\Big(s\,'(t)\Big)^2 \end{aligned}$$

This last equation shows that the component of acceleration that changes the object's direction is dependent on two things: the curvature of the path and the speed of the object.

Imagine driving a car in a clockwise circle. You will naturally feel a force pushing you towards the door (more accurately, the door is pushing you as the car is turning and you want to travel in a straight line). If you keep the radius of the circle constant but speed up (i.e., increasing $s\,'(t)$), the door pushes harder against you (a_{N} has increased). If you keep your speed constant but tighten the turn (i.e., increase κ), once again the door will push harder against you.

Putting our new formulas for a_{T} and a_{N} together, we have

$$\vec a(t) = s\,''(t)\vec T(t) + \kappa||\,\vec v(t)\,||^2\vec N(t).$$

This is not a particularly practical way of finding a_{T} and a_{N}, but it reveals some great concepts about how acceleration interacts with speed and the shape of a curve.

Example 11.5.6 Curvature and road design

The minimum radius of the curve in a highway cloverleaf is determined by the operating speed, as given in the table in Figure 11.5.7. For each curve and speed, compute a_{N}.

Operating Speed (mph)	Minimum Radius (ft)
35	310
40	430
45	540

Figure 11.5.7: Operating speed and minimum radius in highway cloverleaf design.

Solution Using Equation (11.5), we can compute the acceleration normal to the curve in each case. We start by converting each speed from "miles per hour" to "feet per second" by multiplying by 5280/3600.

Notes:

$$\begin{aligned} \text{35mph, 310ft} &\Rightarrow 51.33\text{ft/s}, \quad \kappa = 1/310 \\ a_{\text{N}} &= \kappa \,||\, \vec{v}(t) \,||^2 \\ &= \frac{1}{310}\big(51.33\big)^2 \\ &= 8.50\text{ft/s}^2. \end{aligned}$$

$$\begin{aligned} \text{40mph, 430ft} &\Rightarrow 58.67\text{ft/s}, \quad \kappa = 1/430 \\ a_{\text{N}} &= \frac{1}{430}\big(58.67\big)^2 \\ &= 8.00\text{ft/s}^2. \end{aligned}$$

$$\begin{aligned} \text{45mph,540ft} &\Rightarrow 66\text{ft/s}, \quad \kappa = 1/540 \\ a_{\text{N}} &= \frac{1}{540}\big(66\big)^2 \\ &= 8.07\text{ft/s}^2. \end{aligned}$$

Note that each acceleration is similar; this is by design. Considering the classic "Force $=$ mass $\times$ acceleration" formula, this acceleration must be kept small in order for the tires of a vehicle to keep a "grip" on the road. If one travels on a turn of radius 310ft at a rate of 50mph, the acceleration is double, at 17.35ft/s^2. If the acceleration is too high, the frictional force created by the tires may not be enough to keep the car from sliding. Civil engineers routinely compute a "safe" design speed, then subtract 5-10mph to create the posted speed limit for additional safety.

We end this chapter with a reflection on what we've covered. We started with vector–valued functions, which may have seemed at the time to be just another way of writing parametric equations. However, we have seen that the vector perspective has given us great insight into the behavior of functions and the study of motion. Vector–valued position functions convey displacement, distance traveled, speed, velocity, acceleration and curvature information, each of which has great importance in science and engineering.

Notes:

Exercises 11.5

Terms and Concepts

1. It is common to describe position in terms of both ________ and/or ________.

2. A measure of the "curviness" of a curve is ________.

3. Give two shapes with constant curvature.

4. Describe in your own words what an "osculating circle" is.

5. Complete the identity: $\vec{T}\,'(s) =$ ________$\vec{N}(s)$.

6. Given a position function $\vec{r}(t)$, how are a_{T} and a_{N} affected by the curvature?

Problems

In Exercises 7 – 10 , a position function $\vec{r}(t)$ is given, where $t = 0$ corresponds to the initial position. Find the arc length parameter s, and rewrite $\vec{r}(t)$ in terms of s; that is, find $\vec{r}(s)$.

7. $\vec{r}(t) = \langle 2t, t, -2t\rangle$

8. $\vec{r}(t) = \langle 7\cos t, 7\sin t\rangle$

9. $\vec{r}(t) = \langle 3\cos t, 3\sin t, 2t\rangle$

10. $\vec{r}(t) = \langle 5\cos t, 13\sin t, 12\cos t\rangle$

In Exercises 11 – 22 , a curve C is described along with 2 points on C.

(a) Using a sketch, determine at which of these points the curvature is greater.

(b) Find the curvature κ of C, and evaluate κ at each of the 2 given points.

11. C is defined by $y = x^3 - x$; points given at $x = 0$ and $x = 1/2$.

12. C is defined by $y = \dfrac{1}{x^2+1}$; points given at $x = 0$ and $x = 2$.

13. C is defined by $y = \cos x$; points given at $x = 0$ and $x = \pi/2$.

14. C is defined by $y = \sqrt{1-x^2}$ on $(-1, 1)$; points given at $x = 0$ and $x = 1/2$.

15. C is defined by $\vec{r}(t) = \langle \cos t, \sin(2t)\rangle$; points given at $t = 0$ and $t = \pi/4$.

16. C is defined by $\vec{r}(t) = \left\langle \cos^2 t, \sin t\cos t\right\rangle$; points given at $t = 0$ and $t = \pi/3$.

17. C is defined by $\vec{r}(t) = \left\langle t^2 - 1, t^3 - t\right\rangle$; points given at $t = 0$ and $t = 5$.

18. C is defined by $\vec{r}(t) = \langle \tan t, \sec t\rangle$; points given at $t = 0$ and $t = \pi/6$.

19. C is defined by $\vec{r}(t) = \langle 4t+2, 3t-1, 2t+5\rangle$; points given at $t = 0$ and $t = 1$.

20. C is defined by $\vec{r}(t) = \left\langle t^3 - t, t^3 - 4, t^2 - 1\right\rangle$; points given at $t = 0$ and $t = 1$.

21. C is defined by $\vec{r}(t) = \langle 3\cos t, 3\sin t, 2t\rangle$; points given at $t = 0$ and $t = \pi/2$.

22. C is defined by $\vec{r}(t) = \langle 5\cos t, 13\sin t, 12\cos t\rangle$; points given at $t = 0$ and $t = \pi/2$.

In Exercises 23 – 26 , find the value of x or t where curvature is maximized.

23. $y = \dfrac{1}{6}x^3$

24. $y = \sin x$

25. $\vec{r}(t) = \left\langle t^2 + 2t, 3t - t^2\right\rangle$

26. $\vec{r}(t) = \langle t, 4/t, 3/t\rangle$

In Exercises 27 – 30 , find the radius of curvature at the indicated value.

27. $y = \tan x$, at $x = \pi/4$

28. $y = x^2 + x - 3$, at $x = \pi/4$

29. $\vec{r}(t) = \langle \cos t, \sin(3t)\rangle$, at $t = 0$

30. $\vec{r}(t) = \langle 5\cos(3t), t\rangle$, at $t = 0$

In Exercises 31 – 34 , find the equation of the osculating circle to the curve at the indicated t-value.

31. $\vec{r}(t) = \left\langle t, t^2\right\rangle$, at $t = 0$

32. $\vec{r}(t) = \langle 3\cos t, \sin t\rangle$, at $t = 0$

33. $\vec{r}(t) = \langle 3\cos t, \sin t\rangle$, at $t = \pi/2$

34. $\vec{r}(t) = \left\langle t^2 - t, t^2 + t\right\rangle$, at $t = 0$