5.5 Indefinite Integrals and the Substitution Rule

A definite integral is a number defined by taking the limit of Riemann sums associated with partitions of a finite closed interval whose norms go to zero. The Fundamental Theorem of Calculus says that a definite integral of a continuous function can be computed easily if we can find an antiderivative of the function. Antiderivatives generally turn out to be more difficult to find than derivatives. However, it is well worth the effort to learn techniques for computing them.

Recall from Section 4.8 that the set of *all* antiderivatives of the function f is called the **indefinite integral** of f with respect to x, and is symbolized by

$$\int f(x)\,dx.$$

The connection between antiderivatives and the definite integral stated in the Fundamental Theorem now explains this notation. When finding the indefinite integral of a function f, remember that it always includes an arbitrary constant C.

We must distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a *number*. An indefinite integral $\int f(x)\,dx$ is a *function* plus an arbitrary constant C.

So far, we have only been able to find antiderivatives of functions that are clearly recognizable as derivatives. In this section we begin to develop more general techniques for finding antiderivatives. The first integration techniques we develop are obtained by inverting rules for finding derivatives, such as the Power Rule and the Chain Rule.

The Power Rule in Integral Form

If u is a differentiable function of x and n is a rational number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

From another point of view, this same equation says that $u^{n+1}/(n+1)$ is one of the antiderivatives of the function $u^n(du/dx)$. Therefore,

$$\int \left(u^n \frac{du}{dx}\right) dx = \frac{u^{n+1}}{n+1} + C.$$

The integral on the left-hand side of this equation is usually written in the simpler "differential" form,

$$\int u^n \, du,$$

obtained by treating the dx's as differentials that cancel. We are thus led to the following rule.

If u is any differentiable function, then

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1, n \text{ rational}). \tag{1}$$

Equation (1) actually holds for any real exponent $n \neq -1$, as we see in Chapter 7.

In deriving Equation (1), we assumed u to be a differentiable function of the variable x, but the name of the variable does not matter and does not appear in the final formula. We could have represented the variable with θ, t, y, or any other letter. Equation (1) says that whenever we can cast an integral in the form

$$\int u^n \, du, \qquad (n \neq -1),$$

with u a differentiable function and du its differential, we can evaluate the integral as $[u^{n+1}/(n+1)] + C$.

EXAMPLE 1 Using the Power Rule

$$\int \sqrt{1+y^2} \cdot 2y \, dy = \int \sqrt{u} \cdot \left(\frac{du}{dy}\right) dy \qquad \text{Let } u = 1 + y^2,\ du/dy = 2y$$

$$= \int u^{1/2} \, du$$

$$= \frac{u^{(1/2)+1}}{(1/2)+1} + C \qquad \text{Integrate, using Eq. (1) with } n = 1/2.$$

$$= \frac{2}{3} u^{3/2} + C \qquad \text{Simpler form}$$

$$= \frac{2}{3}(1+y^2)^{3/2} + C \qquad \text{Replace } u \text{ by } 1 + y^2.$$

■

EXAMPLE 2 Adjusting the Integrand by a Constant

$$\int \sqrt{4t-1}\,dt = \int \frac{1}{4}\cdot\sqrt{4t-1}\cdot 4\,dt$$

$$= \frac{1}{4}\int \sqrt{u}\cdot\left(\frac{du}{dt}\right)dt \qquad \text{Let } u = 4t-1,\ du/dt = 4.$$

$$= \frac{1}{4}\int u^{1/2}\,du \qquad \text{With the 1/4 out front, the integral is now in standard form.}$$

$$= \frac{1}{4}\cdot\frac{u^{3/2}}{3/2} + C \qquad \text{Integrate, using Eq. (1) with } n = 1/2.$$

$$= \frac{1}{6}u^{3/2} + C \qquad \text{Simpler form}$$

$$= \frac{1}{6}(4t-1)^{3/2} + C \qquad \text{Replace } u \text{ by } 4t-1.$$

■

Substitution: Running the Chain Rule Backwards

The substitutions in Examples 1 and 2 are instances of the following general rule.

THEOREM 5 The Substitution Rule
If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof The rule is true because, by the Chain Rule, $F(g(x))$ is an antiderivative of $f(g(x))\cdot g'(x)$ whenever F is an antiderivative of f:

$$\frac{d}{dx}F(g(x)) = F'(g(x))\cdot g'(x) \qquad \text{Chain Rule}$$

$$= f(g(x))\cdot g'(x). \qquad \text{Because } F' = f$$

If we make the substitution $u = g(x)$ then

$$\int f(g(x))g'(x)\,dx = \int \frac{d}{dx}F(g(x))\,dx$$

$$= F(g(x)) + C \qquad \text{Fundamental Theorem}$$

$$= F(u) + C \qquad u = g(x)$$

$$= \int F'(u)\,du \qquad \text{Fundamental Theorem}$$

$$= \int f(u)\,du \qquad F' = f$$

■

The Substitution Rule provides the following method to evaluate the integral

$$\int f(g(x))g'(x)\,dx,$$

when f and g' are continuous functions:

1. Substitute $u = g(x)$ and $du = g'(x)\,dx$ to obtain the integral
$$\int f(u)\,du.$$
2. Integrate with respect to u.
3. Replace u by $g(x)$ in the result.

EXAMPLE 3 Using Substitution

$$\int \cos(7\theta + 5)\,d\theta = \int \cos u \cdot \frac{1}{7}\,du$$ Let $u = 7\theta + 5$, $du = 7\,d\theta$, $(1/7)\,du = d\theta$.

$$= \frac{1}{7}\int \cos u\,du$$ With the (1/7) out front, the integral is now in standard form.

$$= \frac{1}{7}\sin u + C$$ Integrate with respect to u, Table 4.2.

$$= \frac{1}{7}\sin(7\theta + 5) + C$$ Replace u by $7\theta + 5$.

We can verify this solution by differentiating and checking that we obtain the original function $\cos(7\theta + 5)$. ■

EXAMPLE 4 Using Substitution

$$\int x^2 \sin(x^3)\,dx = \int \sin(x^3)\cdot x^2\,dx$$

$$= \int \sin u \cdot \frac{1}{3}\,du$$ Let $u = x^3$, $du = 3x^2\,dx$, $(1/3)\,du = x^2\,dx$.

$$= \frac{1}{3}\int \sin u\,du$$

$$= \frac{1}{3}(-\cos u) + C$$ Integrate with respect to u.

$$= -\frac{1}{3}\cos(x^3) + C$$ Replace u by x^3. ■

EXAMPLE 5 Using Identities and Substitution

$$
\begin{aligned}
\int \frac{1}{\cos^2 2x}\,dx &= \int \sec^2 2x\,dx && \frac{1}{\cos 2x} = \sec 2x \\
&= \int \sec^2 u \cdot \frac{1}{2}\,du && u = 2x,\ du = 2\,dx,\ dx = (1/2)\,du \\
&= \frac{1}{2}\int \sec^2 u\,du \\
&= \frac{1}{2}\tan u + C && \frac{d}{du}\tan u = \sec^2 u \\
&= \frac{1}{2}\tan 2x + C && u = 2x
\end{aligned}
$$

■

The success of the substitution method depends on finding a substitution that changes an integral we cannot evaluate directly into one that we can. If the first substitution fails, try to simplify the integrand further with an additional substitution or two (see Exercises 49 and 50). Alternatively, we can start fresh. There can be more than one good way to start, as in the next example.

EXAMPLE 6 Using Different Substitutions

Evaluate

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}.$$

Solution We can use the substitution method of integration as an exploratory tool: Substitute for the most troublesome part of the integrand and see how things work out. For the integral here, we might try $u = z^2 + 1$ or we might even press our luck and take u to be the entire cube root. Here is what happens in each case.

Solution 1: Substitute $u = z^2 + 1$.

$$
\begin{aligned}
\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} &= \int \frac{du}{u^{1/3}} && \text{Let } u = z^2+1,\ du = 2z\,dz. \\
&= \int u^{-1/3}\,du && \text{In the form } \textstyle\int u^n\,du \\
&= \frac{u^{2/3}}{2/3} + C && \text{Integrate with respect to } u. \\
&= \frac{3}{2}u^{2/3} + C \\
&= \frac{3}{2}(z^2+1)^{2/3} + C && \text{Replace } u \text{ by } z^2+1.
\end{aligned}
$$

Solution 2: Substitute $u = \sqrt[3]{z^2 + 1}$ instead.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\,du}{u} \qquad \text{Let } u = \sqrt[3]{z^2+1},\ u^3 = z^2 + 1,\ 3u^2\,du = 2z\,dz.$$

$$= 3\int u\,du$$

$$= 3 \cdot \frac{u^2}{2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } (z^2+1)^{1/3}.$$ ■

The Integrals of $\sin^2 x$ and $\cos^2 x$

Sometimes we can use trigonometric identities to transform integrals we do not know how to evaluate into ones we can using the substitution rule. Here is an example giving the integral formulas for $\sin^2 x$ and $\cos^2 x$ which arise frequently in applications.

EXAMPLE 7

(a) $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C \qquad \text{As in part (a), but with a sign change}$$ ■

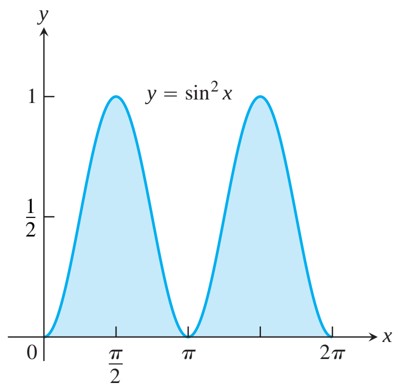


FIGURE 5.24 The area beneath the curve $y = \sin^2 x$ over $[0, 2\pi]$ equals π square units (Example 8).

EXAMPLE 8 Area Beneath the Curve $y = \sin^2 x$

Figure 5.24 shows the graph of $g(x) = \sin^2 x$ over the interval $[0, 2\pi]$. Find

(a) the definite integral of $g(x)$ over $[0, 2\pi]$.

(b) the area between the graph of the function and the x-axis over $[0, 2\pi]$.

Solution

(a) From Example 7(a), the definite integral is

$$\int_0^{2\pi} \sin^2 x\,dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{2\pi} = \left[\frac{2\pi}{2} - \frac{\sin 4\pi}{4}\right] - \left[\frac{0}{2} - \frac{\sin 0}{4}\right]$$

$$= [\pi - 0] - [0 - 0] = \pi.$$

(b) The function $\sin^2 x$ is nonnegative, so the area is equal to the definite integral, or π. ■

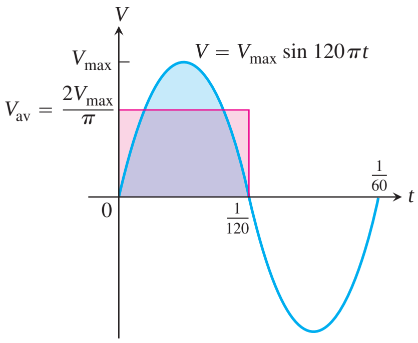


FIGURE 5.25 The graph of the voltage $V = V_{max} \sin 120\pi t$ over a full cycle. Its average value over a half-cycle is $2V_{max}/\pi$. Its average value over a full cycle is zero (Example 9).

EXAMPLE 9 Household Electricity

We can model the voltage in our home wiring with the sine function

$$V = V_{max} \sin 120\pi t,$$

which expresses the voltage V in volts as a function of time t in seconds. The function runs through 60 cycles each second (its frequency is 60 hertz, or 60 Hz). The positive constant V_{max} ("vee max") is the **peak voltage**.

The average value of V over the half-cycle from 0 to 1/120 sec (see Figure 5.25) is

$$\begin{aligned} V_{av} &= \frac{1}{(1/120) - 0} \int_0^{1/120} V_{max} \sin 120\pi t \, dt \\ &= 120 V_{max} \left[-\frac{1}{120\pi} \cos 120\pi t \right]_0^{1/120} \\ &= \frac{V_{max}}{\pi} [-\cos \pi + \cos 0] \\ &= \frac{2V_{max}}{\pi}. \end{aligned}$$

The average value of the voltage over a full cycle is zero, as we can see from Figure 5.25. (Also see Exercise 63.) If we measured the voltage with a standard moving-coil galvanometer, the meter would read zero.

To measure the voltage effectively, we use an instrument that measures the square root of the average value of the square of the voltage, namely

$$V_{rms} = \sqrt{(V^2)_{av}}.$$

The subscript "rms" (read the letters separately) stands for "root mean square." Since the average value of $V^2 = (V_{max})^2 \sin^2 120\pi t$ over a cycle is

$$(V^2)_{av} = \frac{1}{(1/60) - 0} \int_0^{1/60} (V_{max})^2 \sin^2 120\pi t \, dt = \frac{(V_{max})^2}{2},$$

(Exercise 63, part c), the rms voltage is

$$V_{rms} = \sqrt{\frac{(V_{max})^2}{2}} = \frac{V_{max}}{\sqrt{2}}.$$

The values given for household currents and voltages are always rms values. Thus, "115 volts ac" means that the rms voltage is 115. The peak voltage, obtained from the last equation, is

$$V_{max} = \sqrt{2}\, V_{rms} = \sqrt{2} \cdot 115 \approx 163 \text{ volts},$$

which is considerably higher. ■

EXERCISES 5.5

Evaluating Integrals

Evaluate the indefinite integrals in Exercises 1–12 by using the given substitutions to reduce the integrals to standard form.

1. $\int \sin 3x \, dx, \quad u = 3x$

2. $\int x \sin(2x^2) \, dx, \quad u = 2x^2$

3. $\int \sec 2t \tan 2t \, dt, \quad u = 2t$

4. $\int \left(1 - \cos \frac{t}{2}\right)^2 \sin \frac{t}{2} \, dt, \quad u = 1 - \cos \frac{t}{2}$

5. $\int 28(7x-2)^{-5}\,dx, \quad u = 7x - 2$

6. $\int x^3(x^4-1)^2\,dx, \quad u = x^4 - 1$

7. $\int \frac{9r^2\,dr}{\sqrt{1-r^3}}, \quad u = 1 - r^3$

8. $\int 12(y^4+4y^2+1)^2(y^3+2y)\,dy, \quad u = y^4 + 4y^2 + 1$

9. $\int \sqrt{x}\,\sin^2(x^{3/2}-1)\,dx, \quad u = x^{3/2} - 1$

10. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx, \quad u = -\frac{1}{x}$

11. $\int \csc^2 2\theta \cot 2\theta\,d\theta$

a. Using $u = \cot 2\theta$ **b.** Using $u = \csc 2\theta$

12. $\int \frac{dx}{\sqrt{5x+8}}$

a. Using $u = 5x + 8$ **b.** Using $u = \sqrt{5x+8}$

Evaluate the integrals in Exercises 13–48.

13. $\int \sqrt{3-2s}\,ds$

14. $\int (2x+1)^3\,dx$

15. $\int \frac{1}{\sqrt{5s+4}}\,ds$

16. $\int \frac{3\,dx}{(2-x)^2}$

17. $\int \theta\sqrt[4]{1-\theta^2}\,d\theta$

18. $\int 8\theta\sqrt[3]{\theta^2-1}\,d\theta$

19. $\int 3y\sqrt{7-3y^2}\,dy$

20. $\int \frac{4y\,dy}{\sqrt{2y^2+1}}$

21. $\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx$

22. $\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}\,dx$

23. $\int \cos(3z+4)\,dz$

24. $\int \sin(8z-5)\,dz$

25. $\int \sec^2(3x+2)\,dx$

26. $\int \tan^2 x \sec^2 x\,dx$

27. $\int \sin^5\frac{x}{3}\cos\frac{x}{3}\,dx$

28. $\int \tan^7\frac{x}{2}\sec^2\frac{x}{2}\,dx$

29. $\int r^2\left(\frac{r^3}{18}-1\right)^5 dr$

30. $\int r^4\left(7-\frac{r^5}{10}\right)^3 dr$

31. $\int x^{1/2}\sin(x^{3/2}+1)\,dx$

32. $\int x^{1/3}\sin(x^{4/3}-8)\,dx$

33. $\int \sec\left(v+\frac{\pi}{2}\right)\tan\left(v+\frac{\pi}{2}\right)dv$

34. $\int \csc\left(\frac{v-\pi}{2}\right)\cot\left(\frac{v-\pi}{2}\right)dv$

35. $\int \frac{\sin(2t+1)}{\cos^2(2t+1)}\,dt$

36. $\int \frac{6\cos t}{(2+\sin t)^3}\,dt$

37. $\int \sqrt{\cot y}\,\csc^2 y\,dy$

38. $\int \frac{\sec z\tan z}{\sqrt{\sec z}}\,dz$

39. $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt$

40. $\int \frac{1}{\sqrt{t}}\cos(\sqrt{t}+3)\,dt$

41. $\int \frac{1}{\theta^2}\sin\frac{1}{\theta}\cos\frac{1}{\theta}\,d\theta$

42. $\int \frac{\cos\sqrt{\theta}}{\sqrt{\theta}\sin^2\sqrt{\theta}}\,d\theta$

43. $\int (s^3+2s^2-5s+5)(3s^2+4s-5)\,ds$

44. $\int (\theta^4-2\theta^2+8\theta-2)(\theta^3-\theta+2)\,d\theta$

45. $\int t^3(1+t^4)^3\,dt$

46. $\int \sqrt{\frac{x-1}{x^5}}\,dx$

47. $\int x^3\sqrt{x^2+1}\,dx$

48. $\int 3x^5\sqrt{x^3+1}\,dx$

Simplifying Integrals Step by Step

If you do not know what substitution to make, try reducing the integral step by step, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions in Exercises 49 and 50.

49. $\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$

a. $u = \tan x$, followed by $v = u^3$, then by $w = 2 + v$

b. $u = \tan^3 x$, followed by $v = 2 + u$

c. $u = 2 + \tan^3 x$

50. $\int \sqrt{1+\sin^2(x-1)}\,\sin(x-1)\cos(x-1)\,dx$

a. $u = x - 1$, followed by $v = \sin u$, then by $w = 1 + v^2$

b. $u = \sin(x-1)$, followed by $v = 1 + u^2$

c. $u = 1 + \sin^2(x-1)$

Evaluate the integrals in Exercises 51 and 52.

51. $\int \frac{(2r-1)\cos\sqrt{3(2r-1)^2+6}}{\sqrt{3(2r-1)^2+6}}\,dr$

52. $\int \frac{\sin\sqrt{\theta}}{\sqrt{\theta\cos^3\sqrt{\theta}}}\,d\theta$

Initial Value Problems

Solve the initial value problems in Exercises 53–58.

53. $\frac{ds}{dt} = 12t(3t^2-1)^3, \quad s(1) = 3$

54. $\frac{dy}{dx} = 4x(x^2+8)^{-1/3}, \quad y(0) = 0$

55. $\frac{ds}{dt} = 8\sin^2\left(t+\frac{\pi}{12}\right), \quad s(0) = 8$

56. $\frac{dr}{d\theta} = 3\cos^2\left(\frac{\pi}{4}-\theta\right), \quad r(0) = \frac{\pi}{8}$

57. $\dfrac{d^2s}{dt^2} = -4\sin\left(2t - \dfrac{\pi}{2}\right), \quad s'(0) = 100, \quad s(0) = 0$

58. $\dfrac{d^2y}{dx^2} = 4\sec^2 2x \tan 2x, \quad y'(0) = 4, \quad y(0) = -1$

59. The velocity of a particle moving back and forth on a line is $v = ds/dt = 6\sin 2t$ m/sec for all t. If $s = 0$ when $t = 0$, find the value of s when $t = \pi/2$ sec.

60. The acceleration of a particle moving back and forth on a line is $a = d^2s/dt^2 = \pi^2 \cos \pi t$ m/sec^2 for all t. If $s = 0$ and $v = 8$ m/sec when $t = 0$, find s when $t = 1$ sec.

Theory and Examples

61. It looks as if we can integrate $2\sin x \cos x$ with respect to x in three different ways:

a. $\displaystyle\int 2\sin x\cos x\,dx = \int 2u\,du \qquad u = \sin x,$

$\qquad = u^2 + C_1 = \sin^2 x + C_1$

b. $\displaystyle\int 2\sin x\cos x\,dx = \int -2u\,du \qquad u = \cos x,$

$\qquad = -u^2 + C_2 = -\cos^2 x + C_2$

c. $\displaystyle\int 2\sin x\cos x\,dx = \int \sin 2x\,dx \qquad 2\sin x\cos x = \sin 2x$

$\qquad = -\dfrac{\cos 2x}{2} + C_3.$

Can all three integrations be correct? Give reasons for your answer.

62. The substitution $u = \tan x$ gives

$$\int \sec^2 x \tan x\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{\tan^2 x}{2} + C.$$

The substitution $u = \sec x$ gives

$$\int \sec^2 x \tan x\,dx = \int u\,du = \frac{u^2}{2} + C = \frac{\sec^2 x}{2} + C.$$

Can both integrations be correct? Give reasons for your answer.

63. (*Continuation of Example 9.*)

a. Show by evaluating the integral in the expression

$$\frac{1}{(1/60) - 0}\int_0^{1/60} V_{\max}\sin 120\,\pi t\,dt$$

that the average value of $V = V_{\max}\sin 120\,\pi t$ over a full cycle is zero.

b. The circuit that runs your electric stove is rated 240 volts rms. What is the peak value of the allowable voltage?

c. Show that

$$\int_0^{1/60} (V_{\max})^2 \sin^2 120\,\pi t\,dt = \frac{(V_{\max})^2}{120}.$$

5.6 Substitution and Area Between Curves

There are two methods for evaluating a definite integral by substitution. The first method is to find an antiderivative using substitution, and then to evaluate the definite integral by applying the Fundamental Theorem. We used this method in Examples 8 and 9 of the preceding section. The second method extends the process of substitution directly to *definite* integrals. We apply the new formula introduced here to the problem of computing the area between two curves.

Substitution Formula

In the following formula, the limits of integration change when the variable of integration is changed by substitution.

THEOREM 6 Substitution in Definite Integrals

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Proof Let F denote any antiderivative of f. Then,

$$\begin{aligned}\int_a^b f(g(x))\cdot g'(x)\,dx &= F(g(x))\Big]_{x=a}^{x=b} \\ &= F(g(b)) - F(g(a)) \\ &= F(u)\Big]_{u=g(a)}^{u=g(b)} \\ &= \int_{g(a)}^{g(b)} f(u)\,du.\end{aligned}$$

$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$

Fundamental Theorem, Part 2 ■

To use the formula, make the same u-substitution $u = g(x)$ and $du = g'(x)\,dx$ you would use to evaluate the corresponding indefinite integral. Then integrate the transformed integral with respect to u from the value $g(a)$ (the value of u at $x = a$) to the value $g(b)$ (the value of u at $x = b$).

Video You Try It

EXAMPLE 1 Substitution by Two Methods

Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

Solution We have two choices.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 6.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

$$= \int_0^2 \sqrt{u}\,du$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.

$$= \frac{2}{3}u^{3/2} + C$$

Integrate with respect to u.

$$= \frac{2}{3}(x^3+1)^{3/2} + C$$

Replace u by $x^3 + 1$.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1}$$

Use the integral just found, with limits of integration for x.

$$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$ ■

Which method is better—evaluating the transformed definite integral with transformed limits using Theorem 6, or transforming the integral, integrating, and transforming back to use the original limits of integration? In Example 1, the first method seems easier, but that is not always the case. Generally, it is best to know both methods and to use whichever one seems better at the time.

EXAMPLE 2 Using the Substitution Formula

$$\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \, d\theta = \int_1^0 u \cdot (-du) \qquad \begin{array}{l}\text{Let } u = \cot\theta,\ du = -\csc^2\theta\, d\theta, \\ \quad -du = \csc^2\theta\, d\theta. \\ \text{When } \theta = \pi/4,\ u = \cot(\pi/4) = 1. \\ \text{When } \theta = \pi/2,\ u = \cot(\pi/2) = 0.\end{array}$$

$$= -\int_1^0 u\, du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

Definite Integrals of Symmetric Functions

The Substitution Formula in Theorem 6 simplifies the calculation of definite integrals of even and odd functions (Section 1.4) over a symmetric interval $[-a, a]$ (Figure 5.26).

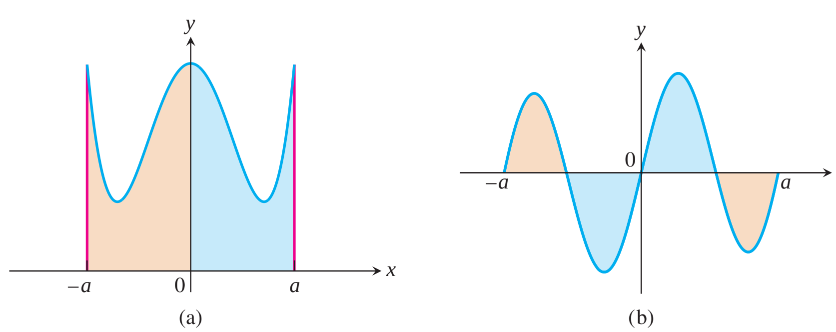


FIGURE 5.26 (a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$ (b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

Theorem 7

Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(b) If f is odd, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Proof of Part (a)

$$\begin{aligned}
\int_{-a}^{a} f(x)\,dx &= \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx && \text{Additivity Rule for Definite Integrals}\\
&= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx && \text{Order of Integration Rule}\\
&= -\int_{0}^{a} f(-u)(-du) + \int_{0}^{a} f(x)\,dx && \text{Let } u=-x,\ du=-dx.\ \text{When } x=0,\ u=0.\ \text{When } x=-a,\ u=a.\\
&= \int_{0}^{a} f(-u)\,du + \int_{0}^{a} f(x)\,dx\\
&= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)\,dx && f \text{ is even, so } f(-u)=f(u).\\
&= 2\int_{0}^{a} f(x)\,dx
\end{aligned}$$

The proof of part (b) is entirely similar and you are asked to give it in Exercise 86. ■

The assertions of Theorem 7 remain true when f is an integrable function (rather than having the stronger property of being continuous), but the proof is somewhat more difficult and best left to a more advanced course.

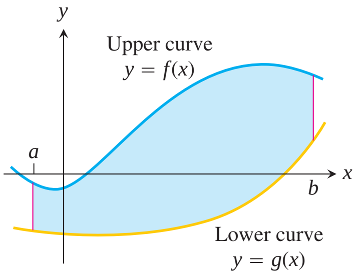


FIGURE 5.27 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

EXAMPLE 3 Integral of an Even Function

Evaluate $\displaystyle\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx.$

Solution Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval $[-2, 2]$, so

$$\begin{aligned}
\int_{-2}^{2} (x^4 - 4x^2 + 6)\,dx &= 2\int_{0}^{2} (x^4 - 4x^2 + 6)\,dx\\
&= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2\\
&= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.
\end{aligned}$$
■

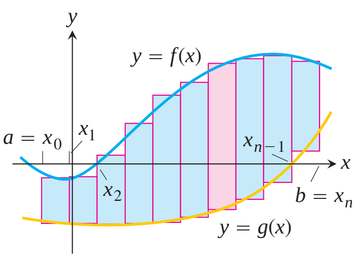


FIGURE 5.28 We approximate the region with rectangles perpendicular to the x-axis.

Areas Between Curves

Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$ (Figure 5.27). The region might accidentally have a shape whose area we could find with geometry, but if f and g are arbitrary continuous functions, we usually have to find the area with an integral.

To see what the integral should be, we first approximate the region with n vertical rectangles based on a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ (Figure 5.28). The area of the kth rectangle (Figure 5.29) is

$$\Delta A_k = \text{height} \times \text{width} = [f(c_k) - g(c_k)]\,\Delta x_k.$$

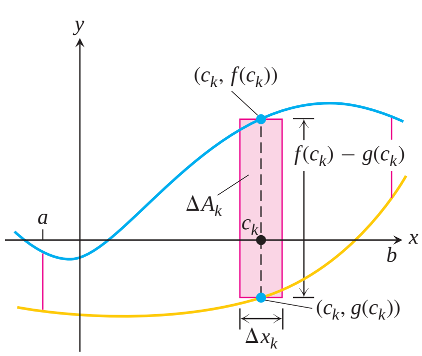


FIGURE 5.29 The area ΔA_k of the kth rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k.

We then approximate the area of the region by adding the areas of the n rectangles:

$$A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} [f(c_k) - g(c_k)]\,\Delta x_k. \qquad \text{Riemann Sum}$$

As $\|P\| \to 0$, the sums on the right approach the limit $\int_a^b [f(x) - g(x)]\,dx$ because f and g are continuous. We take the area of the region to be the value of this integral. That is,

$$A = \lim_{\|P\|\to 0} \sum_{k=1}^{n} [f(c_k) - g(c_k)]\,\Delta x_k = \int_a^b [f(x) - g(x)]\,dx.$$

DEFINITION Area Between Curves

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

When applying this definition it is helpful to graph the curves. The graph reveals which curve is the upper curve f and which is the lower curve g. It also helps you find the limits of integration if they are not already known. You may need to find where the curves intersect to determine the limits of integration, and this may involve solving the equation $f(x) = g(x)$ for values of x. Then you can integrate the function $f - g$ for the area between the intersections.

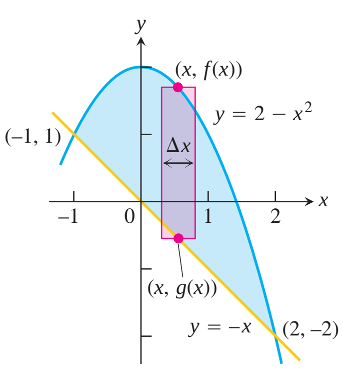


FIGURE 5.30 The region in Example 4 with a typical approximating rectangle.

EXAMPLE 4 Area Between Intersecting Curves

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves (Figure 5.30). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$$\begin{aligned} 2 - x^2 &= -x && \text{Equate } f(x) \text{ and } g(x). \\ x^2 - x - 2 &= 0 && \text{Rewrite.} \\ (x + 1)(x - 2) &= 0 && \text{Factor.} \\ x = -1, \quad x &= 2. && \text{Solve.} \end{aligned}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

The area between the curves is

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\,dx \\ &= \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} \\ &= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2} \end{aligned}$$

HISTORICAL BIOGRAPHY

Richard Dedekind
(1831–1916)

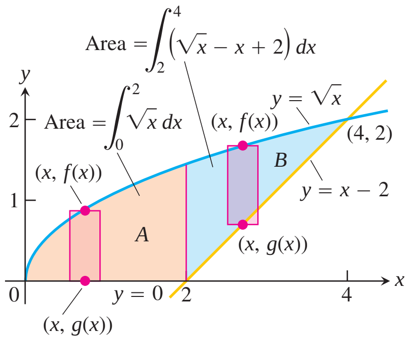


FIGURE 5.31 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 5.

If the formula for a bounding curve changes at one or more points, we subdivide the region into subregions that correspond to the formula changes and apply the formula for the area between curves to each subregion.

EXAMPLE 5 Changing the Integral to Match a Boundary Change

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution The sketch (Figure 5.31) shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (there is agreement at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B, shown in Figure 5.31.

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$\begin{aligned} \sqrt{x} &= x - 2 && \text{Equate } f(x) \text{ and } g(x). \\ x &= (x-2)^2 = x^2 - 4x + 4 && \text{Square both sides.} \\ x^2 - 5x + 4 &= 0 && \text{Rewrite.} \\ (x-1)(x-4) &= 0 && \text{Factor.} \\ x = 1, &\quad x = 4. && \text{Solve.} \end{aligned}$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

$$\begin{aligned} &\text{For } 0 \le x \le 2: && f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x} \\ &\text{For } 2 \le x \le 4: && f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2 \end{aligned}$$

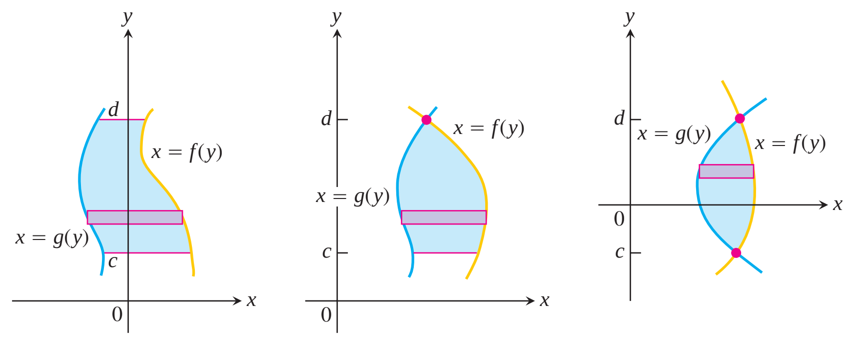
We add the area of subregions A and B to find the total area:

$$\begin{aligned} \text{Total area} &= \underbrace{\int_0^2 \sqrt{x}\, dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\, dx}_{\text{area of } B} \\ &= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4 \\ &= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right) \\ &= \frac{2}{3}(8) - 2 = \frac{10}{3}. \end{aligned}$$

Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these



use the formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

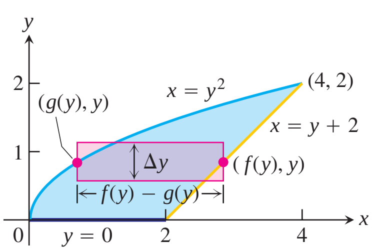


FIGURE 5.32 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 6).

EXAMPLE 6 Find the area of the region in Example 5 by integrating with respect to y.

You Try It

Video

Solution We first sketch the region and a typical *horizontal* rectangle based on a partition of an interval of y-values (Figure 5.32). The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y:

$$y + 2 = y^2 \qquad \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2.$$

$$y^2 - y - 2 = 0 \qquad \text{Rewrite.}$$

$$(y + 1)(y - 2) = 0 \qquad \text{Factor.}$$

$$y = -1, \quad y = 2 \qquad \text{Solve.}$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x-axis.)

The area of the region is

$$\begin{aligned} A = \int_a^b [f(y) - g(y)]\,dy &= \int_0^2 [y + 2 - y^2]\,dy \\ &= \int_0^2 [2 + y - y^2]\,dy \\ &= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 \\ &= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}. \end{aligned}$$

This is the result of Example 5, found with less work. ■

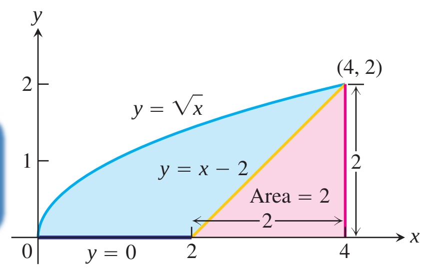


FIGURE 5.33 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle (Example 7).

Combining Integrals with Formulas from Geometry

The fastest way to find an area may be to combine calculus and geometry.

EXAMPLE 7 The Area of the Region in Example 5 Found the Fastest Way

Find the area of the region in Example 5.

Solution The area we want is the area between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$, and the x-axis, *minus* the area of a triangle with base 2 and height 2 (Figure 5.33):

$$\begin{aligned} \text{Area} &= \int_0^4 \sqrt{x}\, dx - \frac{1}{2}(2)(2) \\ &= \frac{2}{3}x^{3/2}\bigg]_0^4 - 2 \\ &= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}. \end{aligned}$$

■

Conclusion from Examples 5–7 It is sometimes easier to find the area between two curves by integrating with respect to y instead of x. Also, it may help to combine geometry and calculus. After sketching the region, take a moment to think about the best way to proceed.

EXERCISES 5.6

Evaluating Definite Integrals

Use the Substitution Formula in Theorem 6 to evaluate the integrals in Exercises 1–24.

1. a. $\int_0^3 \sqrt{y+1}\, dy$ b. $\int_{-1}^0 \sqrt{y+1}\, dy$

2. a. $\int_0^1 r\sqrt{1-r^2}\, dr$ b. $\int_{-1}^1 r\sqrt{1-r^2}\, dr$

3. a. $\int_0^{\pi/4} \tan x \sec^2 x\, dx$ b. $\int_{-\pi/4}^0 \tan x \sec^2 x\, dx$

4. a. $\int_0^{\pi} 3\cos^2 x \sin x\, dx$ b. $\int_{2\pi}^{3\pi} 3\cos^2 x \sin x\, dx$

5. a. $\int_0^1 t^3(1+t^4)^3\, dt$ b. $\int_{-1}^1 t^3(1+t^4)^3\, dt$

6. a. $\int_0^{\sqrt{7}} t(t^2+1)^{1/3}\, dt$ b. $\int_{-\sqrt{7}}^0 t(t^2+1)^{1/3}\, dt$

7. a. $\int_{-1}^1 \frac{5r}{(4+r^2)^2}\, dr$ b. $\int_0^1 \frac{5r}{(4+r^2)^2}\, dr$

8. a. $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\, dv$ b. $\int_1^4 \frac{10\sqrt{v}}{(1+v^{3/2})^2}\, dv$

9. a. $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\, dx$ b. $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\, dx$

10. a. $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\, dx$ b. $\int_{-1}^0 \frac{x^3}{\sqrt{x^4+9}}\, dx$

11. a. $\int_0^{\pi/6} (1-\cos 3t)\sin 3t\, dt$ b. $\int_{\pi/6}^{\pi/3} (1-\cos 3t)\sin 3t\, dt$

12. a. $\int_{-\pi/2}^0 \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\, dt$ b. $\int_{-\pi/2}^{\pi/2} \left(2+\tan\frac{t}{2}\right)\sec^2\frac{t}{2}\, dt$

13. a. $\int_0^{2\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\, dz$ b. $\int_{-\pi}^{\pi} \frac{\cos z}{\sqrt{4+3\sin z}}\, dz$

14. a. $\int_{-\pi/2}^0 \frac{\sin w}{(3+2\cos w)^2}\, dw$ b. $\int_0^{\pi/2} \frac{\sin w}{(3+2\cos w)^2}\, dw$

15. $\int_0^1 \sqrt{t^5+2t}\,(5t^4+2)\, dt$

16. $\int_1^4 \frac{dy}{2\sqrt{y}\,(1+\sqrt{y})^2}$

17. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta\, d\theta$

18. $\int_{\pi}^{3\pi/2} \cot^5\left(\frac{\theta}{6}\right)\sec^2\left(\frac{\theta}{6}\right) d\theta$

19. $\int_0^{\pi} 5(5-4\cos t)^{1/4}\sin t\, dt$

20. $\int_0^{\pi/4} (1-\sin 2t)^{3/2}\cos 2t\, dt$

21. $\int_0^1 (4y-y^2+4y^3+1)^{-2/3}(12y^2-2y+4)\, dy$

22. $\int_0^1 (y^3+6y^2-12y+9)^{-1/2}(y^2+4y-4)\, dy$

23. $\displaystyle\int_0^{\sqrt[3]{\pi^2}} \sqrt{\theta}\cos^2(\theta^{3/2})\,d\theta$

24. $\displaystyle\int_{-1}^{-1/2} t^{-2}\sin^2\left(1 + \frac{1}{t}\right)dt$

Area

Find the total areas of the shaded regions in Exercises 25–40.

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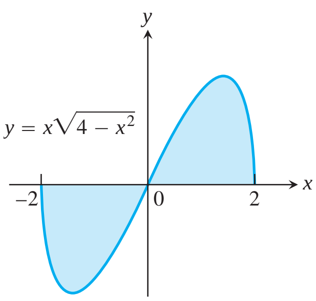

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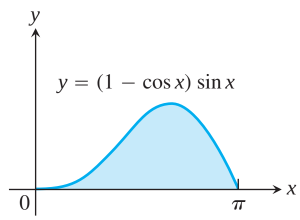

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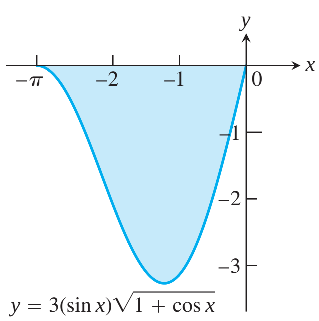

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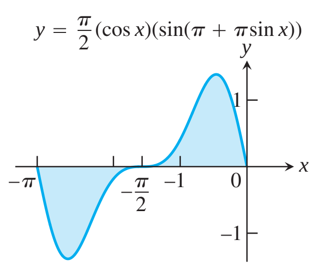

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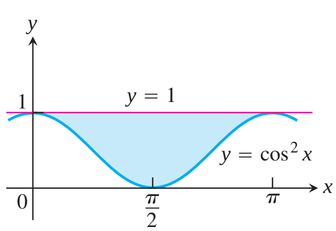

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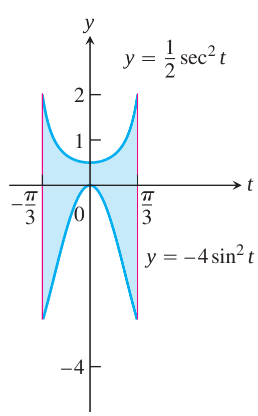

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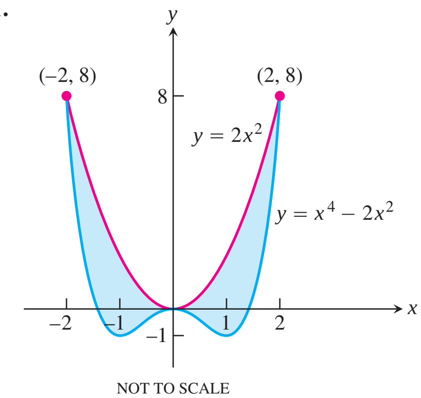

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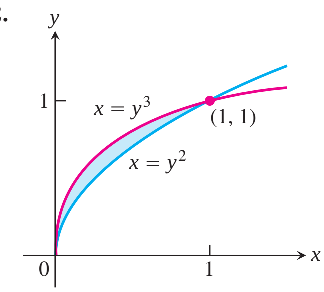

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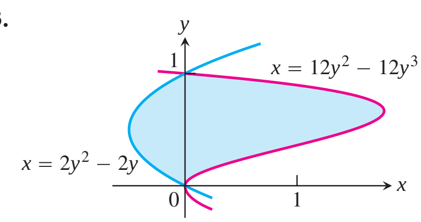

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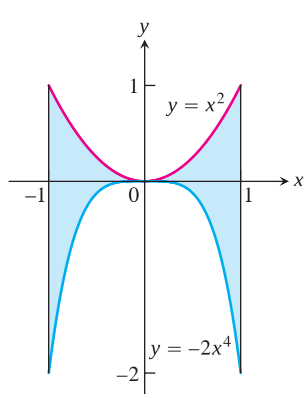

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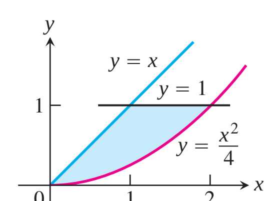

Exercises

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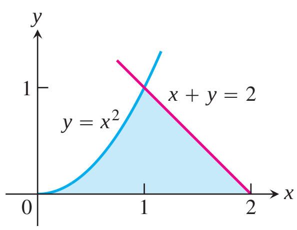

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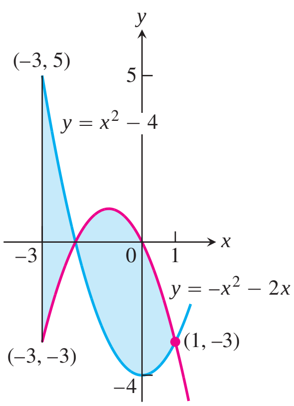

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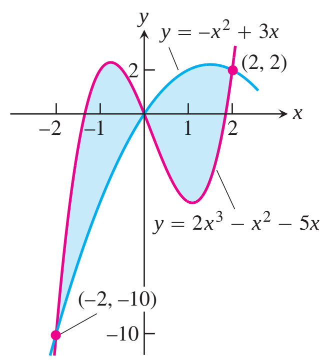

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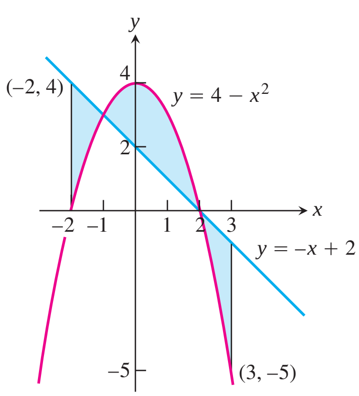

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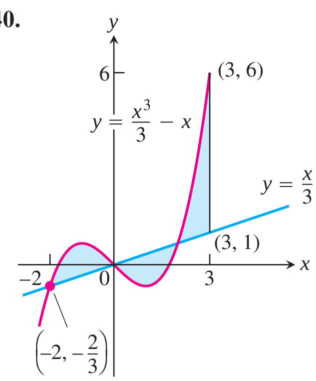


Find the areas of the regions enclosed by the lines and curves in Exercises 41–50.

Exercises

41. $y = x^2 - 2$ and $y = 2$

42. $y = 2x - x^2$ and $y = -3$

43. $y = x^4$ and $y = 8x$

44. $y = x^2 - 2x$ and $y = x$

45. $y = x^2$ and $y = -x^2 + 4x$

46. $y = 7 - 2x^2$ and $y = x^2 + 4$

47. $y = x^4 - 4x^2 + 4$ and $y = x^2$

48. $y = x\sqrt{a^2 - x^2}$, $a > 0$, and $y = 0$

Exercises

49. $y = \sqrt{|x|}$ and $5y = x + 6$ (How many intersection points are there?)

50. $y = |x^2 - 4|$ and $y = (x^2/2) + 4$

Find the areas of the regions enclosed by the lines and curves in Exercises 51–58.

Exercises

51. $x = 2y^2$, $x = 0$, and $y = 3$

52. $x = y^2$ and $x = y + 2$

53. $y^2 - 4x = 4$ and $4x - y = 16$

54. $x - y^2 = 0$ and $x + 2y^2 = 3$

55. $x + y^2 = 0$ and $x + 3y^2 = 2$

56. $x - y^{2/3} = 0$ and $x + y^4 = 2$

57. $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$

58. $x = y^3 - y^2$ and $x = 2y$

Find the areas of the regions enclosed by the curves in Exercises 59–62.

Exercises

59. $4x^2 + y = 4$ and $x^4 - y = 1$

60. $x^3 - y = 0$ and $3x^2 - y = 4$

61. $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

62. $x + y^2 = 3$ and $4x + y^2 = 0$

Find the areas of the regions enclosed by the lines and curves in Exercises 63–70.

Exercises

63. $y = 2 \sin x$ and $y = \sin 2x$, $0 \leq x \leq \pi$

64. $y = 8 \cos x$ and $y = \sec^2 x$, $-\pi/3 \leq x \leq \pi/3$

65. $y = \cos(\pi x/2)$ and $y = 1 - x^2$

66. $y = \sin(\pi x/2)$ and $y = x$

67. $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$

68. $x = \tan^2 y$ and $x = -\tan^2 y$, $-\pi/4 \leq y \leq \pi/4$

69. $x = 3 \sin y \sqrt{\cos y}$ and $x = 0$, $0 \leq y \leq \pi/2$

70. $y = \sec^2(\pi x/3)$ and $y = x^{1/3}$, $-1 \leq x \leq 1$

71. Find the area of the propeller-shaped region enclosed by the curve $x - y^3 = 0$ and the line $x - y = 0$.

72. Find the area of the propeller-shaped region enclosed by the curves $x - y^{1/3} = 0$ and $x - y^{1/5} = 0$.

Exercises

73. Find the area of the region in the first quadrant bounded by the line $y = x$, the line $x = 2$, the curve $y = 1/x^2$, and the x-axis.

74. Find the area of the "triangular" region in the first quadrant bounded on the left by the y-axis and on the right by the curves $y = \sin x$ and $y = \cos x$.

75. The region bounded below by the parabola $y = x^2$ and above by the line $y = 4$ is to be partitioned into two subsections of equal area by cutting across it with the horizontal line $y = c$.

a. Sketch the region and draw a line $y = c$ across it that looks about right. In terms of c, what are the coordinates of the points where the line and parabola intersect? Add them to your figure.

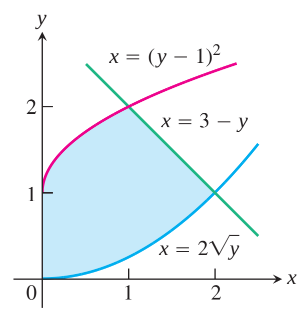
b. Find c by integrating with respect to y. (This puts c in the limits of integration.)

c. Find c by integrating with respect to x. (This puts c into the integrand as well.)

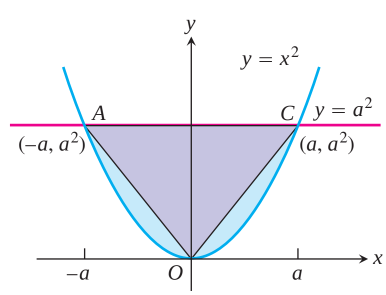
76. Find the area of the region between the curve $y = 3 - x^2$ and the line $y = -1$ by integrating with respect to **a.** x, **b.** y.

77. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

78. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.



79. The figure here shows triangle AOC inscribed in the region cut from the parabola $y = x^2$ by the line $y = a^2$. Find the limit of the ratio of the area of the triangle to the area of the parabolic region as a approaches zero.

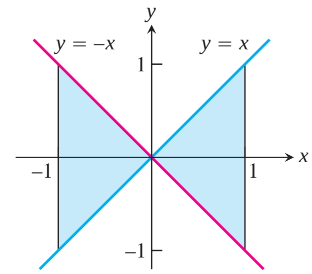


80. Suppose the area of the region between the graph of a positive continuous function f and the x-axis from $x = a$ to $x = b$ is 4 square units. Find the area between the curves $y = f(x)$ and $y = 2f(x)$ from $x = a$ to $x = b$.

81. Which of the following integrals, if either, calculates the area of the shaded region shown here? Give reasons for your answer.

a. $\displaystyle\int_{-1}^{1} (x - (-x))\, dx = \int_{-1}^{1} 2x\, dx$

b. $\displaystyle\int_{-1}^{1} (-x - (x))\, dx = \int_{-1}^{1} -2x\, dx$



82. True, sometimes true, or never true? The area of the region between the graphs of the continuous functions $y = f(x)$ and $y = g(x)$ and the vertical lines $x = a$ and $x = b$ $(a < b)$ is

$$\int_a^b [f(x) - g(x)]\, dx.$$

Give reasons for your answer.

Theory and Examples

83. Suppose that $F(x)$ is an antiderivative of $f(x) = (\sin x)/x,\ x > 0$. Express

$$\int_1^3 \frac{\sin 2x}{x}\, dx$$

in terms of F.

84. Show that if f is continuous, then

$$\int_0^1 f(x)\, dx = \int_0^1 f(1 - x)\, dx.$$

85. Suppose that

$$\int_0^1 f(x)\, dx = 3.$$

Find

$$\int_{-1}^0 f(x)\, dx$$

if **a.** f is odd, **b.** f is even.

86. **a.** Show that if f is odd on $[-a, a]$, then

$$\int_{-a}^a f(x)\, dx = 0.$$

b. Test the result in part (a) with $f(x) = \sin x$ and $a = \pi/2$.

87. If f is a continuous function, find the value of the integral

$$I = \int_0^a \frac{f(x)\, dx}{f(x) + f(a - x)}$$

by making the substitution $u = a - x$ and adding the resulting integral to I.

88. By using a substitution, prove that for all positive numbers x and y,

$$\int_x^{xy} \frac{1}{t}\,dt = \int_1^y \frac{1}{t}\,dt.$$

The Shift Property for Definite Integrals

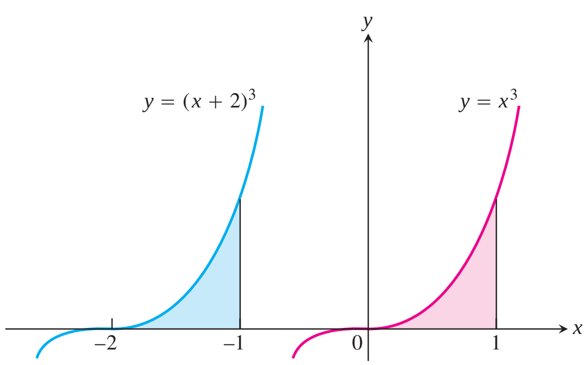
A basic property of definite integrals is their invariance under translation, as expressed by the equation.

$$\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx. \qquad (1)$$

The equation holds whenever f is integrable and defined for the necessary values of x. For example in the accompanying figure, show that

$$\int_{-2}^{-1} (x+2)^3\,dx = \int_0^1 x^3\,dx$$

because the areas of the shaded regions are congruent.



89. Use a substitution to verify Equation (1).

90. For each of the following functions, graph $f(x)$ over $[a, b]$ and $f(x+c)$ over $[a-c, b-c]$ to convince yourself that Equation (1) is reasonable.

a. $f(x) = x^2, \quad a = 0, \quad b = 1, \quad c = 1$

b. $f(x) = \sin x, \quad a = 0, \quad b = \pi, \quad c = \pi/2$

c. $f(x) = \sqrt{x-4}, \quad a = 4, \quad b = 8, \quad c = 5$

COMPUTER EXPLORATIONS

In Exercises 91–94, you will find the area between curves in the plane when you cannot find their points of intersection using simple algebra. Use a CAS to perform the following steps:

a. Plot the curves together to see what they look like and how many points of intersection they have.

b. Use the numerical equation solver in your CAS to find all the points of intersection.

c. Integrate $|f(x) - g(x)|$ over consecutive pairs of intersection values.

d. Sum together the integrals found in part (c).

91. $f(x) = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}, \quad g(x) = x - 1$

92. $f(x) = \frac{x^4}{2} - 3x^3 + 10, \quad g(x) = 8 - 12x$

93. $f(x) = x + \sin(2x), \quad g(x) = x^3$

94. $f(x) = x^2 \cos x, \quad g(x) = x^3 - x$

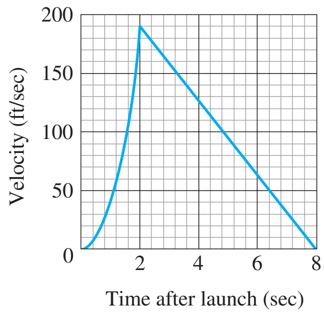
Chapter 5 Questions to Guide Your Review

1. How can you sometimes estimate quantities like distance traveled, area, and average value with finite sums? Why might you want to do so?
2. What is sigma notation? What advantage does it offer? Give examples.
3. What is a Riemann sum? Why might you want to consider such a sum?
4. What is the norm of a partition of a closed interval?
5. What is the definite integral of a function f over a closed interval $[a, b]$? When can you be sure it exists?
6. What is the relation between definite integrals and area? Describe some other interpretations of definite integrals.
7. What is the average value of an integrable function over a closed interval? Must the function assume its average value? Explain.
8. Describe the rules for working with definite integrals (Table 5.3). Give examples.
9. What is the Fundamental Theorem of Calculus? Why is it so important? Illustrate each part of the theorem with an example.
10. How does the Fundamental Theorem provide a solution to the initial value problem $dy/dx = f(x), y(x_0) = y_0$, when f is continuous?
11. How is integration by substitution related to the Chain Rule?
12. How can you sometimes evaluate indefinite integrals by substitution? Give examples.
13. How does the method of substitution work for definite integrals? Give examples.
14. How do you define and calculate the area of the region between the graphs of two continuous functions? Give an example.

Chapter 5 Practice Exercises

Finite Sums and Estimates

1. The accompanying figure shows the graph of the velocity (ft/sec) of a model rocket for the first 8 sec after launch. The rocket accelerated straight up for the first 2 sec and then coasted to reach its maximum height at $t = 8$ sec.

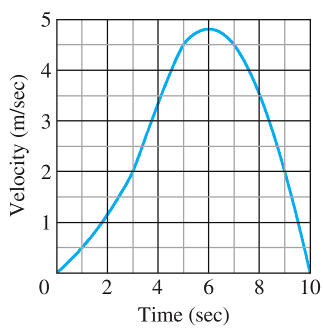


   a. Assuming that the rocket was launched from ground level, about how high did it go? (This is the rocket in Section 3.3, Exercise 17, but you do not need to do Exercise 17 to do the exercise here.)

   b. Sketch a graph of the rocket's height aboveground as a function of time for $0 \le t \le 8$.

2. a. The accompanying figure shows the velocity (m/sec) of a body moving along the s-axis during the time interval from $t = 0$ to $t = 10$ sec. About how far did the body travel during those 10 sec?

   b. Sketch a graph of s as a function of t for $0 \le t \le 10$ assuming $s(0) = 0$.



3. Suppose that $\sum_{k=1}^{10} a_k = -2$ and $\sum_{k=1}^{10} b_k = 25$. Find the value of

   a. $\sum_{k=1}^{10} \frac{a_k}{4}$

   b. $\sum_{k=1}^{10} (b_k - 3a_k)$

   c. $\sum_{k=1}^{10} (a_k + b_k - 1)$

   d. $\sum_{k=1}^{10} \left(\frac{5}{2} - b_k\right)$

4. Suppose that $\sum_{k=1}^{20} a_k = 0$ and $\sum_{k=1}^{20} b_k = 7$. Find the values of

   a. $\sum_{k=1}^{20} 3a_k$

   b. $\sum_{k=1}^{20} (a_k + b_k)$

   c. $\sum_{k=1}^{20} \left(\frac{1}{2} - \frac{2b_k}{7}\right)$

   d. $\sum_{k=1}^{20} (a_k - 2)$

Definite Integrals

In Exercises 5–8, express each limit as a definite integral. Then evaluate the integral to find the value of the limit. In each case, P is a partition of the given interval and the numbers c_k are chosen from the subintervals of P.

5. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (2c_k - 1)^{-1/2}\, \Delta x_k$, where P is a partition of $[1, 5]$

6. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} c_k(c_k^2 - 1)^{1/3}\, \Delta x_k$, where P is a partition of $[1, 3]$

7. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} \left(\cos\left(\frac{c_k}{2}\right)\right) \Delta x_k$, where P is a partition of $[-\pi, 0]$

8. $\lim_{\|P\| \to 0} \sum_{k=1}^{n} (\sin c_k)(\cos c_k)\, \Delta x_k$, where P is a partition of $[0, \pi/2]$

9. If $\int_{-2}^{2} 3f(x)\, dx = 12$, $\int_{-2}^{5} f(x)\, dx = 6$, and $\int_{-2}^{5} g(x)\, dx = 2$, find the values of the following.

   a. $\int_{-2}^{2} f(x)\, dx$

   b. $\int_{2}^{5} f(x)\, dx$

   c. $\int_{5}^{-2} g(x)\, dx$

   d. $\int_{-2}^{5} (-\pi g(x))\, dx$

   e. $\int_{-2}^{5} \left(\frac{f(x) + g(x)}{5}\right) dx$

10. If $\int_{0}^{2} f(x)\, dx = \pi$, $\int_{0}^{2} 7g(x)\, dx = 7$, and $\int_{0}^{1} g(x)\, dx = 2$, find the values of the following.

   a. $\int_{0}^{2} g(x)\, dx$

   b. $\int_{1}^{2} g(x)\, dx$

   c. $\int_{2}^{0} f(x)\, dx$

   d. $\int_{0}^{2} \sqrt{2}\, f(x)\, dx$

   e. $\int_{0}^{2} (g(x) - 3f(x))\, dx$

Area

In Exercise 11–14, find the total area of the region between the graph of f and the x-axis.

11. $f(x) = x^2 - 4x + 3, \quad 0 \le x \le 3$

12. $f(x) = 1 - (x^2/4), \quad -2 \le x \le 3$

13. $f(x) = 5 - 5x^{2/3}, \quad -1 \le x \le 8$

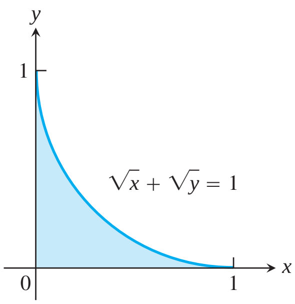
14. $f(x) = 1 - \sqrt{x}, \quad 0 \le x \le 4$

Find the areas of the regions enclosed by the curves and lines in Exercises 15–26.

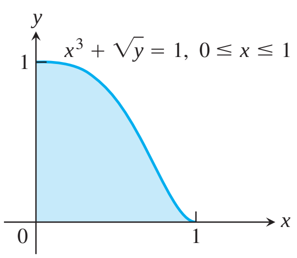
15. $y = x, \quad y = 1/x^2, \quad x = 2$

16. $y = x, \quad y = 1/\sqrt{x}, \quad x = 2$

17. $\sqrt{x} + \sqrt{y} = 1, \quad x = 0, \quad y = 0$



18. $x^3 + \sqrt{y} = 1, \quad x = 0, \quad y = 0, \quad \text{for} \quad 0 \le x \le 1$



19. $x = 2y^2, \quad x = 0, \quad y = 3$

20. $x = 4 - y^2, \quad x = 0$

21. $y^2 = 4x, \quad y = 4x - 2$

22. $y^2 = 4x + 4, \quad y = 4x - 16$

23. $y = \sin x, \quad y = x, \quad 0 \le x \le \pi/4$

24. $y = |\sin x|, \quad y = 1, \quad -\pi/2 \le x \le \pi/2$

25. $y = 2\sin x, \quad y = \sin 2x, \quad 0 \le x \le \pi$

26. $y = 8\cos x, \quad y = \sec^2 x, \quad -\pi/3 \le x \le \pi/3$

27. Find the area of the "triangular" region bounded on the left by $x + y = 2$, on the right by $y = x^2$, and above by $y = 2$.

28. Find the area of the "triangular" region bounded on the left by $y = \sqrt{x}$, on the right by $y = 6 - x$, and below by $y = 1$.

29. Find the extreme values of $f(x) = x^3 - 3x^2$ and find the area of the region enclosed by the graph of f and the x-axis.

30. Find the area of the region cut from the first quadrant by the curve $x^{1/2} + y^{1/2} = a^{1/2}$.

31. Find the total area of the region enclosed by the curve $x = y^{2/3}$ and the lines $x = y$ and $y = -1$.

32. Find the total area of the region between the curves $y = \sin x$ and $y = \cos x$ for $0 \le x \le 3\pi/2$.

Initial Value Problems

33. Show that $y = x^2 + \int_1^x \frac{1}{t}\,dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = 2 - \frac{1}{x^2}; \quad y'(1) = 3, \quad y(1) = 1.$$

34. Show that $y = \int_0^x \left(1 + 2\sqrt{\sec t}\right) dt$ solves the initial value problem

$$\frac{d^2y}{dx^2} = \sqrt{\sec x}\tan x; \quad y'(0) = 3, \quad y(0) = 0.$$

Express the solutions of the initial value problems in Exercises 35 and 36 in terms of integrals.

35. $\dfrac{dy}{dx} = \dfrac{\sin x}{x}, \quad y(5) = -3$

36. $\dfrac{dy}{dx} = \sqrt{2 - \sin^2 x}, \quad y(-1) = 2$

Evaluating Indefinite Integrals

Evaluate the integrals in Exercises 37–44.

37. $\int 2(\cos x)^{-1/2}\sin x\,dx$

38. $\int (\tan x)^{-3/2}\sec^2 x\,dx$

39. $\int (2\theta + 1 + 2\cos(2\theta + 1))\,d\theta$

40. $\int \left(\dfrac{1}{\sqrt{2\theta - \pi}} + 2\sec^2(2\theta - \pi)\right) d\theta$

41. $\int \left(t - \dfrac{2}{t}\right)\left(t + \dfrac{2}{t}\right) dt$

42. $\int \dfrac{(t + 1)^2 - 1}{t^4}\,dt$

43. $\int \sqrt{t}\sin(2t^{3/2})\,dt$

44. $\int \sec\theta\tan\theta\sqrt{1 + \sec\theta}\,d\theta$

Evaluating Definite Integrals

Evaluate the integrals in Exercises 45–70.

45. $\int_{-1}^{1} (3x^2 - 4x + 7)\,dx$

46. $\int_0^1 (8s^3 - 12s^2 + 5)\,ds$

47. $\int_1^2 \dfrac{4}{v^2}\,dv$

48. $\int_1^{27} x^{-4/3}\,dx$

49. $\int_1^4 \dfrac{dt}{t\sqrt{t}}$

50. $\int_1^4 \dfrac{\left(1 + \sqrt{u}\right)^{1/2}}{\sqrt{u}}\,du$

51. $\int_0^1 \dfrac{36\,dx}{(2x + 1)^3}$

52. $\int_0^1 \dfrac{dr}{\sqrt[3]{(7 - 5r)^2}}$

53. $\int_{1/8}^{1} x^{-1/3}(1 - x^{2/3})^{3/2}\,dx$

54. $\int_0^{1/2} x^3(1 + 9x^4)^{-3/2}\,dx$

55. $\int_0^{\pi} \sin^2 5r\,dr$

56. $\int_0^{\pi/4} \cos^2\left(4t - \dfrac{\pi}{4}\right) dt$

57. $\displaystyle\int_0^{\pi/3} \sec^2\theta\, d\theta$

58. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc^2 x\, dx$

59. $\displaystyle\int_{\pi}^{3\pi} \cot^2 \frac{x}{6}\, dx$

60. $\displaystyle\int_0^{\pi} \tan^2 \frac{\theta}{3}\, d\theta$

61. $\displaystyle\int_{-\pi/3}^{0} \sec x \tan x\, dx$

62. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc z \cot z\, dz$

63. $\displaystyle\int_0^{\pi/2} 5(\sin x)^{3/2} \cos x\, dx$

64. $\displaystyle\int_{-1}^{1} 2x \sin(1 - x^2)\, dx$

65. $\displaystyle\int_{-\pi/2}^{\pi/2} 15 \sin^4 3x \cos 3x\, dx$

66. $\displaystyle\int_0^{2\pi/3} \cos^{-4}\left(\frac{x}{2}\right) \sin\left(\frac{x}{2}\right) dx$

67. $\displaystyle\int_0^{\pi/2} \frac{3 \sin x \cos x}{\sqrt{1 + 3\sin^2 x}}\, dx$

68. $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{(1 + 7\tan x)^{2/3}}\, dx$

69. $\displaystyle\int_0^{\pi/3} \frac{\tan\theta}{\sqrt{2\sec\theta}}\, d\theta$

70. $\displaystyle\int_{\pi^2/36}^{\pi^2/4} \frac{\cos\sqrt{t}}{\sqrt{t}\sin\sqrt{t}}\, dt$

Average Values

71. Find the average value of $f(x) = mx + b$
   a. over $[-1, 1]$
   b. over $[-k, k]$

72. Find the average value of
   a. $y = \sqrt{3x}$ over $[0, 3]$
   b. $y = \sqrt{ax}$ over $[0, a]$

73. Let f be a function that is differentiable on $[a, b]$. In Chapter 2 we defined the average rate of change of f over $[a, b]$ to be

$$\frac{f(b) - f(a)}{b - a}$$

and the instantaneous rate of change of f at x to be $f'(x)$. In this chapter we defined the average value of a function. For the new definition of average to be consistent with the old one, we should have

$$\frac{f(b) - f(a)}{b - a} = \text{average value of } f' \text{ on } [a, b].$$

Is this the case? Give reasons for your answer.

74. Is it true that the average value of an integrable function over an interval of length 2 is half the function's integral over the interval? Give reasons for your answer.

T 75. Compute the average value of the temperature function

$$f(x) = 37 \sin\left(\frac{2\pi}{365}(x - 101)\right) + 25$$

for a 365-day year. This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is 25.7°F, which is slightly higher than the average value of $f(x)$. Figure 3.33 shows why.

T 76. **Specific heat of a gas** Specific heat C_v is the amount of heat required to raise the temperature of a given mass of gas with constant volume by 1°C, measured in units of cal/deg-mole (calories per degree gram molecule). The specific heat of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Find the average value of C_v for $20° \le T \le 675°\text{C}$ and the temperature at which it is attained.

Differentiating Integrals

In Exercises 77–80, find dy/dx.

77. $\displaystyle y = \int_2^x \sqrt{2 + \cos^3 t}\, dt$

78. $\displaystyle y = \int_2^{7x^2} \sqrt{2 + \cos^3 t}\, dt$

79. $\displaystyle y = \int_x^1 \frac{6}{3 + t^4}\, dt$

80. $\displaystyle y = \int_{\sec x}^{2} \frac{1}{t^2 + 1}\, dt$

Theory and Examples

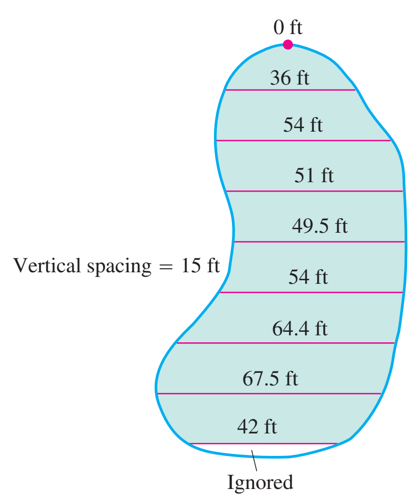
81. Is it true that every function $y = f(x)$ that is differentiable on $[a, b]$ is itself the derivative of some function on $[a, b]$? Give reasons for your answer.

82. Suppose that $F(x)$ is an antiderivative of $f(x) = \sqrt{1 + x^4}$. Express $\int_0^1 \sqrt{1 + x^4}\, dx$ in terms of F and give a reason for your answer.

83. Find dy/dx if $y = \int_x^1 \sqrt{1 + t^2}\, dt$. Explain the main steps in your calculation.

84. Find dy/dx if $y = \int_{\cos x}^0 (1/(1 - t^2))\, dt$. Explain the main steps in your calculation.

85. **A new parking lot** To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for $10,000. The cost to clear the land will be $0.10 a square foot, and the lot will cost $2.00 a square foot to pave. Can the job be done for $10,000? Use a lower sum estimate to see. (Answers may vary slightly, depending on the estimate used.)



86. Skydivers A and B are in a helicopter hovering at 6400 ft. Skydiver A jumps and descends for 4 sec before opening her parachute. The helicopter then climbs to 7000 ft and hovers there. Forty-five seconds after A leaves the aircraft, B jumps and descends for 13 sec before opening his parachute. Both skydivers descend at 16 ft/sec with parachutes open. Assume that the skydivers fall freely (no effective air resistance) before their parachutes open.

a. At what altitude does A's parachute open?

b. At what altitude does B's parachute open?

c. Which skydiver lands first?

Average Daily Inventory

Average value is used in economics to study such things as average daily inventory. If $I(t)$ is the number of radios, tires, shoes, or whatever product a firm has on hand on day t (we call I an **inventory function**), the average value of I over a time period $[0, T]$ is called the firm's average daily inventory for the period.

$$\textbf{Average daily inventory} = \text{av}(I) = \frac{1}{T}\int_0^T I(t)\,dt.$$

If h is the dollar cost of holding one item per day, the product $\text{av}(I) \cdot h$ is the **average daily holding cost** for the period.

87. As a wholesaler, Tracey Burr Distributors receives a shipment of 1200 cases of chocolate bars every 30 days. TBD sells the chocolate to retailers at a steady rate, and t days after a shipment arrives, its inventory of cases on hand is $I(t) = 1200 - 40t$, $0 \le t \le 30$. What is TBD's average daily inventory for the 30-day period? What is its average daily holding cost if the cost of holding one case is 3¢ a day?

88. Rich Wholesale Foods, a manufacturer of cookies, stores its cases of cookies in an air-conditioned warehouse for shipment every 14 days. Rich tries to keep 600 cases on reserve to meet occasional peaks in demand, so a typical 14-day inventory function is $I(t) = 600 + 600t$, $0 \le t \le 14$. The daily holding cost for each case is 4¢ per day. Find Rich's average daily inventory and average daily holding cost.

89. Solon Container receives 450 drums of plastic pellets every 30 days. The inventory function (drums on hand as a function of days) is $I(t) = 450 - t^2/2$. Find the average daily inventory. If the holding cost for one drum is 2¢ per day, find the average daily holding cost.

90. Mitchell Mailorder receives a shipment of 600 cases of athletic socks every 60 days. The number of cases on hand t days after the shipment arrives is $I(t) = 600 - 20\sqrt{15t}$. Find the average daily inventory. If the holding cost for one case is $1/2$¢ per day, find the average daily holding cost.

Chapter 5 Additional and Advanced Exercises

Theory and Examples

1. a. If $\int_0^1 7f(x)\,dx = 7$, does $\int_0^1 f(x)\,dx = 1$?

b. If $\int_0^1 f(x)\,dx = 4$ and $f(x) \geq 0$, does

$$\int_0^1 \sqrt{f(x)}\,dx = \sqrt{4} = 2?$$

Give reasons for your answers.

2. Suppose $\int_{-2}^2 f(x)\,dx = 4$, $\int_2^5 f(x)\,dx = 3$, $\int_{-2}^5 g(x)\,dx = 2$.

Which, if any, of the following statements are true?

a. $\int_5^2 f(x)\,dx = -3$ **b.** $\int_{-2}^5 (f(x) + g(x)) = 9$

c. $f(x) \leq g(x)$ on the interval $-2 \leq x \leq 5$

3. Initial value problem Show that

$$y = \frac{1}{a}\int_0^x f(t)\sin a(x - t)\,dt$$

solves the initial value problem

$$\frac{d^2y}{dx^2} + a^2y = f(x), \qquad \frac{dy}{dx} = 0 \quad \text{and} \quad y = 0 \text{ when } x = 0.$$

(*Hint:* $\sin(ax - at) = \sin ax \cos at - \cos ax \sin at$.)

4. Proportionality Suppose that x and y are related by the equation

$$x = \int_0^y \frac{1}{\sqrt{1 + 4t^2}}\,dt.$$

Show that d^2y/dx^2 is proportional to y and find the constant of proportionality.

5. Find $f(4)$ if

a. $\int_0^{x^2} f(t)\,dt = x\cos\pi x$ **b.** $\int_0^{f(x)} t^2\,dt = x\cos\pi x$.

6. Find $f(\pi/2)$ from the following information.

i. f is positive and continuous.

ii. The area under the curve $y = f(x)$ from $x = 0$ to $x = a$ is

$$\frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a.$$

7. The area of the region in the xy-plane enclosed by the x-axis, the curve $y = f(x)$, $f(x) \geq 0$, and the lines $x = 1$ and $x = b$ is equal to $\sqrt{b^2 + 1} - \sqrt{2}$ for all $b > 1$. Find $f(x)$.

8. Prove that

$$\int_0^x \left(\int_0^u f(t)\, dt \right) du = \int_0^x f(u)(x - u)\, du.$$

(*Hint:* Express the integral on the right-hand side as the difference of two integrals. Then show that both sides of the equation have the same derivative with respect to x.)

9. **Finding a curve** Find the equation for the curve in the xy-plane that passes through the point $(1, -1)$ if its slope at x is always $3x^2 + 2$.

10. **Shoveling dirt** You sling a shovelful of dirt up from the bottom of a hole with an initial velocity of 32 ft/sec. The dirt must rise 17 ft above the release point to clear the edge of the hole. Is that enough speed to get the dirt out, or had you better duck?

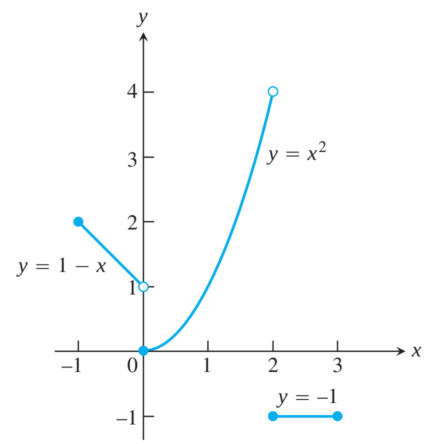


FIGURE 5.34 Piecewise continuous functions like this are integrated piece by piece.

Piecewise Continuous Functions

Although we are mainly interested in continuous functions, many functions in applications are piecewise continuous. A function $f(x)$ is **piecewise continuous on a closed interval *I*** if f has only finitely many discontinuities in I, the limits

$$\lim_{x \to c^-} f(x) \text{ and } \lim_{x \to c^+} f(x)$$

exist and are finite at every interior point of I, and the appropriate one-sided limits exist and are finite at the endpoints of I. All piecewise continuous functions are integrable. The points of discontinuity subdivide I into open and half-open subintervals on which f is continuous, and the limit criteria above guarantee that f has a continuous extension to the closure of each subinterval. To integrate a piecewise continuous function, we integrate the individual extensions and add the results. The integral of

$$f(x) = \begin{cases} 1 - x, & -1 \leq x < 0 \\ x^2, & 0 \leq x < 2 \\ -1, & 2 \leq x \leq 3 \end{cases}$$

(Figure 5.34) over $[-1, 3]$ is

$$\begin{aligned} \int_{-1}^{3} f(x)\, dx &= \int_{-1}^{0} (1 - x)\, dx + \int_{0}^{2} x^2\, dx + \int_{2}^{3} (-1)\, dx \\ &= \left[x - \frac{x^2}{2} \right]_{-1}^{0} + \left[\frac{x^3}{3} \right]_{0}^{2} + \Big[-x \Big]_{2}^{3} \\ &= \frac{3}{2} + \frac{8}{3} - 1 = \frac{19}{6}. \end{aligned}$$

The Fundamental Theorem applies to piecewise continuous functions with the restriction that $(d/dx)\int_a^x f(t)\, dt$ is expected to equal $f(x)$ only at values of x at which f is continuous. There is a similar restriction on Leibniz's Rule below.

Graph the functions in Exercises 11–16 and integrate them over their domains.

11. $f(x) = \begin{cases} x^{2/3}, & -8 \leq x < 0 \\ -4, & 0 \leq x \leq 3 \end{cases}$

12. $f(x) = \begin{cases} \sqrt{-x}, & -4 \leq x < 0 \\ x^2 - 4, & 0 \leq x \leq 3 \end{cases}$

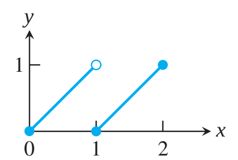
13. $g(t) = \begin{cases} t, & 0 \leq t < 1 \\ \sin \pi t, & 1 \leq t \leq 2 \end{cases}$

14. $h(z) = \begin{cases} \sqrt{1 - z}, & 0 \leq z < 1 \\ (7z - 6)^{-1/3}, & 1 \leq z \leq 2 \end{cases}$

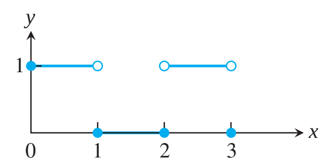
15. $f(x) = \begin{cases} 1, & -2 \leq x < -1 \\ 1 - x^2, & -1 \leq x < 1 \\ 2, & 1 \leq x \leq 2 \end{cases}$

16. $h(r) = \begin{cases} r, & -1 \leq r < 0 \\ 1 - r^2, & 0 \leq r < 1 \\ 1, & 1 \leq r \leq 2 \end{cases}$

17. Find the average value of the function graphed in the accompanying figure.



18. Find the average value of the function graphed in the accompanying figure.



Leibniz's Rule

In applications, we sometimes encounter functions like

$$f(x) = \int_{\sin x}^{x^2} (1 + t)\, dt \qquad \text{and} \qquad g(x) = \int_{\sqrt{x}}^{2\sqrt{x}} \sin t^2\, dt,$$

defined by integrals that have variable upper limits of integration and variable lower limits of integration at the same time. The first integral can be evaluated directly, but the second cannot. We may find the derivative of either integral, however, by a formula called **Leibniz's Rule**.

Leibniz's Rule

If f is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\, dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

Figure 5.35 gives a geometric interpretation of Leibniz's Rule. It shows a carpet of variable width $f(t)$ that is being rolled up at the left at the same time x as it is being unrolled at the right. (In this interpretation, time is x, not t.) At time x, the floor is covered from $u(x)$ to $v(x)$. The rate du/dx at which the carpet is being rolled up need not be the same as the rate dv/dx at which the carpet is being laid down. At any given time x, the area covered by carpet is

$$A(x) = \int_{u(x)}^{v(x)} f(t)\, dt.$$

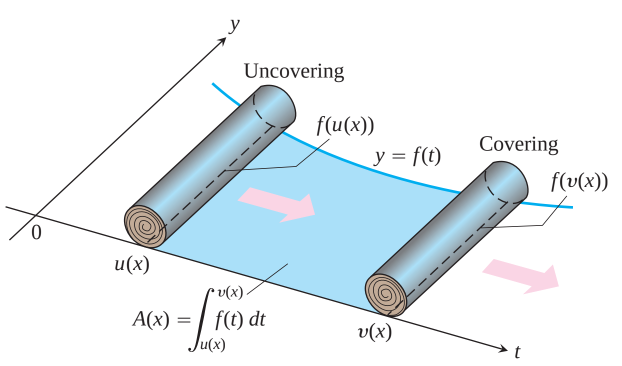


FIGURE 5.35 Rolling and unrolling a carpet: a geometric interpretation of Leibniz's Rule:

$$\frac{dA}{dx} = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

At what rate is the covered area changing? At the instant x, $A(x)$ is increasing by the width $f(v(x))$ of the unrolling carpet times the rate dv/dx at which the carpet is being unrolled. That is, $A(x)$ is being increased at the rate

$$f(v(x))\frac{dv}{dx}.$$

At the same time, A is being decreased at the rate

$$f(u(x))\frac{du}{dx},$$

the width at the end that is being rolled up times the rate du/dx. The net rate of change in A is

$$\frac{dA}{dx} = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx},$$

which is precisely Leibniz's Rule.

To prove the rule, let F be an antiderivative of f on $[a, b]$. Then

$$\int_{u(x)}^{v(x)} f(t)\, dt = F(v(x)) - F(u(x)).$$

Differentiating both sides of this equation with respect to x gives the equation we want:

$$\begin{aligned}\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\, dt &= \frac{d}{dx}\Big[F(v(x)) - F(u(x))\Big] \\ &= F'(v(x))\frac{dv}{dx} - F'(u(x))\frac{du}{dx} \qquad \text{Chain Rule} \\ &= f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.\end{aligned}$$

Use Leibniz's Rule to find the derivatives of the functions in Exercises 19–21.

19. $f(x) = \int_{1/x}^{x} \frac{1}{t}\, dt$

20. $f(x) = \int_{\cos x}^{\sin x} \frac{1}{1 - t^2}\, dt$

21. $g(y) = \int_{\sqrt{y}}^{2\sqrt{y}} \sin t^2\, dt$

22. Use Leibniz's Rule to find the value of x that maximizes the value of the integral

$$\int_{x}^{x+3} t(5 - t)\, dt.$$

Problems like this arise in the mathematical theory of political elections. See "The Entry Problem in a Political Race," by Steven J. Brams and Philip D. Straffin, Jr., in *Political Equilibrium*, Peter Ordeshook and Kenneth Shepfle, Editors, Kluwer-Nijhoff, Boston, 1982, pp. 181–195.

Approximating Finite Sums with Integrals

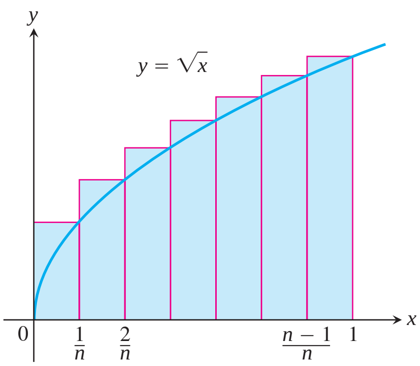
In many applications of calculus, integrals are used to approximate finite sums—the reverse of the usual procedure of using finite sums to approximate integrals.

For example, let's estimate the sum of the square roots of the first n positive integers, $\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$. The integral

$$\int_0^1 \sqrt{x}\, dx = \frac{2}{3}x^{3/2}\Big]_0^1 = \frac{2}{3}$$

is the limit of the upper sums

$$S_n = \sqrt{\frac{1}{n}} \cdot \frac{1}{n} + \sqrt{\frac{2}{n}} \cdot \frac{1}{n} + \cdots + \sqrt{\frac{n}{n}} \cdot \frac{1}{n}$$
$$= \frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n^{3/2}}.$$



Therefore, when n is large, S_n will be close to $2/3$ and we will have

$$\text{Root sum} = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} = S_n \cdot n^{3/2} \approx \frac{2}{3}n^{3/2}.$$

The following table shows how good the approximation can be.

n	Root sum	$(2/3)n^{3/2}$	Relative error
10	22.468	21.082	$1.386/22.468 \approx 6\%$
50	239.04	235.70	1.4%
100	671.46	666.67	0.7%
1000	21,097	21,082	0.07%

23. Evaluate

$$\lim_{n\to\infty} \frac{1^5 + 2^5 + 3^5 + \cdots + n^5}{n^6}$$

by showing that the limit is

$$\int_0^1 x^5\, dx$$

and evaluating the integral.

24. See Exercise 23. Evaluate

$$\lim_{n\to\infty} \frac{1}{n^4}(1^3 + 2^3 + 3^3 + \cdots + n^3).$$

25. Let $f(x)$ be a continuous function. Express

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right]$$

as a definite integral.

26. Use the result of Exercise 25 to evaluate

a. $\displaystyle\lim_{n\to\infty} \frac{1}{n^2}(2 + 4 + 6 + \cdots + 2n)$,

b. $\displaystyle\lim_{n\to\infty} \frac{1}{n^{16}}(1^{15} + 2^{15} + 3^{15} + \cdots + n^{15})$,

c. $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{n\pi}{n}\right)$.

What can be said about the following limits?

d. $\displaystyle\lim_{n\to\infty} \frac{1}{n^{17}}(1^{15} + 2^{15} + 3^{15} + \cdots + n^{15})$

e. $\displaystyle\lim_{n\to\infty} \frac{1}{n^{15}}(1^{15} + 2^{15} + 3^{15} + \cdots + n^{15})$

27. a. Show that the area A_n of an n-sided regular polygon in a circle of radius r is

$$A_n = \frac{nr^2}{2}\sin\frac{2\pi}{n}.$$

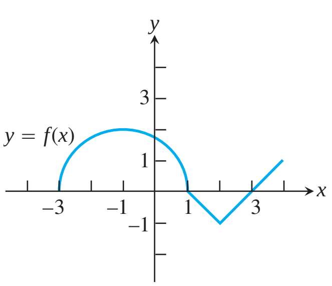
b. Find the limit of A_n as $n \to \infty$. Is this answer consistent with what you know about the area of a circle?

28. A differential equation Show that $y = \sin x + \int_x^{\pi} \cos 2t\, dt + 1$ satisfies both of the following conditions:

i. $y'' = -\sin x + 2\sin 2x$

ii. $y = 1$ and $y' = -2$ when $x = \pi$.

29. A function defined by an integral The graph of a function f consists of a semicircle and two line segments as shown. Let $g(x) = \int_1^x f(t)\, dt$.



a. Find $g(1)$. **b.** Find $g(3)$. **c.** Find $g(-1)$.

d. Find all values of x on the open interval $(-3, 4)$ at which g has a relative maximum.

e. Write an equation for the line tangent to the graph of g at $x = -1$.

f. Find the x-coordinate of each point of inflection of the graph of g on the open interval $(-3, 4)$.

g. Find the range of g.