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Elementary Analysis

The Theory of Calculus

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Contents

9.15 Show $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

9.16 Use Theorems 9.9 and 9.10 or Exercises 9.9–9.15 to prove the following:

(a) $\lim \frac{n^4+8n}{n^2+9} = +\infty$

(b) $\lim[\frac{2^n}{n^2} + (-1)^n] = +\infty$

(c) $\lim[\frac{3^n}{n^3} - \frac{3^n}{n!}] = +\infty$

9.17 Give a formal proof that $\lim n^2 = +\infty$ using only Definition 9.8.

9.18 **(a)** Verify $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ for $a \neq 1$.

(b) Find $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.

(c) Calculate $\lim_{n\to\infty}(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n})$.

(d) What is $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $a \geq 1$?

§10 Monotone Sequences and Cauchy Sequences

In this section we obtain two theorems [Theorems 10.2 and 10.11] that will allow us to conclude certain sequences converge *without* knowing the limit in advance. These theorems are important because in practice the limits are not usually known in advance.

10.1 Definition.
A sequence (s_n) of real numbers is called an *increasing sequence* if $s_n \leq s_{n+1}$ for all n, and (s_n) is called a *decreasing sequence* if $s_n \geq s_{n+1}$ for all n. Note that if (s_n) is increasing, then $s_n \leq s_m$ whenever $n < m$. A sequence that is increasing or decreasing[4] will be called a *monotone sequence* or a *monotonic sequence*.

Example 1
The sequences defined by $a_n = 1 - \frac{1}{n}$, $b_n = n^3$ and $c_n = (1+\frac{1}{n})^n$ are increasing sequences, although this is not obvious for the

[4]In the First Edition of this book, increasing and decreasing sequences were referred to as "nondecreasing" and "nonincreasing" sequences, respectively.

sequence (c_n). The sequence $d_n = \frac{1}{n^2}$ is decreasing. The sequences $s_n = (-1)^n$, $t_n = \cos(\frac{n\pi}{3})$, $u_n = (-1)^n n$ and $v_n = \frac{(-1)^n}{n}$ are not monotonic sequences. Also $x_n = n^{1/n}$ is not monotonic, as can be seen by examining the first four values; see Example 1(d) on page 33 in §7.

Of the sequences above, (a_n), (c_n), (d_n), (s_n), (t_n), (v_n) and (x_n) are bounded sequences. The remaining sequences, (b_n) and (u_n), are unbounded sequences. □

10.2 Theorem.
All bounded monotone sequences converge.

Proof
Let (s_n) be a bounded increasing sequence. Let S denote the set $\{s_n : n \in \mathbb{N}\}$, and let $u = \sup S$. Since S is bounded, u represents a real number. We show $\lim s_n = u$. Let $\epsilon > 0$. Since $u - \epsilon$ is not an upper bound for S, there exists N such that $s_N > u - \epsilon$. Since (s_n) is increasing, we have $s_N \leq s_n$ for all $n \geq N$. Of course, $s_n \leq u$ for all n, so $n > N$ implies $u - \epsilon < s_n \leq u$, which implies $|s_n - u| < \epsilon$. This shows $\lim s_n = u$.

The proof for bounded decreasing sequences is left to Exercise 10.2. ■

Note the Completeness Axiom 4.4 is a vital ingredient in the proof of Theorem 10.2.

Example 2
Consider the sequence (s_n) defined *recursively* by

$$s_1 = 5 \quad \text{and} \quad s_n = \frac{s_{n-1}^2 + 5}{2s_{n-1}} \quad \text{for} \quad n \geq 2. \tag{1}$$

Thus $s_2 = 3$ and $s_3 = \frac{7}{3} \approx 2.333$. First, note a simple induction argument shows $s_n > 0$ for all n. We will show $\lim_n s_n$ exists by showing the sequence is decreasing and bounded; see Theorem 10.2. In fact, we will prove the following by induction:

$$\sqrt{5} < s_{n+1} < s_n \leq 5 \quad \text{for} \quad n \geq 1. \tag{2}$$

Since $\sqrt{5} \approx 2.236$, our computations show (2) holds for $n \leq 2$. For the induction step, assume (2) holds for some $n \geq 2$. To show $s_{n+2} < s_{n+1}$, we need

$$\frac{s_{n+1}^2 + 5}{2s_{n+1}} < s_{n+1} \quad \text{or} \quad s_{n+1}^2 + 5 < 2s_{n+1}^2 \quad \text{or} \quad 5 < s_{n+1}^2,$$

but this holds because $s_{n+1} > \sqrt{5}$ by the assumption (2) for n. To show $s_{n+2} > \sqrt{5}$, we need

$$\frac{s_{n+1}^2 + 5}{2s_{n+1}} > \sqrt{5} \quad \text{or} \quad s_{n+1}^2 + 5 > 2\sqrt{5}s_{n+1}$$

or $s_{n+1}^2 - 2\sqrt{5}s_{n+1} + 5 > 0$, which is true because $s_{n+1}^2 - 2\sqrt{5}s_{n+1} + 5 = (s_{n+1} - \sqrt{5})^2 > 0$. Thus (2) holds for $n+1$ whenever (2) holds for n. Hence (2) holds for all n by induction. Thus $s = \lim_n s_n$ exists.

If one looks at $s_4 = \frac{47}{21} \approx 2.238095$ and compares with $\sqrt{5} \approx 2.236068$, one might suspect $s = \sqrt{5}$. To verify this, we apply the limit Theorems 9.2–9.4 and the fact $s = \lim_n s_{n+1}$ to the equation $2 \cdot s_{n+1}s_n = s_n^2 + 5$ to obtain $2s^2 = s^2 + 5$. Thus $s^2 = 5$ and $s = \sqrt{5}$, since the limit is certainly not $-\sqrt{5}$. □

10.3 Discussion of Decimals.

We have not given much attention to the notion that real numbers are simply decimal expansions. This notion is substantially correct, but there are subtleties to be faced. For example, different decimal expansions can represent the same real number. The somewhat more abstract developments of the set $\mathbb{R}$ of real numbers discussed in §6 turn out to be more satisfactory.

We restrict our attention to nonnegative decimal expansions and nonnegative real numbers. From our point of view, every nonnegative decimal expansion is shorthand for the limit of a bounded increasing sequence of real numbers. Suppose we are given a decimal expansion $K.d_1d_2d_3d_4\cdots$, where K is a nonnegative integer and each d_j belongs to $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let

$$s_n = K + \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}. \tag{1}$$

Then (s_n) is an increasing sequence of real numbers, and (s_n) is bounded [by $K + 1$, in fact]. So by Theorem 10.2, (s_n) converges to a real number we traditionally write as $K.d_1d_2d_3d_4\cdots$. For example, $3.3333\cdots$ represents

$$\lim_{n\to\infty}\left(3+\frac{3}{10}+\frac{3}{10^2}+\cdots+\frac{3}{10^n}\right).$$

To calculate this limit, we borrow the following fact about geometric series from Example 1 on page 96 in §14:

$$\lim_{n\to\infty} a(1+r+r^2+\cdots+r^n)=\frac{a}{1-r}\quad\text{for}\quad |r|<1; \tag{2}$$

see also Exercise 9.18. In our case, $a=3$ and $r=\frac{1}{10}$, so $3.3333\cdots$ represents $\frac{3}{1-\frac{1}{10}}=\frac{10}{3}$, as expected. Similarly, $0.9999\cdots$ represents

$$\lim_{n\to\infty}\left(\frac{9}{10}+\frac{9}{10^2}+\cdots+\frac{9}{10^n}\right)=\frac{\frac{9}{10}}{1-\frac{1}{10}}=1.$$

Thus $0.9999\cdots$ and $1.0000\cdots$ are different decimal expansions that represent the same real number!

The converse of the preceding discussion also holds. That is, every nonnegative real number x has at least one decimal expansion. This will be proved, along with some related results, in §16. □

Unbounded monotone sequences also have limits.

10.4 Theorem.

(i) *If (s_n) is an unbounded increasing sequence, then $\lim s_n = +\infty$.*

(ii) *If (s_n) is an unbounded decreasing sequence, then $\lim s_n = -\infty$.*

Proof

(i) Let (s_n) be an unbounded increasing sequence. Let $M>0$. Since the set $\{s_n : n\in\mathbb{N}\}$ is unbounded and it is bounded below by s_1, it must be unbounded above. Hence for some N in $\mathbb{N}$ we have $s_N > M$. Clearly $n>N$ implies $s_n \ge s_N > M$, so $\lim s_n = +\infty$.

(ii) The proof is similar and is left to Exercise 10.5. ■

10.5 Corollary.
If (s_n) *is a monotone sequence, then the sequence either converges, diverges to* $+\infty$*, or diverges to* $-\infty$*. Thus* $\lim s_n$ *is always meaningful for monotone sequences.*

Proof
Apply Theorems 10.2 and 10.4. ■

Let (s_n) be a bounded sequence in $\mathbb{R}$; it may or may not converge. It is apparent from the definition of limit in 7.1 that the limiting behavior of (s_n) depends only on sets of the form $\{s_n : n > N\}$. For example, if $\lim s_n$ exists, clearly it lies in the interval $[u_N, v_N]$ where

$$u_N = \inf\{s_n : n > N\} \quad \text{and} \quad v_N = \sup\{s_n : n > N\};$$

see Exercise 8.9. As N increases, the sets $\{s_n : n > N\}$ get smaller, so we have

$$u_1 \le u_2 \le u_3 \le \cdots \quad \text{and} \quad v_1 \ge v_2 \ge v_3 \ge \cdots;$$

see Exercise 4.7(a). By Theorem 10.2 the limits $u = \lim_{N\to\infty} u_N$ and $v = \lim_{N\to\infty} v_N$ both exist, and $u \le v$ since $u_N \le v_N$ for all N. If $\lim s_n$ exists then, as noted above, $u_N \le \lim s_n \le v_N$ for all N, so we must have $u \le \lim s_n \le v$. The numbers u and v are useful whether $\lim s_n$ exists or not and are denoted $\liminf s_n$ and $\limsup s_n$, respectively.

10.6 Definition.
Let (s_n) be a sequence in $\mathbb{R}$. We define

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\} \tag{1}$$

and

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\}. \tag{2}$$

Note that in this definition we do not restrict (s_n) to be bounded. However, we adopt the following conventions. If (s_n) is not bounded above, $\sup\{s_n : n > N\} = +\infty$ for all N and we decree $\limsup s_n = +\infty$. Likewise, if (s_n) is not bounded below, $\inf\{s_n : n > N\} = -\infty$ for all N and we decree $\liminf s_n = -\infty$.

We emphasize $\limsup s_n$ need not equal $\sup\{s_n : n \in \mathbb{N}\}$, but $\limsup s_n \le \sup\{s_n : n \in \mathbb{N}\}$. Some of the values s_n may be much larger than $\limsup s_n$; $\limsup s_n$ is the largest value that *infinitely many* s_n's can get close to. Similar remarks apply to $\liminf s_n$. These remarks will be clarified in Theorem 11.8 and §12, where we will give a thorough treatment of lim inf's and lim sup's. For now, we need a theorem that shows (s_n) has a limit if and only if $\liminf s_n = \limsup s_n$.

10.7 Theorem.
Let (s_n) *be a sequence in* $\mathbb{R}$.

(i) *If* $\lim s_n$ *is defined* [*as a real number,* $+\infty$ *or* $-\infty$], *then* $\liminf s_n = \lim s_n = \limsup s_n$.

(ii) *If* $\liminf s_n = \limsup s_n$, *then* $\lim s_n$ *is defined and* $\lim s_n = \liminf s_n = \limsup s_n$.

Proof
We use the notation $u_N = \inf\{s_n : n > N\}$, $v_N = \sup\{s_n : n > N\}$, $u = \lim u_N = \liminf s_n$ and $v = \lim v_N = \limsup s_n$.

(i) Suppose $\lim s_n = +\infty$. Let M be a positive real number. Then there is a positive integer N so that

$$n > N \quad \text{implies} \quad s_n > M.$$

Then $u_N = \inf\{s_n : n > N\} \ge M$. It follows that $m > N$ implies $u_m \ge M$. In other words, the sequence (u_N) satisfies the condition defining $\lim u_N = +\infty$, i.e., $\liminf s_n = +\infty$. Likewise $\limsup s_n = +\infty$.

The case $\lim s_n = -\infty$ is handled in a similar manner.

Now suppose $\lim s_n = s$ where s is a real number. Consider $\epsilon > 0$. There exists a positive integer N such that $|s_n - s| < \epsilon$ for $n > N$. Thus $s_n < s + \epsilon$ for $n > N$, so

$$v_N = \sup\{s_n : n > N\} \le s + \epsilon.$$

Also, $m > N$ implies $v_m \le s+\epsilon$, so $\limsup s_n = \lim v_m \le s+\epsilon$. Since $\limsup s_n \le s + \epsilon$ for all $\epsilon > 0$, no matter how small, we conclude $\limsup s_n \le s = \lim s_n$. A similar argument shows $\lim s_n \le \liminf s_n$. Since $\liminf s_n \le \limsup s_n$, we infer all three numbers are equal:

$$\liminf s_n = \lim s_n = \limsup s_n.$$

(ii) If $\liminf s_n = \limsup s_n = +\infty$ it is easy to show $\lim s_n = +\infty$. And if $\liminf s_n = \limsup s_n = -\infty$ it is easy to show $\lim s_n = -\infty$. We leave these two special cases to the reader.

Suppose, finally, that $\liminf s_n = \limsup s_n = s$ where s is a real number. We need to prove $\lim s_n = s$. Let $\epsilon > 0$. Since $s = \lim v_N$ there exists a positive integer N_0 such that

$$|s - \sup\{s_n : n > N_0\}| < \epsilon.$$

Thus $\sup\{s_n : n > N_0\} < s + \epsilon$, so

$$s_n < s + \epsilon \quad \text{for all} \quad n > N_0. \tag{1}$$

Similarly, there exists N_1 such that $|s - \inf\{s_n : n > N_1\}| < \epsilon$, hence $\inf\{s_n : n > N_1\} > s - \epsilon$, hence

$$s_n > s - \epsilon \quad \text{for all} \quad n > N_1. \tag{2}$$

From (1) and (2) we conclude

$$s - \epsilon < s_n < s + \epsilon \quad \text{for} \quad n > \max\{N_0, N_1\},$$

equivalently

$$|s_n - s| < \epsilon \quad \text{for} \quad n > \max\{N_0, N_1\}.$$

This proves $\lim s_n = s$ as desired. ■

If (s_n) converges, then $\liminf s_n = \limsup s_n$ by the theorem just proved, so for large N the numbers $\sup\{s_n : n > N\}$ and $\inf\{s_n : n > N\}$ are close together. This implies that all the numbers in the set $\{s_n : n > N\}$ are close to each other. This leads us to a concept of great theoretical importance that will be used throughout the book.

10.8 Definition.
A sequence (s_n) of real numbers is called a *Cauchy sequence* if

$$\text{for each } \epsilon > 0 \text{ there exists a number } N \text{ such that}$$
$$m, n > N \text{ implies } |s_n - s_m| < \epsilon. \tag{1}$$

Compare this definition with Definition 7.1.

10.9 Lemma.
Convergent sequences are Cauchy sequences.

Proof
Suppose $\lim s_n = s$. The idea is that, since the terms s_n are close to s for large n, they also must be close to each other; indeed

$$|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s - s_m|.$$

To be precise, let $\epsilon > 0$. Then there exists N such that

$$n > N \quad \text{implies} \quad |s_n - s| < \frac{\epsilon}{2}.$$

Clearly we may also write

$$m > N \quad \text{implies} \quad |s_m - s| < \frac{\epsilon}{2},$$

so

$$m, n > N \quad \text{implies} \quad |s_n - s_m| \leq |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus (s_n) is a Cauchy sequence. ■

10.10 Lemma.
Cauchy sequences are bounded.

Proof
The proof is similar to that of Theorem 9.1. Applying Definition 10.8 with $\epsilon = 1$ we obtain N in $\mathbb{N}$ so that

$$m, n > N \quad \text{implies} \quad |s_n - s_m| < 1.$$

In particular, $|s_n - s_{N+1}| < 1$ for $n > N$, so $|s_n| < |s_{N+1}| + 1$ for $n > N$. If $M = \max\{|s_{N+1}| + 1, |s_1|, |s_2|, \ldots, |s_N|\}$, then $|s_n| \leq M$ for all $n \in \mathbb{N}$. ■

The next theorem is very important because it shows that to verify that a sequence converges it suffices to check it is a Cauchy sequence, a property that does not involve the limit itself.

10.11 Theorem.
A sequence is a convergent sequence if and only if it is a Cauchy sequence.

Proof

The expression "if and only if" indicates that we have two assertions to verify: (i) convergent sequences are Cauchy sequences, and (ii) Cauchy sequences are convergent sequences. We already verified (i) in Lemma 10.9. To check (ii), consider a Cauchy sequence (s_n) and note (s_n) is bounded by Lemma 10.10. By Theorem 10.7 we need only show

$$\liminf s_n = \limsup s_n. \tag{1}$$

Let $\epsilon > 0$. Since (s_n) is a Cauchy sequence, there exists N so that

$$m, n > N \quad \text{implies} \quad |s_n - s_m| < \epsilon.$$

In particular, $s_n < s_m + \epsilon$ for all $m, n > N$. This shows $s_m + \epsilon$ is an upper bound for $\{s_n : n > N\}$, so $v_N = \sup\{s_n : n > N\} \leq s_m + \epsilon$ for $m > N$. This, in turn, shows $v_N - \epsilon$ is a lower bound for $\{s_m : m > N\}$, so $v_N - \epsilon \leq \inf\{s_m : m > N\} = u_N$. Thus

$$\limsup s_n \leq v_N \leq u_N + \epsilon \leq \liminf s_n + \epsilon.$$

Since this holds for all $\epsilon > 0$, we have $\limsup s_n \leq \liminf s_n$. The opposite inequality always holds, so we have established (1). ■

The proof of Theorem 10.11 uses Theorem 10.7, and Theorem 10.7 relies implicitly on the Completeness Axiom 4.4, since without the completeness axiom it is not clear that $\liminf s_n$ and $\limsup s_n$ are meaningful. The completeness axiom assures us that the expressions $\sup\{s_n : n > N\}$ and $\inf\{s_n : n > N\}$ in Definition 10.6 are meaningful, and Theorem 10.2 [which itself relies on the completeness axiom] assures us that the limits in Definition 10.6 also are meaningful.

Exercises on lim sup's and lim inf's appear in §§11 and 12.

Exercises

10.1 Which of the following sequences are increasing? decreasing? bounded?

(a) $\frac{1}{n}$

(b) $\frac{(-1)^n}{n^2}$

(c) n^5

(d) $\sin(\frac{n\pi}{7})$

(e) $(-2)^n$

(f) $\frac{n}{3^n}$

10.2 Prove Theorem 10.2 for bounded decreasing sequences.

10.3 For a decimal expansion $K.d_1d_2d_3d_4\cdots$, let (s_n) be defined as in Discussion 10.3. Prove $s_n < K+1$ for all $n \in \mathbb{N}$. *Hint*: $\frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = 1 - \frac{1}{10^n}$ for all n.

10.4 Discuss why Theorems 10.2 and 10.11 would fail if we restricted our world of numbers to the set $\mathbb{Q}$ of rational numbers.

10.5 Prove Theorem 10.4(ii).

10.6 **(a)** Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| < 2^{-n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Prove (s_n) is a Cauchy sequence and hence a convergent sequence.

(b) Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?

10.7 Let S be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S$ is not in S. Prove there is a sequence (s_n) of points in S such that $\lim s_n = \sup S$. See also Exercise 11.11.

10.8 Let (s_n) be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove (σ_n) is an increasing sequence.

10.9 Let $s_1 = 1$ and $s_{n+1} = (\frac{n}{n+1})s_n^2$ for $n \geq 1$.

(a) Find s_2, s_3 and s_4.

(b) Show $\lim s_n$ exists.

(c) Prove $\lim s_n = 0$.

10.10 Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

(a) Find s_2, s_3 and s_4.

(b) Use induction to show $s_n > \frac{1}{2}$ for all n.

(c) Show (s_n) is a decreasing sequence.

(d) Show $\lim s_n$ exists and find $\lim s_n$.

10.11 Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{4n^2}] \cdot t_n$ for $n \geq 1$.

(a) Show $\lim t_n$ exists.

(b) What do you think $\lim t_n$ is?

10.12 Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{(n+1)^2}] \cdot t_n$ for $n \geq 1$.

(a) Show $\lim t_n$ exists.

(b) What do you think $\lim t_n$ is?

(c) Use induction to show $t_n = \frac{n+1}{2n}$.

(d) Repeat part (b).

§11 Subsequences

11.1 Definition.
Suppose $(s_n)_{n \in \mathbb{N}}$ is a sequence. A *subsequence* of this sequence is a sequence of the form $(t_k)_{k \in \mathbb{N}}$ where for each k there is a positive integer n_k such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots \tag{1}$$

and

$$t_k = s_{n_k}. \tag{2}$$

Thus (t_k) is just a selection of some [possibly all] of the s_n's taken in order.

Here are some alternative ways to approach this concept. Note that (1) defines an infinite subset of $\mathbb{N}$, namely $\{n_1, n_2, n_3, \ldots\}$. Conversely, every infinite subset of $\mathbb{N}$ can be described by (1). Thus a subsequence of (s_n) is a sequence obtained by selecting, in order, an infinite subset of the terms.

For a more precise definition, recall we can view the sequence $(s_n)_{n \in \mathbb{N}}$ as a function s with domain $\mathbb{N}$; see §7. For the subset $\{n_1, n_2, n_3, \ldots\}$, there is a natural function σ [lower case Greek sigma] given by $\sigma(k) = n_k$ for $k \in \mathbb{N}$. The function σ "selects" an infinite subset of $\mathbb{N}$, in order. The subsequence of s corresponding to σ is simply the composite function $t = s \circ \sigma$. That is,

$$t_k = t(k) = s \circ \sigma(k) = s(\sigma(k)) = s(n_k) = s_{n_k} \quad \text{for} \quad k \in \mathbb{N}. \tag{3}$$

Thus a sequence t is a subsequence of a sequence s if and only if $t = s \circ \sigma$ for some increasing function σ mapping $\mathbb{N}$ into $\mathbb{N}$. We will usually suppress the notation σ and often suppress the notation t

6 CHAPTER Integration

This chapter serves two purposes. It contains a careful development of the Riemann integral, which is the integral studied in standard calculus courses. It also contains an introduction to a generalization of the Riemann integral called the Riemann-Stieltjes integral. The generalization is easy and natural. Moreover, the Riemann-Stieltjes integral is an important tool in probability and statistics, and other areas of mathematics.

§32 The Riemann Integral

The theory of the Riemann integral is no more difficult than several other topics dealt with in this book. The one drawback is that it involves some technical notation and terminology.

32.1 Definition.
Let f be a bounded function on a closed interval $[a, b]$.[1] For $S \subseteq [a, b]$, we adopt the notation

[1]Here and elsewhere in this chapter, we assume $a < b$.

K.A. Ross, *Elementary Analysis: The Theory of Calculus*, Undergraduate Texts in Mathematics, DOI 10.1007/978-1-4614-6271-2_6,

$$M(f,S) = \sup\{f(x) : x \in S\} \quad \text{and} \quad m(f,S) = \inf\{f(x) : x \in S\}.$$

A *partition of* $[a,b]$ is any finite ordered subset P having the form

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}.$$

The *upper Darboux sum* $U(f,P)$ of f with respect to P is the sum

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1})$$

and the *lower Darboux sum* $L(f,P)$ is

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k]) \cdot (t_k - t_{k-1}).$$

Note

$$U(f,P) \le \sum_{k=1}^{n} M(f,[a,b]) \cdot (t_k - t_{k-1}) = M(f,[a,b]) \cdot (b-a);$$

likewise $L(f,P) \ge m(f,[a,b]) \cdot (b-a)$, so

$$m(f,[a,b]) \cdot (b-a) \le L(f,P) \le U(f,P) \le M(f,[a,b]) \cdot (b-a). \quad (1)$$

The *upper Darboux integral* $U(f)$ of f over $[a,b]$ is defined by

$$U(f) = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$

and the *lower Darboux integral* is

$$L(f) = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}.$$

In view of (1), $U(f)$ and $L(f)$ are real numbers.

We will prove in Theorem 32.4 that $L(f) \le U(f)$. This is not obvious from (1). [Why?] We say f is *integrable* on $[a,b]$ provided $L(f) = U(f)$. In this case, we write $\int_a^b f$ or $\int_a^b f(x)\,dx$ for this common value:

$$\int_a^b f = \int_a^b f(x)\,dx = L(f) = U(f). \quad (2)$$

Specialists call this integral the *Darboux integral.* Riemann's definition of the integral is a little different [Definition 32.8], but we will show in Theorem 32.9 that the definitions are equivalent. For this

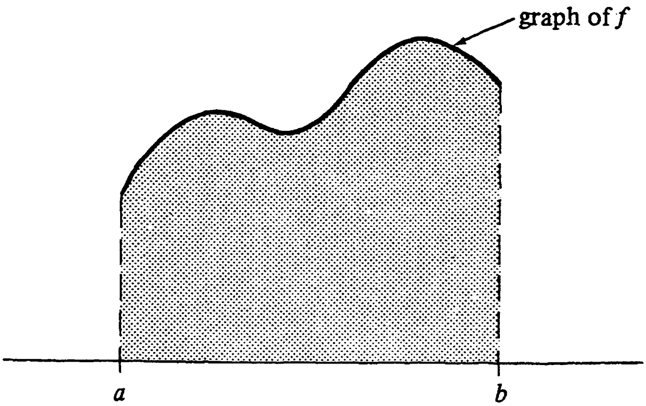

FIGURE 32.1

reason, we will follow customary usage and call the integral defined above the *Riemann integral*.

For nonnegative functions, $\int_a^b f$ is interpreted as the area of the region under the graph of f [see Fig. 32.1] for the following reason. Each lower Darboux sum represents the area of a union of rectangles inside the region, and each upper Darboux sum represents the area of a union of rectangles that contains the region. Moreover, $\int_a^b f$ is the unique number that is larger than or equal to all lower Darboux sums and smaller than or equal to all upper Darboux sums. Figure 19.2 on page 145 illustrates the situation for $[a, b] = [0, 1]$ and

$$P = \left\{0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1\right\}.$$

Example 1

The simplest function whose integral is not obvious is $f(x) = x^2$. Consider f on the interval $[0, b]$ where $b > 0$. For a partition

$$P = \{0 = t_0 < t_1 < \cdots < t_n = b\},$$

we have

$$U(f, P) = \sum_{k=1}^{n} \sup\{x^2 : x \in [t_{k-1}, t_k]\} \cdot (t_k - t_{k-1}) = \sum_{k=1}^{n} t_k^2 (t_k - t_{k-1}).$$

If we choose $t_k = \frac{kb}{n}$, then we can use Exercise 1.1 to calculate

$$U(f,P) = \sum_{k=1}^{n} \frac{k^2b^2}{n^2}\left(\frac{b}{n}\right) = \frac{b^3}{n^3}\sum_{k=1}^{n} k^2 = \frac{b^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}.$$

For large n, this is close to $\frac{b^3}{3}$, so we conclude $U(f) \le \frac{b^3}{3}$. For the same partition we find

$$L(f,P) = \sum_{k=1}^{n} \frac{(k-1)^2b^2}{n^2}\left(\frac{b}{n}\right) = \frac{b^3}{n^3}\cdot\frac{(n-1)(n)(2n-1)}{6},$$

so $L(f) \ge \frac{b^3}{3}$. Therefore $f(x) = x^2$ is integrable on $[0,b]$ and

$$\int_0^b x^2\,dx = \frac{b^3}{3}.$$

Of course, any calculus student could have calculated this integral using a formula that is based on the Fundamental Theorem of Calculus; see Example 1 in §34. □

Example 2

Consider the interval $[a,b]$, where $a < b$, and let $f(x) = 1$ for rational x in $[a,b]$, and let $f(x) = 0$ for irrational x in $[a,b]$. For any partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\},$$

we have

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])\cdot(t_k - t_{k-1}) = \sum_{k=1}^{n} 1\cdot(t_k - t_{k-1}) = b-a$$

and

$$L(f,P) = \sum_{k=1}^{n} 0\cdot(t_k - t_{k-1}) = 0.$$

It follows that $U(f) = b - a$ and $L(f) = 0$. The upper and lower Darboux integrals for f do not agree, so f is not integrable! □

We next develop some properties of the integral.

32.2 Lemma.
Let f be a bounded function on $[a, b]$. If P and Q are partitions of $[a, b]$ and $P \subseteq Q$, then

$$L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P). \tag{1}$$

Proof
The middle inequality is obvious. The proofs of the first and third inequalities are similar, so we will prove

$$L(f, P) \le L(f, Q). \tag{2}$$

An induction argument [Exercise 32.4] shows we may assume
Q has only one more point, say u, than P. If

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\},$$

then

$$Q = \{a = t_0 < t_1 < \cdots < t_{k-1} < u < t_k < \cdots < t_n = b\}$$

for some $k \in \{1, 2, \ldots, n\}$. The lower Darboux sums for P and Q are the same except for the terms involving t_{k-1} or t_k. In fact, their difference is

$$\begin{aligned} L(f, Q) - L(f, P) = m(f, [t_{k-1}, u]) \cdot (u - t_{k-1}) + m(f, [u, t_k]) \cdot (t_k - u) \\ -m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}). \end{aligned} \tag{3}$$

To establish (2) it suffices to show this quantity is nonnegative. Using Exercise 4.7(a), we see

$$\begin{aligned} &m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1}) \\ &\quad = m(f, [t_{k-1}, t_k]) \cdot \{(t_k - u) + (u - t_{k-1})\} \\ &\quad \le m(f, [u, t_k]) \cdot (t_k - u) + m(f, [t_{k-1}, u]) \cdot (u - t_{k-1}). \end{aligned}$$

■

32.3 Lemma.
If f is a bounded function on $[a, b]$, and if P and Q are partitions of $[a, b]$, then $L(f, P) \le U(f, Q)$.

Proof
The set $P \cup Q$ is also a partition of $[a, b]$. Since $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$, we can apply Lemma 32.2 to obtain

$$L(f, P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f, Q).$$

■

32.4 Theorem.
If f is a bounded function on $[a,b]$, then $L(f) \leq U(f)$.

Proof
Fix a partition P of $[a,b]$. Lemma 32.3 shows $L(f,P)$ is a lower bound for the set

$$\{U(f,Q) : Q \text{ is a partition of } [a,b]\}.$$

Therefore $L(f,P)$ is less than or equal to the greatest lower bound [infimum!] of this set. That is

$$L(f,P) \leq U(f). \tag{1}$$

Now (1) shows that $U(f)$ is an upper bound for the set

$$\{L(f,P) : P \text{ is a partition of } [a,b]\},$$

so $U(f) \geq L(f)$. ■

Note that Theorem 32.4 also follows from Lemma 32.3 and Exercise 4.8; see Exercise 32.5. The next theorem gives a "Cauchy criterion" for integrability.

32.5 Theorem.
A bounded function f on $[a,b]$ is integrable if and only if for each $\epsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \epsilon. \tag{1}$$

Proof
Suppose first that f is integrable and consider $\epsilon > 0$. There exist partitions P_1 and P_2 of $[a,b]$ satisfying

$$L(f,P_1) > L(f) - \frac{\epsilon}{2} \quad \text{and} \quad U(f,P_2) < U(f) + \frac{\epsilon}{2}.$$

For $P = P_1 \cup P_2$, we apply Lemma 32.2 to obtain

$$\begin{aligned} U(f,P) - L(f,P) &\leq U(f,P_2) - L(f,P_1) \\ &< U(f) + \frac{\epsilon}{2} - \left[L(f) - \frac{\epsilon}{2}\right] = U(f) - L(f) + \epsilon. \end{aligned}$$

Since f is integrable, $U(f) = L(f)$, so (1) holds.

Conversely, suppose for $\epsilon > 0$ the inequality (1) holds for some partition P. Then we have

$$\begin{aligned} U(f) \le U(f,P) &= U(f,P) - L(f,P) + L(f,P) \\ &< \epsilon + L(f,P) \le \epsilon + L(f). \end{aligned}$$

Since ϵ is arbitrary, we conclude $U(f) \le L(f)$. Hence we have $U(f) = L(f)$ by Theorem 32.4, i.e., f is integrable. ■

The remainder of this section is devoted to establishing the equivalence of Riemann's and Darboux's definitions of integrability. Subsequent sections will depend only on items Definition 32.1 through Theorem 32.5. Therefore the reader who is content with the Darboux integral in Definition 32.1 can safely proceed directly to the next section.

32.6 Definition.
The *mesh* of a partition P is the maximum length of the subintervals comprising P. Thus if

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\},$$

then

$$\text{mesh}(P) = \max\{t_k - t_{k-1} : k = 1, 2, \ldots, n\}.$$

Here is another "Cauchy criterion" for integrability.

32.7 Theorem.
A bounded function f on $[a,b]$ is integrable if and only if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{mesh}(P) < \delta \quad \textit{implies} \quad U(f,P) - L(f,P) < \epsilon \tag{1}$$

for all partitions P of $[a,b]$.

Proof
The ϵ–δ condition in (1) implies integrability by Theorem 32.5.

Conversely, suppose f is integrable on $[a,b]$. Let $\epsilon > 0$ and select a partition

$$P_0 = \{a = u_0 < u_1 < \cdots < u_m = b\}$$

of $[a, b]$ such that

$$U(f, P_0) - L(f, P_0) < \frac{\epsilon}{2}. \tag{2}$$

Since f is bounded, there exists $B > 0$ such that $|f(x)| \leq B$ for all $x \in [a, b]$. Let $\delta = \frac{\epsilon}{8mB}$; m is the number of intervals comprising P_0.

To verify (1), we consider any partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

with $\text{mesh}(P) < \delta$. Let $Q = P \cup P_0$. If Q has one more element than P, then a glance at (3) in the proof of Lemma 32.2 leads us to

$$L(f, Q) - L(f, P) \leq B \cdot \text{mesh}(P) - (-B) \cdot \text{mesh}(P) = 2B \cdot \text{mesh}(P).$$

Since Q has at most m elements that are not in P, an induction argument shows

$$L(f, Q) - L(f, P) \leq 2mB \cdot \text{mesh}(P) < 2mB\delta = \frac{\epsilon}{4}.$$

By Lemma 32.2 we have $L(f, P_0) \leq L(f, Q)$, so

$$L(f, P_0) - L(f, P) < \frac{\epsilon}{4}.$$

Similarly

$$U(f, P) - U(f, P_0) < \frac{\epsilon}{4},$$

so

$$U(f, P) - L(f, P) < U(f, P_0) - L(f, P_0) + \frac{\epsilon}{2}.$$

Now (2) implies $U(f, P) - L(f, P) < \epsilon$ and we have verified (1). ■

Now we give Riemann's definition of integrability.

32.8 Definition.
Let f be a bounded function on $[a, b]$, and let $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ be a partition of $[a, b]$. A *Riemann sum* of f associated with the partition P is a sum of the form

$$\sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$$

where $x_k \in [t_{k-1}, t_k]$ for $k = 1, 2, \ldots, n$. The choice of x_k's is quite arbitrary, so there are infinitely many Riemann sums associated with a single function and partition.

The function f is *Riemann integrable* on $[a, b]$ if there exists a number r with the following property. For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$|S - r| < \epsilon \tag{1}$$

for every Riemann sum S of f associated with a partition P having $\text{mesh}(P) < \delta$. The number r is the *Riemann integral* of f on $[a, b]$ and will be provisionally written as $\mathcal{R}\int_a^b f$.

32.9 Theorem.
A bounded function f on $[a, b]$ is Riemann integrable if and only if it is [Darboux] integrable, in which case the values of the integrals agree.

Proof
Suppose first that f is [Darboux] integrable on $[a, b]$ in the sense of Definition 32.1. Let $\epsilon > 0$, and let $\delta > 0$ be chosen so that (1) of Theorem 32.7 holds. We show

$$\left| S - \int_a^b f \right| < \epsilon \tag{1}$$

for every Riemann sum

$$S = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$$

associated with a partition P having $\text{mesh}(P) < \delta$. Clearly we have $L(f, P) \le S \le U(f, P)$, so (1) follows from the inequalities

$$U(f, P) < L(f, P) + \epsilon \le L(f) + \epsilon = \int_a^b f + \epsilon$$

and

$$L(f, P) > U(f, P) - \epsilon \ge U(f) - \epsilon = \int_a^b f - \epsilon.$$

This proves (1); hence f is Riemann integrable and

$$\mathcal{R}\int_a^b f = \int_a^b f.$$

Now suppose f is Riemann integrable in the sense of Definition 32.8, and consider $\epsilon > 0$. Let $\delta > 0$ and r be as given in Definition 32.8. Select any partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

with $\text{mesh}(P) < \delta$, and for each $k = 1, 2, \ldots, n$, select x_k in $[t_{k-1}, t_k]$ so that

$$f(x_k) < m(f, [t_{k-1}, t_k]) + \epsilon.$$

The Riemann sum S for this choice of x_k's satisfies

$$S \leq L(f, P) + \epsilon(b - a)$$

as well as

$$|S - r| < \epsilon.$$

It follows that

$$L(f) \geq L(f, P) \geq S - \epsilon(b - a) > r - \epsilon - \epsilon(b - a).$$

Since ϵ is arbitrary, we have $L(f) \geq r$. A similar argument shows $U(f) \leq r$. Since $L(f) \leq U(f)$, we see $L(f) = U(f) = r$. This shows f is [Darboux] integrable and

$$\int_a^b f = r = \mathcal{R}\int_a^b f.$$

■

32.10 Corollary.
Let f be a bounded Riemann integrable function on $[a, b]$. Suppose (S_n) is a sequence of Riemann sums, with corresponding partitions P_n, satisfying $\lim_n \text{mesh}(P_n) = 0$. Then the sequence (S_n) converges to $\int_a^b f$.

Proof
Let $\epsilon > 0$. There is a $\delta > 0$ so that if S is a Riemann sum with corresponding partition P, and if $\text{mesh}(P) < \delta$, then

$$\left| S - \int_a^b f \right| < \epsilon.$$

Choose N so that $\text{mesh}(P_n) < \delta$ for $n > N$. Then

$$\left| S_n - \int_a^b f \right| < \epsilon \quad \text{for} \quad n > N.$$

Since $\epsilon > 0$ is arbitrary, this shows $\lim_n S_n = \int_a^b f$. ■

32.11 Remark.
I recently had occasion to use the following simple observation. If one ignores the end intervals of the partitions, the "almost Riemann sums" so obtained still converge to the integral; see [59]. This arose because the intervals had the form $[a, b]$, but the partition points had the form $\frac{k}{n}$. Thus the partition points were nice and equally spaced, *except* for the end ones.

Exercises

32.1 Find the upper and lower Darboux integrals for $f(x) = x^3$ on the interval $[0, b]$. *Hint*: Exercise 1.3 and Example 1 in §1 will be useful.

32.2 Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x.

- **(a)** Calculate the upper and lower Darboux integrals for f on the interval $[0, b]$.
- **(b)** Is f integrable on $[0, b]$?

32.3 Repeat Exercise 32.2 for g where $g(x) = x^2$ for rational x and $g(x) = 0$ for irrational x.

32.4 Supply the induction argument needed in the proof of Lemma 32.2.

32.5 Use Exercise 4.8 to prove Theorem 32.4. Specify the sets S and T in this case.

32.6 Let f be a bounded function on $[a,b]$. Suppose there exist sequences (U_n) and (L_n) of upper and lower Darboux sums for f such that $\lim(U_n - L_n) = 0$. Show f is integrable and $\int_a^b f = \lim U_n = \lim L_n$.

32.7 Let f be integrable on $[a,b]$, and suppose g is a function on $[a,b]$ such that $g(x) = f(x)$ except for finitely many x in $[a,b]$. Show g is integrable and $\int_a^b f = \int_a^b g$. *Hint*: First reduce to the case where f is the function identically equal to 0.

32.8 Show that if f is integrable on $[a,b]$, then f is integrable on every interval $[c,d] \subseteq [a,b]$.

§33 Properties of the Riemann Integral

In this section we establish some basic properties of the Riemann integral and we show many familiar functions, including piecewise continuous and piecewise monotonic functions, are Riemann integrable.

A function is *monotonic* on an interval if it is either increasing or decreasing on the interval; see Definition 29.6.

33.1 Theorem.
Every monotonic function f on $[a,b]$ is integrable.

Proof
We assume f is increasing on $[a,b]$ and leave the decreasing case to Exercise 33.1. We also assume $f(a) < f(b)$, since otherwise f would be a constant function. Since $f(a) \le f(x) \le f(b)$ for all $x \in [a,b]$, f is clearly bounded on $[a,b]$. In order to apply Theorem 32.5, let $\epsilon > 0$ and select a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ with mesh less than $\frac{\epsilon}{f(b)-f(a)}$. Then

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n} \{M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\} \cdot (t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} [f(t_k) - f(t_{k-1})] \cdot (t_k - t_{k-1}). \end{aligned}$$

Since $\text{mesh}(P) < \frac{\epsilon}{f(b)-f(a)}$, we have

$$U(f,P) - L(f,P) < \sum_{k=1}^{n} [f(t_k) - f(t_{k-1})] \cdot \frac{\epsilon}{f(b)-f(a)}$$
$$= [f(b) - f(a)] \cdot \frac{\epsilon}{f(b)-f(a)} = \epsilon.$$

Theorem 32.5 now shows f is integrable. ■

33.2 Theorem.
Every continuous function f on $[a,b]$ is integrable.

Proof
Again, in order to apply Theorem 32.5, consider $\epsilon > 0$. Since f is uniformly continuous on $[a,b]$ by Theorem 19.2, there exists $\delta > 0$ such that

$$x, y \in [a,b] \quad \text{and} \quad |x-y| < \delta \quad \text{imply} \quad |f(x) - f(y)| < \frac{\epsilon}{b-a}. \quad (1)$$

Consider any partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ where

$$\max\{t_k - t_{k-1} : k = 1, 2, \ldots, n\} < \delta.$$

Since f assumes its maximum and minimum on each interval $[t_{k-1}, t_k]$ by Theorem 18.1, it follows from (1) above that

$$M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k]) < \frac{\epsilon}{b-a}$$

for each k. Therefore we have

$$U(f,P) - L(f,P) < \sum_{k=1}^{n} \frac{\epsilon}{b-a}(t_k - t_{k-1}) = \epsilon,$$

and Theorem 32.5 shows f is integrable. ■

33.3 Theorem.
Let f and g be integrable functions on $[a,b]$, and let c be a real number. Then
(i) *cf is integrable and $\int_a^b cf = c\int_a^b f$;*
(ii) *$f+g$ is integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.*

Exercise 33.8 shows fg, $\max(f,g)$ and $\min(f,g)$ are also integrable, but there are no formulas giving their integrals in terms of $\int_a^b f$ and $\int_a^b g$.

Proof

The proof of (i) involves three cases: $c > 0$, $c = -1$, and $c < 0$. Of course, (i) is obvious for $c = 0$.

Let $c > 0$ and consider a partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

of $[a, b]$. A simple exercise [Exercise 33.2] shows

$$M(cf, [t_{k-1}, t_k]) = c \cdot M(f, [t_{k-1}, t_k])$$

for all k, so $U(cf, P) = c \cdot U(f, P)$. Another application of the same exercise shows $U(cf) = c \cdot U(f)$. Similar arguments show $L(cf) = c \cdot L(f)$. Since f is integrable, we have $L(cf) = c \cdot L(f) = c \cdot U(f) = U(cf)$. Hence cf is integrable and

$$\int_a^b cf = U(cf) = c \cdot U(f) = c \int_a^b f, \qquad c > 0. \tag{1}$$

Now we deal with the case $c = -1$. Exercise 5.4 implies $U(-f, P) = -L(f, P)$ for all partitions P of $[a, b]$. Hence we have

$$\begin{aligned} U(-f) &= \inf\{U(-f, P) : P \text{ is a partition of } [a, b]\} \\ &= \inf\{-L(f, P) : P \text{ is a partition of } [a, b]\} \\ &= -\sup\{L(f, P) : P \text{ is a partition of } [a, b]\} = -L(f). \end{aligned}$$

Replacing f by $-f$, we also obtain $L(-f) = -U(f)$. Since f is integrable, $U(-f) = -L(f) = -U(f) = L(-f)$; hence $-f$ is integrable and

$$\int_a^b (-f) = -\int_a^b f. \tag{2}$$

The case $c < 0$ is handled by applying (2), and then (1) to $-c$:

$$\int_a^b cf = -\int_a^b (-c)f = -(-c)\int_a^b f = c\int_a^b f.$$

To prove (ii) we will again use Theorem 32.5. Let $\epsilon > 0$. By Theorem 32.5 there exist partitions P_1 and P_2 of $[a, b]$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\epsilon}{2} \quad \text{and} \quad U(g, P_2) - L(g, P_2) < \frac{\epsilon}{2}.$$

Lemma 32.2 shows that if $P = P_1 \cup P_2$, then

$$U(f, P) - L(f, P) < \frac{\epsilon}{2} \quad \text{and} \quad U(g, P) - L(g, P) < \frac{\epsilon}{2}. \tag{3}$$

For any subset S of $[a, b]$, we have

$$\inf\{f(x) + g(x) : x \in S\} \geq \inf\{f(x) : x \in S\} + \inf\{g(x) : x \in S\},$$

i.e., $m(f + g, S) \geq m(f, S) + m(g, S)$. It follows that

$$L(f + g, P) \geq L(f, P) + L(g, P)$$

and similarly we have

$$U(f + g, P) \leq U(f, P) + U(g, P).$$

Therefore from (3) we obtain

$$U(f + g, P) - L(f + g, P) < \epsilon.$$

Theorem 32.5 now shows $f + g$ is integrable. Since

$$\int_a^b (f + g) = U(f + g) \leq U(f + g, P) \leq U(f, P) + U(g, P)$$
$$< L(f, P) + L(g, P) + \epsilon \leq L(f) + L(g) + \epsilon = \int_a^b f + \int_a^b g + \epsilon$$

and

$$\int_a^b (f + g) = L(f + g) \geq L(f + g, P) \geq L(f, P) + L(g, P)$$
$$> U(f, P) + U(g, P) - \epsilon \geq U(f) + U(g) - \epsilon = \int_a^b f + \int_a^b g - \epsilon,$$

we see that

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

■

33.4 Theorem.

(i) *If f and g are integrable on $[a,b]$ and if $f(x) \le g(x)$ for x in $[a,b]$, then $\int_a^b f \le \int_a^b g$.*

(ii) *If g is a continuous nonnegative function on $[a,b]$ and if $\int_a^b g = 0$, then g is identically 0 on $[a,b]$.*

Proof

(i) By Theorem 33.3, $h = g - f$ is integrable on $[a,b]$. Since $h(x) \ge 0$ for all $x \in [a,b]$, it is clear that $L(h,P) \ge 0$ for all partitions P of $[a,b]$, so $\int_a^b h = L(h) \ge 0$. Applying Theorem 33.3 again, we see

$$\int_a^b g = \int_a^b f + \int_a^b h \ \ge \int_a^b f.$$

(ii) Otherwise, since g is continuous, there is a nonempty interval $(c,d) \subseteq [a,b]$ and $\alpha > 0$ satisfying $g(x) \ge \alpha/2$ for $x \in (c,d)$. Then

$$\int_a^b g \ge \int_c^d g \ge \frac{\alpha}{2}(d-c) > 0,$$

contradicting $\int_a^b g = 0$. ■

33.5 Theorem.

If f is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$ and

$$\left|\int_a^b f\right| \le \int_a^b |f|. \tag{1}$$

Proof

This follows easily from Theorem 33.4(i) provided we know $|f|$ is integrable on $[a,b]$. In fact, $-|f| \le f \le |f|$; therefore

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|,$$

which implies (1).

We now show $|f|$ is integrable, a point that was conveniently glossed over in Exercise 25.1. For any subset S of $[a,b]$, we have

$$M(|f|,S) - m(|f|,S) \le M(f,S) - m(f,S) \tag{2}$$

by Exercise 33.6. From (2) it follows that

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) \tag{3}$$

for all partitions P of $[a, b]$. By Theorem 32.5, for each $\epsilon > 0$ there exists a partition P such that

$$U(f, P) - L(f, P) < \epsilon.$$

In view of (3), the same remark applies to $|f|$, so $|f|$ is integrable by Theorem 32.5. ■

33.6 Theorem.
Let f be a function defined on $[a, b]$. If $a < c < b$ and f is integrable on $[a, c]$ and on $[c, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{1}$$

Proof
Since f is bounded on both $[a, c]$ and $[c, b]$, f is bounded on $[a, b]$. In this proof we will decorate upper and lower sums so that it will be clear which intervals we are dealing with. Let $\epsilon > 0$. By Theorem 32.5 there exist partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ such that

$$U_a^c(f, P_1) - L_a^c(f, P_1) < \frac{\epsilon}{2} \quad \text{and} \quad U_c^b(f, P_2) - L_c^b(f, P_2) < \frac{\epsilon}{2}.$$

The set $P = P_1 \cup P_2$ is a partition of $[a, b]$, and it is obvious that

$$U_a^b(f, P) = U_a^c(f, P_1) + U_c^b(f, P_2) \tag{2}$$

with a similar identity for lower sums. It follows that

$$U_a^b(f, P) - L_a^b(f, P) < \epsilon,$$

so f is integrable on $[a, b]$ by Theorem 32.5. Also (1) holds because

$$\begin{aligned} \int_a^b f &\leq U_a^b(f, P) = U_a^c(f, P_1) + U_c^b(f, P_2) \\ &< L_a^c(f, P_1) + L_c^b(f, P_2) + \epsilon \leq \int_a^c f + \int_c^b f + \epsilon \end{aligned}$$

and similarly $\int_a^b f > \int_a^c f + \int_c^b f - \epsilon$. ■

Most functions encountered in calculus and analysis are covered by the next definition. However, see Exercises 33.10–33.12.

33.7 Definition.
A function f on $[a, b]$ is *piecewise monotonic* if there is a partition

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

of $[a, b]$ such that f is monotonic on each interval (t_{k-1}, t_k). The function f is *piecewise continuous* if there is a partition P of $[a, b]$ such that f is uniformly continuous on each interval (t_{k-1}, t_k).

33.8 Theorem.
If f is a piecewise continuous function or a bounded piecewise monotonic function on $[a, b]$, then f is integrable on $[a, b]$.

Proof
Let P be the partition described in Definition 33.7. Consider a fixed interval $[t_{k-1}, t_k]$. If f is piecewise continuous, then its restriction to (t_{k-1}, t_k) can be extended to a continuous function f_k on $[t_{k-1}, t_k]$ by Theorem 19.5. If f is piecewise monotonic, then its restriction to (t_{k-1}, t_k) can be extended to a monotonic function f_k on $[t_{k-1}, t_k]$; for example, if f is increasing on (t_{k-1}, t_k), simply define

$$f_k(t_k) = \sup\{f(x) : x \in (t_{k-1}, t_k)\}$$

and

$$f_k(t_{k-1}) = \inf\{f(x) : x \in (t_{k-1}, t_k)\}.$$

In either case, f_k is integrable on $[t_{k-1}, t_k]$ by Theorem 33.1 or 33.2. Since f agrees with f_k on $[t_{k-1}, t_k]$ except possibly at the endpoints, Exercise 32.7 shows f is also integrable on $[t_{k-1}, t_k]$. Now Theorem 33.6 and a trivial induction argument show f is integrable on $[a, b]$. ∎

We close this section with a simple but useful result.

33.9 Intermediate Value Theorem for Integrals.
If f is a continuous function on $[a,b]$, then for at least one x in (a,b) we have

$$f(x) = \frac{1}{b-a}\int_a^b f.$$

Proof
Let M and m be the maximum and minimum values of f on $[a,b]$. If $m = M$, then f is a constant function and $f(x) = \frac{1}{b-a}\int_a^b f$ for all $x \in [a,b]$. Otherwise, $m < M$ and by Theorem 18.1, there exist distinct x_0 and y_0 in $[a,b]$ satisfying $f(x_0) = m$ and $f(y_0) = M$. Since each function $M - f$ and $f - m$ is nonnegative and not identically 0, Theorem 33.4(ii) shows $\int_a^b m < \int_a^b f < \int_a^b M$. Thus

$$m < \frac{1}{b-a}\int_a^b f < M,$$

and by the Intermediate Value Theorem 18.2 for continuous functions, we have $f(x) = \frac{1}{b-a}\int_a^b f$ for some x between x_0 and y_0. Since x is in (a,b), this completes the proof. ■

33.10 Discussion.
An important question concerns when one can interchange limits and integrals, i.e., when is

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx \qquad (1)$$

true? By Theorems 24.3 and 25.2, if the f_ns are continuous and converge uniformly to $f = \lim_{n\to\infty} f_n$ on $[a,b]$, then f is continuous and (1) holds. It turns out that if each f_n is just Riemann integrable and $f_n \to f$ uniformly, then f is Riemann integrable and (1) holds; see Exercise 33.9. What happens if $f_n \to f$ pointwise on $[a,b]$? One problem is that f need not be integrable even if it is bounded and each f_n is integrable.

Consider, for example, the non-integrable function f in Example 2 on page 272: $f(x) = 1$ for rational x in $[a,b]$ and $f(x) = 0$ for irrational x in $[a,b]$. Let $(x_k)_{k\in\mathbb{N}}$ be an enumeration of the rationals in $[a,b]$, and define $f_n(x_k) = 1$ for $1 \le k \le n$ and $f_n(x) = 0$ for all other x in $[a, b]$. Then $f_n \to f$ pointwise on $[a, b]$, and each f_n is integrable.

This example leaves open the possibility that (1) will hold provided all the functions f_n *and* the limit function f are integrable. However, Exercise 33.15 provides an example of a sequence (f_n) of functions on $[0, 1]$ converging pointwise to a function f, with all the functions integrable, and yet (1) does not hold. Nevertheless, there is an important theorem that does apply to sequences of functions that converge pointwise. □

33.11 Dominated Convergence Theorem.
Suppose (f_n) is a sequence of integrable functions on $[a, b]$ and $f_n \to f$ pointwise where f is an integrable function on $[a, b]$. If there exists an $M > 0$ such that $|f_n(x)| \leq M$ for all n and all x in $[a, b]$, then

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)\, dx.$$

We omit the proof. An elementary proof of the Dominated Convergence Theorem is given by Jonathan W. Lewin [42]. Here is a corollary.

33.12 Monotone Convergence Theorem.
Suppose (f_n) is a sequence of integrable functions on $[a, b]$ such that $f_1(x) \leq f_2(x) \leq \cdots$ for all x in $[a, b]$. Suppose also that $f_n \to f$ pointwise where f is an integrable function on $[a, b]$. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b \lim_{n\to\infty} f_n(x)\, dx.$$

This follows from the Dominated Convergence Theorem, because there exists an $M > 0$ such that $|f_1(x)| \leq M$ and also $|f(x)| \leq M$ for all x in $[a, b]$. This implies $|f_n(x)| \leq M$ for all n and all x in $[a, b]$, since $-M \leq f_1(x) \leq f_n(x) \leq M$ for all x.

Our version of the Dominated Convergence Theorem is a special case of a much more general theorem, which is usually stated and proved for the family of all "Lebesgue integrable functions," not just for Riemann integrable functions. There is also a Monotone Convergence Theorem for Lebesgue integrable functions, but in that generality it does not follow immediately from the Dominated Convergence Theorem, because in that setting integrable functions need not be bounded. An elementary proof of the Monotone Convergence Theorem is proved for Riemann integrable functions, without resort to Lebesgue theory, by Brian S. Thomson [67].

Exercises

33.1 Complete the proof of Theorem 33.1 by showing that a decreasing function on $[a, b]$ is integrable.

33.2 This exercise could have appeared just as easily in §4. Let S be a nonempty bounded subset of $\mathbb{R}$. For fixed $c > 0$, let $cS = \{cs : s \in S\}$. Show $\sup(cS) = c \cdot \sup(S)$ and $\inf(cS) = c \cdot \inf(S)$.

33.3 A function f on $[a, b]$ is called a *step function* if there exists a partition $P = \{a = u_0 < u_1 < \cdots < c_m = b\}$ of $[a, b]$ such that f is constant on each interval (u_{j-1}, u_j), say $f(x) = c_j$ for x in (u_{j-1}, u_j).

(a) Show that a step function f is integrable and evaluate $\int_a^b f$.

(b) Evaluate the integral $\int_0^4 P(x)\,dx$ for the postage-stamp function P in Exercise 17.16.

33.4 Give an example of a function f on $[0, 1]$ that is *not* integrable for which $|f|$ is integrable. *Hint*: Modify Example 2 in §32.

33.5 Show $|\int_{-2\pi}^{2\pi} x^2 \sin^8(e^x)\,dx| \le \frac{16\pi^3}{3}$.

33.6 Prove (2) in the proof of Theorem 33.5. *Hint*: For $x_0, y_0 \in S$, we have $|f(x_0)| - |f(y_0)| \le |f(x_0) - f(y_0)| \le M(f, S) - m(f, S)$.

33.7 Let f be a bounded function on $[a, b]$, so that there exists $B > 0$ such that $|f(x)| \le B$ for all $x \in [a, b]$.

(a) Show

$$U(f^2, P) - L(f^2, P) \le 2B[U(f, P) - L(f, P)]$$

for all partitions P of $[a, b]$. *Hint*: $f(x)^2 - f(y)^2 = [f(x) + f(y)] \cdot [f(x) - f(y)]$.

(b) Show that if f is integrable on $[a, b]$, then f^2 also is integrable on $[a, b]$.

33.8 Let f and g be integrable functions on $[a, b]$.

(a) Show fg is integrable on $[a, b]$. *Hint*: Use Exercise 33.7 and $4fg = (f + g)^2 - (f - g)^2$.

(b) Show $\max(f, g)$ and $\min(f, g)$ are integrable on $[a, b]$. *Hint*: Exercise 17.8.

33.9 Let (f_n) be a sequence of integrable functions on $[a, b]$, and suppose $f_n \to f$ uniformly on $[a, b]$. Prove f is integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Compare this result with Theorems 25.2 and 33.11.

33.10 Let $f(x) = \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Show f is integrable on $[-1, 1]$. *Hint*: See the answer to Exercise 33.11(c).

33.11 Let $f(x) = x\ \mathrm{sgn}(\sin\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$.

(a) Show f is not piecewise continuous on $[-1, 1]$.

(b) Show f is not piecewise monotonic on $[-1, 1]$.

(c) Show f is integrable on $[-1, 1]$.

33.12 Let f be the function described in Exercise 17.14.

(a) Show f is not piecewise continuous or piecewise monotonic on any interval $[a, b]$.

(b) Show f is integrable on every interval $[a, b]$ and $\int_a^b f = 0$.

33.13 Suppose f and g are continuous functions on $[a, b]$ such that $\int_a^b f = \int_a^b g$. Prove there exists x in (a, b) such that $f(x) = g(x)$.

33.14 **(a)** Prove the following generalization of the Intermediate Value Theorem for Integrals. If f and g are continuous functions on $[a, b]$ and $g(t) \geq 0$ for all $t \in [a, b]$, then there exists x in (a, b) such that

$$\int_a^b f(t)g(t)\,dt = f(x)\int_a^b g(t)\,dt.$$

(b) Show Theorem 33.9 is a special case of part (a).

(c) Does the conclusion in part (a) hold if $[a, b] = [-1, 1]$ and $f(t) = g(t) = t$ for all t?

33.15 For integers $n \geq 3$, define the function f_n on $[0,1]$ by the rules:

$$f_n(0) = f_n\left(\frac{2}{n}\right) = f_n(1) = 0 \quad \text{and} \quad f_n\left(\frac{1}{n}\right) = n,$$

and so that its graph is a straight line from $(0,0)$ to $(\frac{1}{n}, n)$, from $(\frac{1}{n}, n)$ to $(\frac{2}{n}, 0)$, and from $(\frac{2}{n}, 0)$ to $(1,0)$.

(a) Graph f_3, f_4 and f_5.

(b) Show $f_n \to 0$ pointwise on $[0,1]$.

(c) Show $\lim_n \int_0^1 f_n(x)\,dx \neq \int_0^1 0\,dx$. Why doesn't this contradict the Dominated Convergence Theorem?

§34 Fundamental Theorem of Calculus

There are two versions of the Fundamental Theorem of Calculus. Each says, roughly speaking, that differentiation and integration are inverse operations. In fact, our first version [Theorem 34.1] says "the integral of the derivative of a function is given by the function," and our second version [Theorem 34.3] says "the derivative of the integral of a continuous function is the function." It is somewhat traditional for books to prove our second version first and use it to prove our first version, although some books do avoid this approach. F. Cunningham, Jr. [18] offers some good reasons for avoiding the traditional approach:

(a) Theorem 34.3 implies Theorem 34.1 only for functions g whose derivative g' is continuous; see Exercise 34.1.

(b) Making Theorem 34.1 depend on Theorem 34.3 obscures the fact that the two theorems say different things, have different applications, and may leave the impression Theorem 34.3 is *the* fundamental theorem.

(c) The need for Theorem 34.1 in calculus is immediate and easily motivated.

In what follows, we say a function h defined on (a,b) is *integrable* on $[a,b]$ if every extension of h to $[a,b]$ is integrable. In view of

Exercise 32.7, the value $\int_a^b h$ will not depend on the values of the extensions at a or b.

34.1 Fundamental Theorem of Calculus I.
If g is a continuous function on $[a,b]$ that is differentiable on (a,b), and if g' is integrable on $[a,b]$, then

$$\int_a^b g' = g(b) - g(a). \tag{1}$$

Proof
Let $\epsilon > 0$. By Theorem 32.5, there exists a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a,b]$ such that

$$U(g', P) - L(g', P) < \epsilon. \tag{2}$$

We apply the Mean Value Theorem 29.3 to each interval $[t_{k-1}, t_k]$ to obtain x_k in (t_{k-1}, t_k) for which

$$(t_k - t_{k-1})g'(x_k) = g(t_k) - g(t_{k-1}).$$

Hence we have

$$g(b) - g(a) = \sum_{k=1}^{n}[g(t_k) - g(t_{k-1})] = \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}).$$

It follows that

$$L(g', P) \le g(b) - g(a) \le U(g', P); \tag{3}$$

see Definition 32.1. Since

$$L(g', P) \le \int_a^b g' \le U(g', P),$$

inequalities (2) and (3) imply

$$\left| \int_a^b g' - [g(b) - g(a)] \right| < \epsilon.$$

Since ϵ is arbitary, (1) holds. ■

The integration formulas in calculus all rely in the end on Theorem 34.1.

Example 1
If $g(x) = \frac{x^{n+1}}{n+1}$, then $g'(x) = x^n$, so

$$\int_a^b x^n\,dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} = \frac{b^{n+1} - a^{n+1}}{n+1}. \qquad (1)$$

In particular,

$$\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}.$$

Formula (1) is valid for any powers n for which $g(x) = \frac{x^{n+1}}{n+1}$ is defined on $[a, b]$. See Examples 3 and 4 in §28 and Exercises 29.15 and 37.5. For example,

$$\int_a^b \sqrt{x}\,dx = \int_a^b x^{1/2}\,dx = \frac{2}{3}[b^{3/2} - a^{3/2}] \quad \text{for} \quad 0 \le a < b.$$

□

34.2 Theorem [Integration by Parts].
If u and v are continuous functions on $[a, b]$ that are differentiable on (a, b), and if u' and v' are integrable on $[a, b]$, then

$$\int_a^b u(x)v'(x)\,dx + \int_a^b u'(x)v(x)\,dx = u(b)v(b) - u(a)v(a). \qquad (1)$$

Proof
Let $g = uv$; then $g' = uv' + u'v$ by Theorem 28.3. Exercise 33.8 shows g' is integrable. Now Theorem 34.1 shows

$$\int_a^b g'(x)\,dx = g(b) - g(a) = u(b)v(b) - u(a)v(a),$$

so (1) holds. ■

Note the use of Exercise 33.8 above can be avoided if u' and v' are continuous, which is normally the case.

Example 2
Here is a simple application of integration by parts. To calculate $\int_0^\pi x \cos x\,dx$, we note the integrand has the form $u(x)v'(x)$ where $u(x) = x$ and $v(x) = \sin x$. Hence

$$\int_0^\pi x\cos x\,dx = u(\pi)v(\pi) - u(0)v(0) - \int_0^\pi 1\cdot\sin x\,dx = -\int_0^\pi \sin x\,dx = -2.$$

□

In what follows we use the convention $\int_a^b f = -\int_b^a f$ for $a > b$.

34.3 Fundamental Theorem of Calculus II.
Let f be an integrable function on $[a,b]$. For x in $[a,b]$, let

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is continuous on $[a,b]$. If f is continuous at x_0 in (a,b), then F is differentiable at x_0 and

$$F'(x_0) = f(x_0).$$

Proof
Select $B > 0$ so that $|f(x)| \le B$ for all $x \in [a,b]$. If $x, y \in [a,b]$ and $|x-y| < \frac{\epsilon}{B}$ where $x < y$, say, then

$$|F(y) - F(x)| = \left|\int_x^y f(t)\,dt\right| \le \int_x^y |f(t)|\,dt \le \int_x^y B\,dt = B(y-x) < \epsilon.$$

This shows F is [uniformly] continuous on $[a,b]$.

Suppose f is continuous at x_0 in (a,b). Observe

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt$$

for $x \ne x_0$. The trick is to observe

$$f(x_0) = \frac{1}{x - x_0}\int_{x_0}^x f(x_0)\,dt,$$

and therefore

$$\frac{F(x) - f(x_0)}{x - x_0} - f(x_0) = \frac{1}{x - x_0}\int_{x_0}^x [f(t) - f(x_0)]\,dt. \qquad (1)$$

Let $\epsilon > 0$. Since f is continuous at x_0, there exists $\delta > 0$ such that

$$t \in (a,b) \quad \text{and} \quad |t - x_0| < \delta \quad \text{imply} \quad |f(t) - f(x_0)| < \epsilon;$$

see Theorem 17.2. It follows from (1) that

$$\left|\frac{F(x)-F(x_0)}{x-x_0}-f(x_0)\right|\le\epsilon$$

for x in (a,b) satisfying $|x-x_0|<\delta$; the cases $x>x_0$ and $x<x_0$ require separate arguments. We have just shown

$$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}=f(x_0).$$

In other words, $F'(x_0)=f(x_0)$. ■

A useful technique of integration is known as "substitution." A more accurate description of the process is "change of variable." The technique is the reverse of the chain rule.

34.4 Theorem [Change of Variable].
Let u be a differentiable function on an open interval J such that u' is continuous, and let I be an open interval such that $u(x)\in I$ for all $x\in J$. If f is continuous on I, then $f\circ u$ is continuous on J and

$$\int_a^b f\circ u(x)u'(x)\,dx=\int_{u(a)}^{u(b)} f(u)\,du \tag{1}$$

for a,b in J.

Note $u(a)$ need not be less than $u(b)$, even if $a<b$.

Proof
The continuity of $f\circ u$ follows from Theorem 17.5. Fix c in I and let $F(u)=\int_c^u f(t)\,dt$. Then $F'(u)=f(u)$ for all $u\in I$ by Theorem 34.3. Let $g=F\circ u$. By the Chain Rule 28.4, we have

$$g'(x)=F'(u(x))\cdot u'(x)=f(u(x))\cdot u'(x),$$

so by Theorem 34.1

$$\begin{aligned}\int_a^b f\circ u(x)u'(x)\,dx&=\int_a^b g'(x)\,dx=g(b)-g(a)=F(u(b))-F(u(a))\\&=\int_c^{u(b)} f(t)\,dt-\int_c^{u(a)} f(t)\,dt=\int_{u(a)}^{u(b)} f(t)\,dt.\end{aligned}$$

This proves (1). ■

Example 3

Let g be a one-to-one differentiable function on an open interval I. Then $J = g(I)$ is an open interval, and the inverse function g^{-1} is differentiable on J by Theorem 29.9. We show

$$\int_a^b g(x)\,dx + \int_{g(a)}^{g(b)} g^{-1}(u)\,du = b \cdot g(b) - a \cdot g(a) \tag{1}$$

for a, b in I.

We put $f = g^{-1}$ and $u = g$ in the change of variable formula to obtain

$$\int_a^b g^{-1} \circ g(x) g'(x)\,dx = \int_{g(a)}^{g(b)} g^{-1}(u)\,du.$$

Since $g^{-1} \circ g(x) = x$ for x in I, we obtain

$$\int_{g(a)}^{g(b)} g^{-1}(u)\,du = \int_a^b x g'(x)\,dx.$$

Now integrate by parts with $u(x) = x$ and $v(x) = g(x)$:

$$\int_{g(a)}^{g(b)} g^{-1}(u)\,du = b \cdot g(b) - a \cdot g(a) - \int_a^b g(x)\,dx.$$

This is formula (1). □

Exercises

34.1 Use Theorem 34.3 to prove Theorem 34.1 for the case g' is continuous. *Hint*: Let $F(x) = \int_a^x g'$; then $F' = g'$. Apply Corollary 29.5.

34.2 Calculate
(a) $\lim_{x \to 0} \frac{1}{x} \int_0^x e^{t^2}\,dt$ **(b)** $\lim_{h \to 0} \frac{1}{h} \int_3^{3+h} e^{t^2}\,dt$.

34.3 Let f be defined as follows: $f(t) = 0$ for $t < 0$; $f(t) = t$ for $0 \le t \le 1$; $f(t) = 4$ for $t > 1$.

(a) Determine the function $F(x) = \int_0^x f(t)\,dt$.

(b) Sketch F. Where is F continuous?

(c) Where is F differentiable? Calculate F' at the points of differentiability.

34.4 Repeat Exercise 34.3 for f where $f(t) = t$ for $t < 0$; $f(t) = t^2 + 1$ for $0 \le t \le 2$; $f(t) = 0$ for $t > 2$.

34.5 Let f be a continuous function on $\mathbb{R}$ and define

$$F(x) = \int_{x-1}^{x+1} f(t)\,dt \quad \text{for} \quad x \in \mathbb{R}.$$

Show F is differentiable on $\mathbb{R}$ and compute F'.

34.6 Let f be a continuous function on $\mathbb{R}$ and define

$$G(x) = \int_{0}^{\sin x} f(t)\,dt \quad \text{for} \quad x \in \mathbb{R}.$$

Show G is differentiable on $\mathbb{R}$ and compute G'.

34.7 Use change of variables to integrate $\int_0^1 x\sqrt{1-x^2}\,dx$.

34.8 **(a)** Use integration by parts to evaluate

$$\int_0^1 x \arctan x\,dx.$$

Hint: Let $u(x) = \arctan x$, so that $u'(x) = \frac{1}{1+x^2}$.

(b) If you used $v(x) = \frac{x^2}{2}$ in part (a), do the computation again with $v(x) = \frac{x^2+1}{2}$. This interesting example is taken from J.L. Borman [10].

34.9 Use Example 3 to show $\int_0^{1/2} \arcsin x\,dx = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$.

34.10 Let g be a strictly increasing continuous function mapping $[0,1]$ onto $[0,1]$. Give a geometric argument showing $\int_0^1 g(x)dx + \int_0^1 g^{-1}(u)du = 1$.

34.11 Suppose f is a continuous function on $[a,b]$. Show that if $\int_a^b f(x)^2 dx = 0$, then $f(x) = 0$ for all x in $[a,b]$. *Hint*: See Theorem 33.4.

34.12 Show that if f is a continuous real-valued function on $[a,b]$ satisfying $\int_a^b f(x)g(x)\,dx = 0$ for every continuous function g on $[a,b]$, then $f(x) = 0$ for all x in $[a,b]$.