# 3.7 Related Rates

In this section we look at problems that ask for the rate at which some variable changes. In each case the rate is a derivative that has to be computed from the rate at which some other variable (or perhaps several variables) is known to change. To find it, we write an equation that relates the variables involved and differentiate it to get an equation that relates the rate we seek to the rates we know. The problem of finding a rate you cannot measure easily from some other rates that you can is called a *related rates problem*.

## Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If $V$ is the volume and $r$ is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3}\pi r^3.$$

Using the Chain Rule, we differentiate to find the related rates equation

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So if we know the radius $r$ of the balloon and the rate $dV/dt$ at which the volume is increasing at a given instant of time, then we can solve this last equation for $dr/dt$ to find how fast the radius is increasing at that instant. Note that it is easier to measure directly the rate of increase of the volume than it is to measure the increase in the radius. The related rates equation allows us to calculate $dr/dt$ from $dV/dt$.

Very often the key to relating the variables in a related rates problem is drawing a picture that shows the geometric relations between them, as illustrated in the following example.

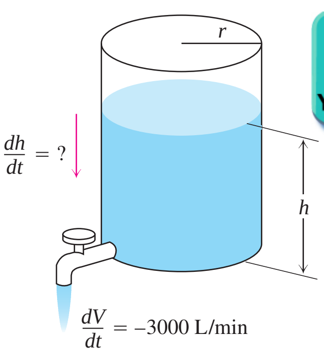



**FIGURE 3.42** The rate of change of fluid volume in a cylindrical tank is related to the rate of change of fluid level in the tank (Example 1).

### EXAMPLE 1 Pumping Out a Tank

How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L/min?

**Solution** We draw a picture of a partially filled vertical cylindrical tank, calling its radius $r$ and the height of the fluid $h$ (Figure 3.42). Call the volume of the fluid $V$.

As time passes, the radius remains constant, but $V$ and $h$ change. We think of $V$ and $h$ as differentiable functions of time and use $t$ to represent time. We are told that

$$\frac{dV}{dt} = -3000.$$

We pump out at the rate of 3000 L/min. The rate is negative because the volume is decreasing.

We are asked to find

$$\frac{dh}{dt}.$$

How fast will the fluid level drop?

To find $dh/dt$, we first write an equation that relates $h$ to $V$. The equation depends on the units chosen for $V$, $r$, and $h$. With $V$ in liters and $r$ and $h$ in meters, the appropriate equation for the cylinder's volume is

$$V = 1000\pi r^2 h$$

because a cubic meter contains 1000 L.

Since $V$ and $h$ are differentiable functions of $t$, we can differentiate both sides of the equation $V = 1000\pi r^2 h$ with respect to $t$ to get an equation that relates $dh/dt$ to $dV/dt$:

$$\frac{dV}{dt} = 1000\pi r^2 \frac{dh}{dt}.$$

$r$ is a constant.

We substitute the known value $dV/dt = -3000$ and solve for $dh/dt$:

$$\frac{dh}{dt} = \frac{-3000}{1000\pi r^2} = -\frac{3}{\pi r^2}.$$

The fluid level will drop at the rate of $3/(\pi r^2)$ m/min.

The equation $dh/dt = -3/\pi r^2$ shows how the rate at which the fluid level drops depends on the tank's radius. If $r$ is small, $dh/dt$ will be large; if $r$ is large, $dh/dt$ will be small.

$$\text{If } r = 1 \text{ m:} \qquad \frac{dh}{dt} = -\frac{3}{\pi} \approx -0.95 \text{ m/min} = -95 \text{ cm/min}.$$

$$\text{If } r = 10 \text{ m:} \qquad \frac{dh}{dt} = -\frac{3}{100\pi} \approx -0.0095 \text{ m/min} = -0.95 \text{ cm/min}.$$

### Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants*. Use $t$ for time. Assume that all variables are differentiable functions of $t$.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables*. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to $t$*. Then express the rate you want in terms of the rate and variables whose values you know.
6. *Evaluate*. Use known values to find the unknown rate.

Video

You Try It

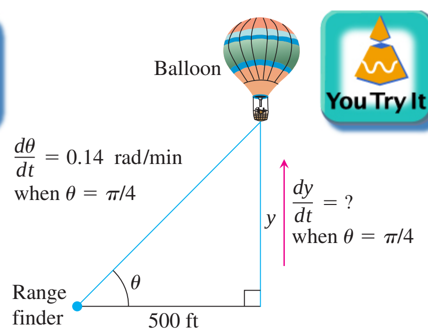



**FIGURE 3.43** The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

## EXAMPLE 2 A Rising Balloon

A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

**Solution** We answer the question in six steps.

1. *Draw a picture and name the variables and constants* (Figure 3.43). The variables in the picture are

   $\theta =$ the angle in radians the range finder makes with the ground.

   $y =$ the height in feet of the balloon.

We let $t$ represent time in minutes and assume that $\theta$ and $y$ are differentiable functions of $t$.

The one constant in the picture is the distance from the range finder to the liftoff point (500 ft). There is no need to give it a special symbol.

2. *Write down the additional numerical information.*

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \qquad \text{when} \qquad \theta = \frac{\pi}{4}$$

3. *Write down what we are to find*. We want $dy/dt$ when $\theta = \pi/4$.

**4.** *Write an equation that relates the variables y and θ.*

$$\frac{y}{500} = \tan \theta \qquad \text{or} \qquad y = 500 \tan \theta$$

**5.** *Differentiate with respect to t using the Chain Rule.* The result tells how $dy/dt$ (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 500\,(\sec^2 \theta)\,\frac{d\theta}{dt}$$

**6.** *Evaluate with* $\theta = \pi/4$ *and* $d\theta/dt = 0.14$ *to find* $dy/dt$.

$$\frac{dy}{dt} = 500\left(\sqrt{2}\right)^2(0.14) = 140 \qquad \sec\frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 140 ft/min. ■

## EXAMPLE 3 A Highway Chase

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

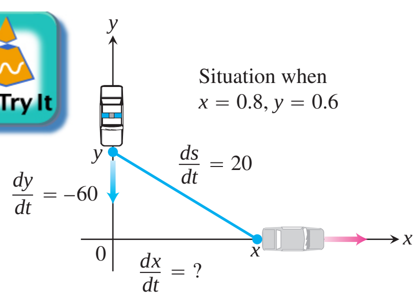



**FIGURE 3.44** The speed of the car is related to the speed of the police cruiser and the rate of change of the distance between them (Example 3).

**Solution** We picture the car and cruiser in the coordinate plane, using the positive $x$-axis as the eastbound highway and the positive $y$-axis as the southbound highway (Figure 3.44). We let $t$ represent time and set

$$\begin{aligned} x &= \text{position of car at time } t \\ y &= \text{position of cruiser at time } t \\ s &= \text{distance between car and cruiser at time } t. \end{aligned}$$

We assume that $x$, $y$, and $s$ are differentiable functions of $t$.

We want to find $dx/dt$ when

$$x = 0.8 \text{ mi}, \qquad y = 0.6 \text{ mi}, \qquad \frac{dy}{dt} = -60 \text{ mph}, \qquad \frac{ds}{dt} = 20 \text{ mph}.$$

Note that $dy/dt$ is negative because $y$ is decreasing.

We differentiate the distance equation

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$\begin{aligned} 2s\frac{ds}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \\ \frac{ds}{dt} &= \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) \\ &= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right). \end{aligned}$$

Finally, use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for $dx/dt$.

$$20 = \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}\left(0.8\,\frac{dx}{dt} + (0.6)(-60)\right)$$

$$\frac{dx}{dt} = \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph. ■

You Try It

Video

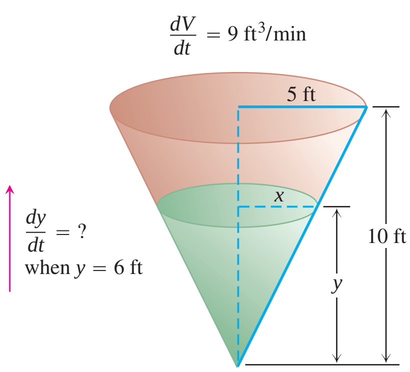



**FIGURE 3.45** The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 4).

## EXAMPLE 4 Filling a Conical Tank

Water runs into a conical tank at the rate of 9 ft$^3$/min. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

**Solution** Figure 3.45 shows a partially filled conical tank. The variables in the problem are

$V$ = volume (ft$^3$) of the water in the tank at time $t$ (min)

$x$ = radius (ft) of the surface of the water at time $t$

$y$ = depth (ft) of water in tank at time $t$.

We assume that $V$, $x$, and $y$ are differentiable functions of $t$. The constants are the dimensions of the tank. We are asked for $dy/dt$ when

$$y = 6 \text{ ft} \quad \text{and} \quad \frac{dV}{dt} = 9 \text{ ft}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

This equation involves $x$ as well as $V$ and $y$. Because no information is given about $x$ and $dx/dt$ at the time in question, we need to eliminate $x$. The similar triangles in Figure 3.45 give us a way to express $x$ in terms of $y$:

$$\frac{x}{y} = \frac{5}{10} \quad \text{or} \quad x = \frac{y}{2}.$$

Therefore,

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

to give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12}\cdot 3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}.$$

Finally, use $y = 6$ and $dV/dt = 9$ to solve for $dy/dt$.

$$9 = \frac{\pi}{4}(6)^2\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about 0.32 ft/min. ■

## EXERCISES 3.7

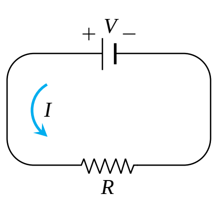
1. **Area** Suppose that the radius $r$ and area $A = \pi r^2$ of a circle are differentiable functions of $t$. Write an equation that relates $dA/dt$ to $dr/dt$.

2. **Surface area** Suppose that the radius $r$ and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of $t$. Write an equation that relates $dS/dt$ to $dr/dt$.

3. **Volume** The radius $r$ and height $h$ of a right circular cylinder are related to the cylinder's volume $V$ by the formula $V = \pi r^2 h$.
   a. How is $dV/dt$ related to $dh/dt$ if $r$ is constant?
   b. How is $dV/dt$ related to $dr/dt$ if $h$ is constant?
   c. How is $dV/dt$ related to $dr/dt$ and $dh/dt$ if neither $r$ nor $h$ is constant?

4. **Volume** The radius $r$ and height $h$ of a right circular cone are related to the cone's volume $V$ by the equation $V = (1/3)\pi r^2 h$.
   a. How is $dV/dt$ related to $dh/dt$ if $r$ is constant?
   b. How is $dV/dt$ related to $dr/dt$ if $h$ is constant?
   c. How is $dV/dt$ related to $dr/dt$ and $dh/dt$ if neither $r$ nor $h$ is constant?

5. **Changing voltage** The voltage $V$ (volts), current $I$ (amperes), and resistance $R$ (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that $V$ is increasing at the rate of 1 volt/sec while $I$ is decreasing at the rate of 1/3 amp/sec. Let $t$ denote time in seconds.




   a. What is the value of $dV/dt$?
   b. What is the value of $dI/dt$?
   c. What equation relates $dR/dt$ to $dV/dt$ and $dI/dt$?
   d. Find the rate at which $R$ is changing when $V = 12$ volts and $I = 2$ amp. Is $R$ increasing, or decreasing?

6. **Electrical power** The power $P$ (watts) of an electric circuit is related to the circuit's resistance $R$ (ohms) and current $I$ (amperes) by the equation $P = RI^2$.
   a. How are $dP/dt$, $dR/dt$, and $dI/dt$ related if none of $P$, $R$, and $I$ are constant?
   b. How is $dR/dt$ related to $dI/dt$ if $P$ is constant?

7. **Distance** Let $x$ and $y$ be differentiable functions of $t$ and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the $xy$-plane.
   a. How is $ds/dt$ related to $dx/dt$ if $y$ is constant?
   b. How is $ds/dt$ related to $dx/dt$ and $dy/dt$ if neither $x$ nor $y$ is constant?
   c. How is $dx/dt$ related to $dy/dt$ if $s$ is constant?

8. **Diagonals** If $x$, $y$, and $z$ are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$.
   a. Assuming that $x$, $y$, and $z$ are differentiable functions of $t$, how is $ds/dt$ related to $dx/dt$, $dy/dt$, and $dz/dt$?
   b. How is $ds/dt$ related to $dy/dt$ and $dz/dt$ if $x$ is constant?
   c. How are $dx/dt$, $dy/dt$, and $dz/dt$ related if $s$ is constant?

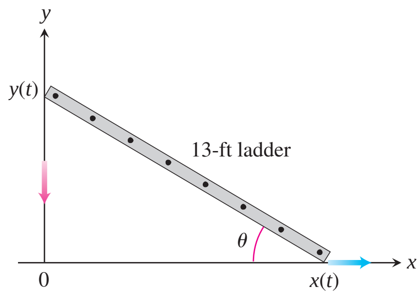
9. **Area** The area $A$ of a triangle with sides of lengths $a$ and $b$ enclosing an angle of measure $\theta$ is
$$A = \frac{1}{2}ab\sin\theta.$$
   a. How is $dA/dt$ related to $d\theta/dt$ if $a$ and $b$ are constant?
   b. How is $dA/dt$ related to $d\theta/dt$ and $da/dt$ if only $b$ is constant?
   c. How is $dA/dt$ related to $d\theta/dt$, $da/dt$, and $db/dt$ if none of $a$, $b$, and $\theta$ are constant?

10. **Heating a plate** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?

11. **Changing dimensions in a rectangle** The length $l$ of a rectangle is decreasing at the rate of 2 cm/sec while the width $w$ is increasing at the rate of 2 cm/sec. When $l = 12$ cm and $w = 5$ cm, find the rates of change of **(a)** the area, **(b)** the perimeter, and **(c)** the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

12. **Changing dimensions in a rectangular box** Suppose that the edge lengths $x$, $y$, and $z$ of a closed rectangular box are changing at the following rates:
$$\frac{dx}{dt} = 1 \text{ m/sec}, \quad \frac{dy}{dt} = -2 \text{ m/sec}, \quad \frac{dz}{dt} = 1 \text{ m/sec}.$$
Find the rates at which the box's **(a)** volume, **(b)** surface area, and **(c)** diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

13. **A sliding ladder** A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.
   a. How fast is the top of the ladder sliding down the wall then?
   b. At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
   c. At what rate is the angle $\theta$ between the ladder and the ground changing then?




**14. Commercial air traffic** Two commercial airplanes are flying at 40,000 ft along straight-line courses that intersect at right angles. Plane $A$ is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 2000 yd). Plane $B$ is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when $A$ is 5 nautical miles from the intersection point and $B$ is 12 nautical miles from the intersection point?

**15. Flying a kite** A girl flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

**16. Boring a cylinder** The mechanics at Lincoln Automotive are reboring a 6-in.-deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.?

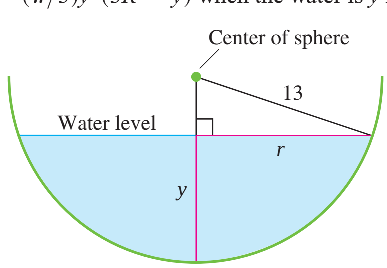
**17. A growing sand pile** Sand falls from a conveyor belt at the rate of 10 $m^3$/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high? Answer in centimeters per minute.

**18. A draining conical reservoir** Water is flowing at the rate of 50 $m^3$/min from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m.

**a.** How fast (centimeters per minute) is the water level falling when the water is 5 m deep?

**b.** How fast is the radius of the water's surface changing then? Answer in centimeters per minute.

**19. A draining hemispherical reservoir** Water is flowing at the rate of 6 $m^3$/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius $R$ is $V = (\pi/3)y^2(3R - y)$ when the water is $y$ meters deep.




**a.** At what rate is the water level changing when the water is 8 m deep?

**b.** What is the radius $r$ of the water's surface when the water is $y$ m deep?

**c.** At what rate is the radius $r$ changing when the water is 8 m deep?

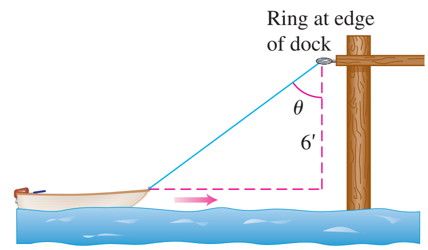
**20. A growing raindrop** Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.

**21. The radius of an inflating balloon** A spherical balloon is inflated with helium at the rate of $100\pi$ $ft^3$/min. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?

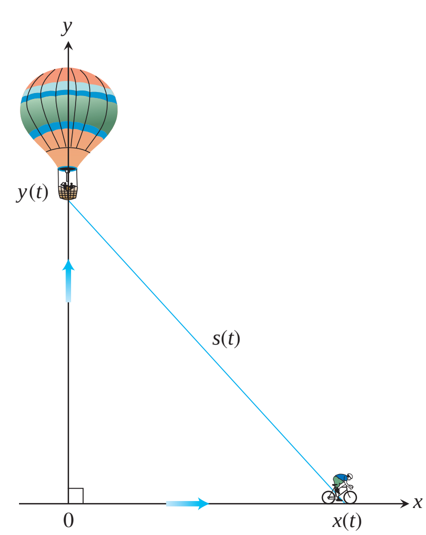
**22. Hauling in a dinghy** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at the rate of 2 ft/sec.

**a.** How fast is the boat approaching the dock when 10 ft of rope are out?

**b.** At what rate is the angle $\theta$ changing then (see the figure)?



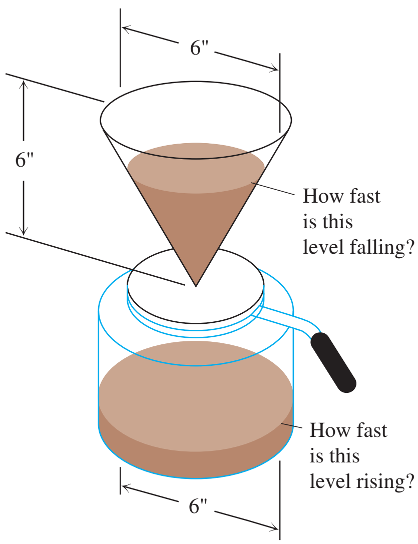

**23. A balloon and a bicycle** A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 sec later?



**24. Making coffee** Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 $\text{in}^3/\text{min}$.

**a.** How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

**b.** How fast is the level in the cone falling then?




**25. Cardiac output** In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 L/min. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where $Q$ is the number of milliliters of $CO_2$ you exhale in a minute and $D$ is the difference between the $CO_2$ concentration (ml/L) in the blood pumped to the lungs and the $CO_2$ concentration in the blood returning from the lungs. With $Q = 233$ ml/min and $D = 97 - 56 = 41$ ml/L,

$$y = \frac{233 \text{ ml/min}}{41 \text{ ml/L}} \approx 5.68 \text{ L/min},$$

fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that $D$ is decreasing at the rate of 2 units a minute but that $Q$ remains unchanged. What is happening to the cardiac output?

**26. Cost, revenue, and profit** A company can manufacture $x$ items at a cost of $c(x)$ thousand dollars, a sales revenue of $r(x)$ thousand dollars, and a profit of $p(x) = r(x) - c(x)$ thousand dollars. Find $dc/dt$, $dr/dt$, and $dp/dt$ for the following values of $x$ and $dx/dt$.

**a.** $r(x) = 9x$, $c(x) = x^3 - 6x^2 + 15x$, and $dx/dt = 0.1$ when $x = 2$

**b.** $r(x) = 70x$, $c(x) = x^3 - 6x^2 + 45/x$, and $dx/dt = 0.05$ when $x = 1.5$

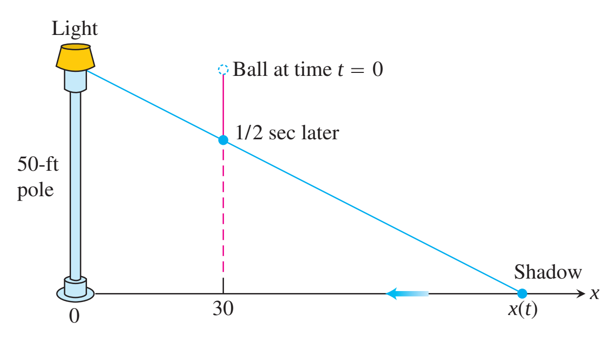
**27. Moving along a parabola** A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its $x$-coordinate (measured in meters) increases at a steady 10 m/sec. How fast is the angle of inclination $\theta$ of the line joining the particle to the origin changing when $x = 3$ m?

**28. Moving along another parabola** A particle moves from right to left along the parabolic curve $y = \sqrt{-x}$ in such a way that its $x$-coordinate (measured in meters) decreases at the rate of 8 m/sec. How fast is the angle of inclination $\theta$ of the line joining the particle to the origin changing when $x = -4$?

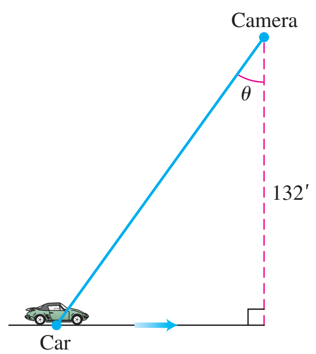
**29. Motion in the plane** The coordinates of a particle in the metric $xy$-plane are differentiable functions of time $t$ with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point $(5, 12)$?

**30. A moving shadow** A man 6 ft tall walks at the rate of 5 ft/sec toward a streetlight that is 16 ft above the ground. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing when he is 10 ft from the base of the light?

**31. Another moving shadow** A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ ft in $t$ sec.)




**32. Videotaping a moving car** You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mi/h (264 ft/sec). How fast will your camera angle $\theta$ be changing when the car is right in front of you? A half second later?

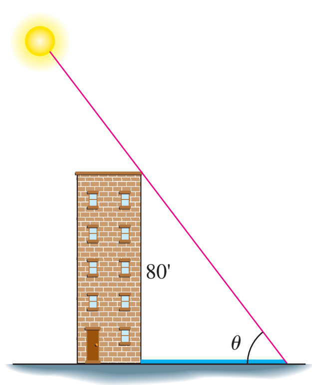



Exercises

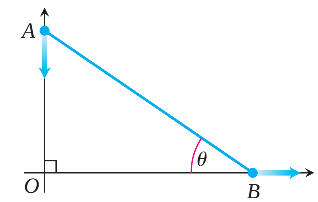
**33. A melting ice layer** A spherical iron ball 8 in. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of $10 \text{ in}^3/\text{min}$, how fast is the thickness of the ice decreasing when it is 2 in. thick? How fast is the outer surface area of ice decreasing?

**34. Highway patrol** A highway patrol plane flies 3 mi above a level, straight road at a steady 120 mi/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi, the line-of-sight distance is decreasing at the rate of 160 mi/h. Find the car's speed along the highway.

**35. A building's shadow** On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long. At the moment in question, the angle $\theta$ the sun makes with the ground is increasing at the rate of $0.27°/\text{min}$. At what rate is the shadow decreasing? (Remember to use radians. Express your answer in inches per minute, to the nearest tenth.)




**36. Walkers** $A$ and $B$ are walking on straight streets that meet at right angles. $A$ approaches the intersection at 2 m/sec; $B$ moves away from the intersection 1 m/sec. At what rate is the angle $\theta$ changing when $A$ is 10 m from the intersection and $B$ is 20 m from the intersection? Express your answer in degrees per second to the nearest degree.

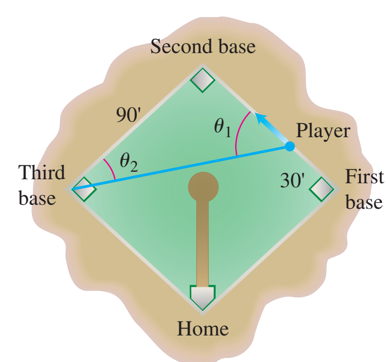



**37. Baseball players** A baseball diamond is a square 90 ft on a side. A player runs from first base to second at a rate of 16 ft/sec.

**a.** At what rate is the player's distance from third base changing when the player is 30 ft from first base?

**b.** At what rates are angles $\theta_1$ and $\theta_2$ (see the figure) changing at that time?

**c.** The player slides into second base at the rate of 15 ft/sec. At what rates are angles $\theta_1$ and $\theta_2$ changing as the player touches base?




**38. Ships** Two ships are steaming straight away from a point $O$ along routes that make a 120° angle. Ship $A$ moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yd). Ship $B$ moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

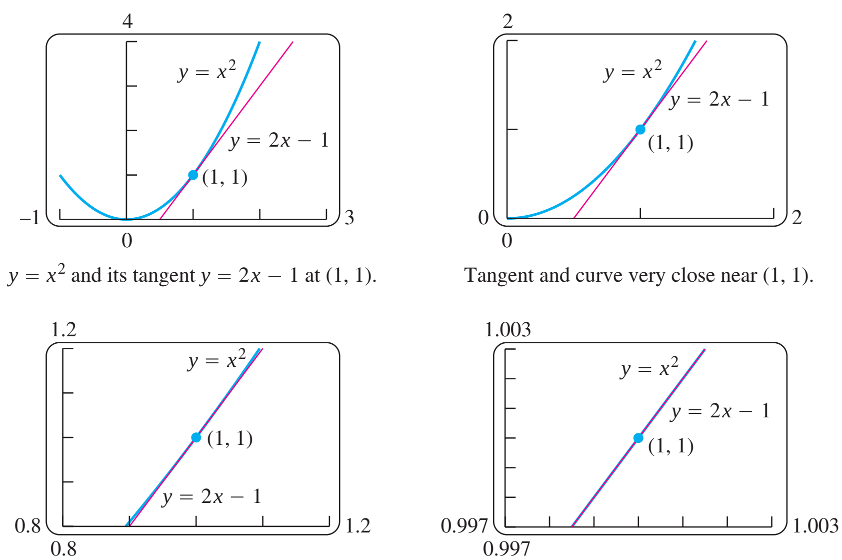
# 3.8 Linearization and Differentials

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and are easier to work with. The approximating functions discussed in this section are called *linearizations*, and they are based on tangent lines. Other approximating functions, such as polynomials, are discussed in Chapter 11.

We introduce new variables $dx$ and $dy$, called *differentials*, and define them in a way that makes Leibniz's notation for the derivative $dy/dx$ a true ratio. We use $dy$ to estimate error in measurement and sensitivity of a function to change. Application of these ideas then provides for a precise proof of the Chain Rule (Section 3.5).

## Linearization

As you can see in Figure 3.46, the tangent to the curve $y = x^2$ lies close to the curve near the point of tangency. For a brief interval to either side, the $y$-values along the tangent line give good approximations to the $y$-values on the curve. We observe this phenomenon by zooming in on the two graphs at the point of tangency or by looking at tables of values for the difference between $f(x)$ and its tangent line near the $x$-coordinate of the point of tangency. Locally, every differentiable curve behaves like a straight line.




$y = x^2$ and its tangent $y = 2x - 1$ at (1, 1).

Tangent and curve very close near (1, 1).

Tangent and curve very close throughout entire $x$-interval shown.

Tangent and curve closer still. Computer screen cannot distinguish tangent from curve on this $x$-interval.

**FIGURE 3.46** The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

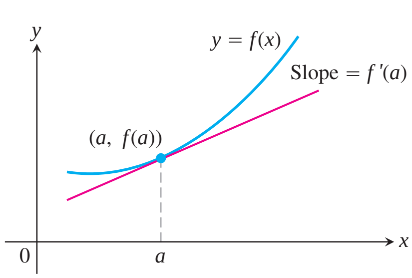



**FIGURE 3.47** The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

In general, the tangent to $y = f(x)$ at a point $x = a$, where $f$ is differentiable (Figure 3.47), passes through the point $(a, f(a))$, so its point-slope equation is

$$y = f(a) + f'(a)(x - a).$$

Thus, this tangent line is the graph of the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

For as long as this line remains close to the graph of $f$, $L(x)$ gives a good approximation to $f(x)$.

**DEFINITIONS Linearization, Standard Linear Approximation**

If $f$ is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of $f$ at $a$. The approximation

$$f(x) \approx L(x)$$

of $f$ by $L$ is the **standard linear approximation** of $f$ at $a$. The point $x = a$ is the **center** of the approximation.

You Try It

**EXAMPLE 1** Finding a Linearization

Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$ (Figure 3.48).

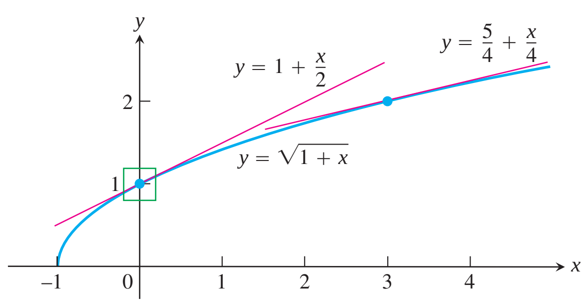



**FIGURE 3.48** The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.49 shows a magnified view of the small window about 1 on the $y$-axis.

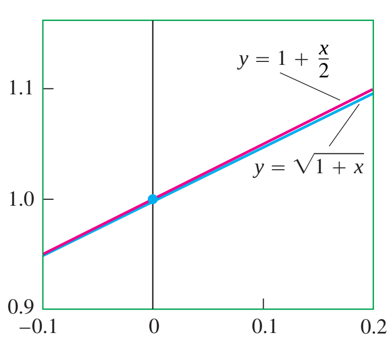



**FIGURE 3.49** Magnified view of the window in Figure 3.48.

**Solution** Since

$$f'(x) = \frac{1}{2}\left(1 + x\right)^{-1/2},$$

we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}\left(x - 0\right) = 1 + \frac{x}{2}.$$

See Figure 3.49. ■

Look at how accurate the approximation $\sqrt{1 + x} \approx 1 + (x/2)$ from Example 1 is for values of $x$ near 0.

As we move away from zero, we lose accuracy. For example, for $x = 2$, the linearization gives 2 as the approximation for $\sqrt{3}$, which is not even accurate to one decimal place.

Do not be misled by the preceding calculations into thinking that whatever we do with a linearization is better done with a calculator. In practice, we would never use a linearization to find a particular square root. The utility of a linearization is its ability to replace a complicated formula by a simpler one over an entire interval of values. If we have to work with $\sqrt{1 + x}$ for $x$ close to 0 and can tolerate the small amount of error involved, we can

| Approximation | True value | $\lvert$True value − approximation$\rvert$ |
| --- | --- | --- |
| $\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$ | 1.095445 | $<10^{-2}$ |
| $\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$ | 1.024695 | $<10^{-3}$ |
| $\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$ | 1.002497 | $<10^{-5}$ |

work with $1 + (x/2)$ instead. Of course, we then need to know how much error there is. We have more to say on the estimation of error in Chapter 11.

A linear approximation normally loses accuracy away from its center. As Figure 3.48 suggests, the approximation $\sqrt{1 + x} \approx 1 + (x/2)$ will probably be too crude to be useful near $x = 3$. There, we need the linearization at $x = 3$.

## EXAMPLE 2 Finding a Linearization at Another Point

Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 3$.

**Solution** We evaluate the equation defining $L(x)$ at $a = 3$. With

$$f(3) = 2, \qquad f'(3) = \frac{1}{2}\left(1 + x\right)^{-1/2}\bigg|_{x=3} = \frac{1}{4},$$

we have

$$L(x) = 2 + \frac{1}{4}\left(x - 3\right) = \frac{5}{4} + \frac{x}{4}.$$ ■

At $x = 3.2$, the linearization in Example 2 gives

$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx \frac{5}{4} + \frac{3.2}{4} = 1.250 + 0.800 = 2.050,$$

which differs from the true value $\sqrt{4.2} \approx 2.04939$ by less than one one-thousandth. The linearization in Example 1 gives

$$\sqrt{1 + x} = \sqrt{1 + 3.2} \approx 1 + \frac{3.2}{2} = 1 + 1.6 = 2.6,$$

a result that is off by more than 25%.

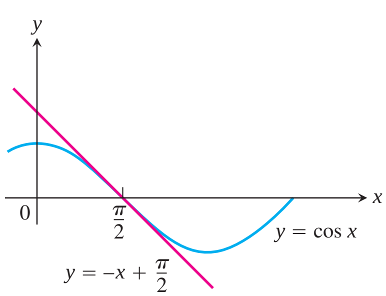



**FIGURE 3.50** The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$ (Example 3).

## EXAMPLE 3 Finding a Linearization for the Cosine Function

Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ (Figure 3.50).

**Solution** Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we have

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned}$$ ■

An important linear approximation for roots and powers is

$$(1 + x)^k \approx 1 + kx \qquad (x \text{ near } 0; \text{ any number } k)$$

(Exercise 15). This approximation, good for values of $x$ sufficiently close to zero, has broad application. For example, when $x$ is small,

$$\sqrt{1 + x} \approx 1 + \frac{1}{2}x \qquad k = 1/2$$

$$\frac{1}{1 - x} = (1 - x)^{-1} \approx 1 + (-1)(-x) = 1 + x \qquad k = -1;\ \text{replace } x \text{ by } -x.$$

$$\sqrt[3]{1 + 5x^4} = (1 + 5x^4)^{1/3} \approx 1 + \frac{1}{3}\left(5x^4\right) = 1 + \frac{5}{3}x^4 \qquad k = 1/3;\ \text{replace } x \text{ by } 5x^4.$$

$$\frac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-x^2) = 1 + \frac{1}{2}x^2 \qquad k = -1/2;\ \text{replace } x \text{ by } -x^2.$$

## Differentials

We sometimes use the Leibniz notation $dy/dx$ to represent the derivative of $y$ with respect to $x$. Contrary to its appearance, it is not a ratio. We now introduce two new variables $dx$ and $dy$ with the property that if their ratio exists, it will be equal to the derivative.

**DEFINITION Differential**

Let $y = f(x)$ be a differentiable function. The **differential $dx$** is an independent variable. The **differential $dy$** is

$$dy = f'(x)\,dx.$$

Unlike the independent variable $dx$, the variable $dy$ is always a dependent variable. It depends on both $x$ and $dx$. If $dx$ is given a specific value and $x$ is a particular number in the domain of the function $f$, then the numerical value of $dy$ is determined.

### EXAMPLE 4 Finding the Differential $dy$

**(a)** Find $dy$ if $y = x^5 + 37x$.

**(b)** Find the value of $dy$ when $x = 1$ and $dx = 0.2$.

**Solution**

**(a)** $dy = (5x^4 + 37)\,dx$

**(b)** Substituting $x = 1$ and $dx = 0.2$ in the expression for $dy$, we have

$$dy = (5 \cdot 1^4 + 37)0.2 = 8.4.$$

■

The geometric meaning of differentials is shown in Figure 3.51. Let $x = a$ and set $dx = \Delta x$. The corresponding change in $y = f(x)$ is

$$\Delta y = f(a + dx) - f(a).$$

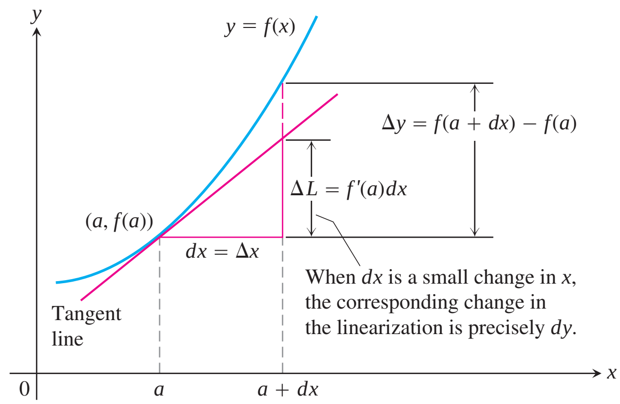



**FIGURE 3.51** Geometrically, the differential $dy$ is the change $\Delta L$ in the linearization of $f$ when $x = a$ changes by an amount $dx = \Delta x$.

The corresponding change in the tangent line $L$ is

$$\begin{aligned}\Delta L &= L(a + dx) - L(a)\\ &= \underbrace{f(a) + f'(a)[(a + dx) - a]}_{L(a+dx)} - \underbrace{f(a)}_{L(a)}\\ &= f'(a)\,dx.\end{aligned}$$

That is, the change in the linearization of $f$ is precisely the value of the differential $dy$ when $x = a$ and $dx = \Delta x$. Therefore, $dy$ represents the amount the tangent line rises or falls when $x$ changes by an amount $dx = \Delta x$.

If $dx \neq 0$, then the quotient of the differential $dy$ by the differential $dx$ is equal to the derivative $f'(x)$ because

$$dy \div dx = \frac{f'(x)\,dx}{dx} = f'(x) = \frac{dy}{dx}.$$

We sometimes write

$$df = f'(x)\,dx$$

in place of $dy = f'(x)\,dx$, calling $df$ the **differential of *f***. For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x\,dx.$$

Every differentiation formula like

$$\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{or} \qquad \frac{d(\sin u)}{dx} = \cos u\,\frac{du}{dx}$$

has a corresponding differential form like

$$d(u + v) = du + dv \qquad \text{or} \qquad d(\sin u) = \cos u\,du.$$

## EXAMPLE 5 Finding Differentials of Functions

**(a)** $d(\tan 2x) = \sec^2(2x)\, d(2x) = 2 \sec^2 2x\, dx$

**(b)** $d\left(\dfrac{x}{x+1}\right) = \dfrac{(x+1)\,dx - x\,d(x+1)}{(x+1)^2} = \dfrac{x\,dx + dx - x\,dx}{(x+1)^2} = \dfrac{dx}{(x+1)^2}$ ■

## Estimating with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point $a$ and want to predict how much this value will change if we move to a nearby point $a + dx$. If $dx$ is small, then we can see from Figure 3.51 that $\Delta y$ is approximately equal to the differential $dy$. Since

$$f(a + dx) = f(a) + \Delta y,$$

the differential approximation gives

$$f(a + dx) \approx f(a) + dy$$

where $dx = \Delta x$. Thus the approximation $\Delta y \approx dy$ can be used to calculate $f(a + dx)$ when $f(a)$ is known and $dx$ is small.

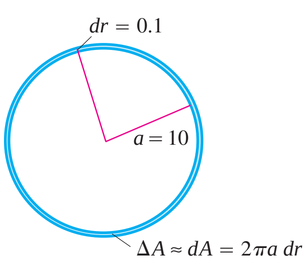



**FIGURE 3.52** When $dr$ is small compared with $a$, as it is when $dr = 0.1$ and $a = 10$, the differential $dA = 2\pi a\, dr$ gives a way to estimate the area of the circle with radius $r = a + dr$ (Example 6).

## EXAMPLE 6 Estimating with Differentials

The radius $r$ of a circle increases from $a = 10$ m to 10.1 m (Figure 3.52). Use $dA$ to estimate the increase in the circle's area $A$. Estimate the area of the enlarged circle and compare your estimate to the true area.

**Solution** Since $A = \pi r^2$, the estimated increase is

$$dA = A'(a)\,dr = 2\pi a\,dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

Thus,

$$\begin{aligned} A(10 + 0.1) &\approx A(10) + 2\pi \\ &= \pi(10)^2 + 2\pi = 102\pi. \end{aligned}$$

The area of a circle of radius 10.1 m is approximately $102\pi$ m$^2$.

The true area is

$$\begin{aligned} A(10.1) &= \pi(10.1)^2 \\ &= 102.01\pi \text{ m}^2. \end{aligned}$$

The error in our estimate is $0.01\pi$ m$^2$, which is the difference $\Delta A - dA$. ■

## Error in Differential Approximation

Let $f(x)$ be differentiable at $x = a$ and suppose that $dx = \Delta x$ is an increment of $x$. We have two ways to describe the change in $f$ as $x$ changes from $a$ to $a + \Delta x$:

The true change: $\Delta f = f(a + \Delta x) - f(a)$

The differential estimate: $df = f'(a)\,\Delta x.$

How well does $df$ approximate $\Delta f$?

We measure the approximation error by subtracting $df$ from $\Delta f$:

$$\begin{aligned}
\text{Approximation error} &= \Delta f - df \\
&= \Delta f - f'(a)\Delta x \\
&= \underbrace{f(a + \Delta x) - f(a)}_{\Delta f} - f'(a)\Delta x \\
&= \underbrace{\left(\frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a)\right)}_{\text{Call this part } \epsilon} \cdot \Delta x \\
&= \epsilon \cdot \Delta x.
\end{aligned}$$

As $\Delta x \to 0$, the difference quotient

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

approaches $f'(a)$ (remember the definition of $f'(a)$), so the quantity in parentheses becomes a very small number (which is why we called it $\epsilon$). In fact, $\epsilon \to 0$ as $\Delta x \to 0$. When $\Delta x$ is small, the approximation error $\epsilon\ \Delta x$ is smaller still.

$$\underbrace{\Delta f}_{\substack{\text{true}\\ \text{change}}} = \underbrace{f'(a)\Delta x}_{\substack{\text{estimated}\\ \text{change}}} + \underbrace{\epsilon\ \Delta x}_{\text{error}}$$

Although we do not know exactly how small the error is and will not be able to make much progress on this front until Chapter 11, there is something worth noting here, namely the *form* taken by the equation.

**Change in $y = f(x)$ near $x = a$**

If $y = f(x)$ is differentiable at $x = a$ and $x$ changes from $a$ to $a + \Delta x$, the change $\Delta y$ in $f$ is given by an equation of the form

$$\Delta y = f'(a)\ \Delta x + \epsilon\ \Delta x \tag{1}$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

In Example 6 we found that

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = (\underbrace{2\pi}_{dA} + \underbrace{0.01\pi}_{\text{error}})\ \text{m}^2$$

so the approximation error is $\Delta A - dA = \epsilon\ \Delta r = 0.01\pi$ and $\epsilon = 0.01\pi/\Delta r = 0.01\pi/0.1 = 0.1\pi$ m.

Equation (1) enables us to bring the proof of the Chain Rule to a successful conclusion.

### Proof of the Chain Rule

Our goal is to show that if $f(u)$ is a differentiable function of $u$ and $u = g(x)$ is a differentiable function of $x$, then the composite $y = f(g(x))$ is a differentiable function of $x$.

More precisely, if $g$ is differentiable at $x_0$ and $f$ is differentiable at $g(x_0)$, then the composite is differentiable at $x_0$ and

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f'(g(x_0)) \cdot g'(x_0).$$

Let $\Delta x$ be an increment in $x$ and let $\Delta u$ and $\Delta y$ be the corresponding increments in $u$ and $y$. Applying Equation (1) we have,

$$\Delta u = g'(x_0)\Delta x + \epsilon_1 \Delta x = (g'(x_0) + \epsilon_1)\Delta x,$$

where $\epsilon_1 \rightarrow 0$ as $\Delta x \rightarrow 0$. Similarly,

$$\Delta y = f'(u_0)\Delta u + \epsilon_2 \Delta u = (f'(u_0) + \epsilon_2)\Delta u,$$

where $\epsilon_2 \rightarrow 0$ as $\Delta u \rightarrow 0$. Notice also that $\Delta u \rightarrow 0$ as $\Delta x \rightarrow 0$. Combining the equations for $\Delta u$ and $\Delta y$ gives

$$\Delta y = (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1)\Delta x,$$

so

$$\frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) + \epsilon_2 g'(x_0) + f'(u_0)\epsilon_1 + \epsilon_2\epsilon_1.$$

Since $\epsilon_1$ and $\epsilon_2$ go to zero as $\Delta x$ goes to zero, three of the four terms on the right vanish in the limit, leaving

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) = f'(g(x_0)) \cdot g'(x_0).$$

This concludes the proof. ■

## Sensitivity to Change

The equation $df = f'(x)\,dx$ tells how *sensitive* the output of $f$ is to a change in input at different values of $x$. The larger the value of $f'$ at $x$, the greater the effect of a given change $dx$. As we move from $a$ to a nearby point $a + dx$, we can describe the change in $f$ in three ways:

| | **True** | **Estimated** |
|---|---|---|
| Absolute change | $\Delta f = f(a + dx) - f(a)$ | $df = f'(a)\,dx$ |
| Relative change | $\dfrac{\Delta f}{f(a)}$ | $\dfrac{df}{f(a)}$ |
| Percentage change | $\dfrac{\Delta f}{f(a)} \times 100$ | $\dfrac{df}{f(a)} \times 100$ |

### EXAMPLE 7 Finding the Depth of a Well

You want to calculate the depth of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculations be to a 0.1-sec error in measuring the time?

**Solution** The size of $ds$ in the equation

$$ds = 32t\,dt$$

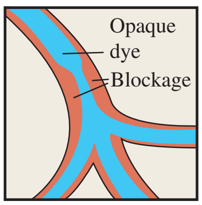
depends on how big $t$ is. If $t = 2$ sec, the change caused by $dt = 0.1$ is about

$$ds = 32(2)(0.1) = 6.4 \text{ ft}.$$

Three seconds later at $t = 5$ sec, the change caused by the same $dt$ is

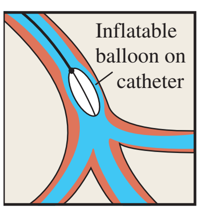
$$ds = 32(5)(0.1) = 16 \text{ ft}.$$

The estimated depth of the well differs from its true depth by a greater distance the longer the time it takes the stone to splash into the water below, for a given error in measuring the time. ■




**Angiography**

An opaque dye is injected into a partially blocked artery to make the inside visible under X-rays. This reveals the location and severity of the blockage.




**Angioplasty**

A balloon-tipped catheter is inflated inside the artery to widen it at the blockage site.

## EXAMPLE 8 Unclogging Arteries

In the late 1830s, French physiologist Jean Poiseuille ("pwa-ZOY") discovered the formula we use today to predict how much the radius of a partially clogged artery has to be expanded to restore normal flow. His formula,

$$V = kr^4,$$

says that the volume $V$ of fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius $r$. How will a 10% increase in $r$ affect $V$?

**Solution** The differentials of $r$ and $V$ are related by the equation

$$dV = \frac{dV}{dr}\,dr = 4kr^3\,dr.$$

The relative change in $V$ is

$$\frac{dV}{V} = \frac{4kr^3\,dr}{kr^4} = 4\,\frac{dr}{r}.$$

The relative change in $V$ is 4 times the relative change in $r$, so a 10% increase in $r$ will produce a 40% increase in the flow. ■

## EXAMPLE 9 Converting Mass to Energy

Newton's second law,

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} = ma,$$

is stated with the assumption that mass is constant, but we know this is not strictly true because the mass of a body increases with velocity. In Einstein's corrected formula, mass has the value

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where the "rest mass" $m_0$ represents the mass of a body that is not moving and $c$ is the speed of light, which is about 300,000 km/sec. Use the approximation

$$\frac{1}{\sqrt{1 - x^2}} \approx 1 + \frac{1}{2}x^2 \tag{2}$$

to estimate the increase $\Delta m$ in mass resulting from the added velocity $v$.

**Solution** When $v$ is very small compared with $c$, $v^2/c^2$ is close to zero and it is safe to use the approximation

$$\frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right) \qquad \text{Eq. (2) with } x = \frac{v}{c}$$

to obtain

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0\left[1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)\right] = m_0 + \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right),$$

or

$$m \approx m_0 + \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right). \tag{3}$$

Equation (3) expresses the increase in mass that results from the added velocity $v$.

### Energy Interpretation

In Newtonian physics, $(1/2)m_0 v^2$ is the kinetic energy (KE) of the body, and if we rewrite Equation (3) in the form

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2,$$

we see that

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2 = \frac{1}{2}m_0 v^2 - \frac{1}{2}m_0(0)^2 = \Delta(\text{KE}),$$

or

$$(\Delta m)c^2 \approx \Delta(\text{KE}).$$

So the change in kinetic energy $\Delta(\text{KE})$ in going from velocity 0 to velocity $v$ is approximately equal to $(\Delta m)c^2$, the change in mass times the square of the speed of light. Using $c \approx 3 \times 10^8$ m/sec, we see that a small change in mass can create a large change in energy. ■

# EXERCISES 3.8

## Finding Linearizations

In Exercises 1–4, find the linearization $L(x)$ of $f(x)$ at $x = a$.

**1.** $f(x) = x^3 - 2x + 3, \quad a = 2$

**2.** $f(x) = \sqrt{x^2 + 9}, \quad a = -4$

**3.** $f(x) = x + \dfrac{1}{x}, \quad a = 1$

**4.** $f(x) = \sqrt[3]{x}, \quad a = -8$

## Linearization for Approximation

You want linearizations that will replace the functions in Exercises 5–10 over intervals that include the given points $x_0$. To make your subsequent work as simple as possible, you want to center each linearization not at $x_0$ but at a nearby integer $x = a$ at which the given function and its derivative are easy to evaluate. What linearization do you use in each case?

**5.** $f(x) = x^2 + 2x, \quad x_0 = 0.1$

**6.** $f(x) = x^{-1}, \quad x_0 = 0.9$

**7.** $f(x) = 2x^2 + 4x - 3, \quad x_0 = -0.9$

**8.** $f(x) = 1 + x, \quad x_0 = 8.1$

**9.** $f(x) = \sqrt[3]{x}, \quad x_0 = 8.5$

**10.** $f(x) = \dfrac{x}{x + 1}, \quad x_0 = 1.3$

## Linearizing Trigonometric Functions

In Exercises 11–14, find the linearization of $f$ at $x = a$. Then graph the linearization and $f$ together.

**11.** $f(x) = \sin x$ at (a) $x = 0$, (b) $x = \pi$

**12.** $f(x) = \cos x$ at (a) $x = 0$, (b) $x = -\pi/2$

**13.** $f(x) = \sec x$ at (a) $x = 0$, (b) $x = -\pi/3$

**14.** $f(x) = \tan x$ at (a) $x = 0$, (b) $x = \pi/4$

## The Approximation $(1 + x)^k \approx 1 + kx$

**15.** Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

**16.** Use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of $x$ near zero.

**a.** $f(x) = (1 - x)^6$ **b.** $f(x) = \dfrac{2}{1 - x}$

**c.** $f(x) = \dfrac{1}{\sqrt{1 + x}}$ **d.** $f(x) = \sqrt{2 + x^2}$

**e.** $f(x) = (4 + 3x)^{1/3}$ **f.** $f(x) = \sqrt[3]{\left(1 - \dfrac{1}{2 + x}\right)^2}$

**17. Faster than a calculator** Use the approximation $(1 + x)^k \approx 1 + kx$ to estimate the following.

**a.** $(1.0002)^{50}$ **b.** $\sqrt[3]{1.009}$

**18.** Find the linearization of $f(x) = \sqrt{x + 1} + \sin x$ at $x = 0$. How is it related to the individual linearizations of $\sqrt{x + 1}$ and $\sin x$ at $x = 0$?

## Derivatives in Differential Form

In Exercises 19–30, find $dy$.

**19.** $y = x^3 - 3\sqrt{x}$

**20.** $y = x\sqrt{1 - x^2}$

**21.** $y = \dfrac{2x}{1 + x^2}$

**22.** $y = \dfrac{2\sqrt{x}}{3(1 + \sqrt{x})}$

**23.** $2y^{3/2} + xy - x = 0$

**24.** $xy^2 - 4x^{3/2} - y = 0$

**25.** $y = \sin(5\sqrt{x})$

**26.** $y = \cos(x^2)$

**27.** $y = 4\tan(x^3/3)$

**28.** $y = \sec(x^2 - 1)$

**29.** $y = 3\csc(1 - 2\sqrt{x})$

**30.** $y = 2\cot\left(\dfrac{1}{\sqrt{x}}\right)$

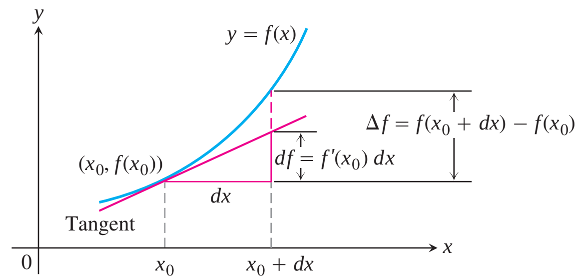
## Approximation Error

In Exercises 31–36, each function $f(x)$ changes value when $x$ changes from $x_0$ to $x_0 + dx$. Find

**a.** the change $\Delta f = f(x_0 + dx) - f(x_0)$;

**b.** the value of the estimate $df = f'(x_0)\,dx$; and

**c.** the approximation error $|\Delta f - df|$.




**31.** $f(x) = x^2 + 2x$, $x_0 = 1$, $dx = 0.1$

**32.** $f(x) = 2x^2 + 4x - 3$, $x_0 = -1$, $dx = 0.1$

**33.** $f(x) = x^3 - x$, $x_0 = 1$, $dx = 0.1$

**34.** $f(x) = x^4$, $x_0 = 1$, $dx = 0.1$

**35.** $f(x) = x^{-1}$, $x_0 = 0.5$, $dx = 0.1$

**36.** $f(x) = x^3 - 2x + 3$, $x_0 = 2$, $dx = 0.1$

## Differential Estimates of Change

In Exercises 37–42, write a differential formula that estimates the given change in volume or surface area.

**37.** The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from $r_0$ to $r_0 + dr$

**38.** The change in the volume $V = x^3$ of a cube when the edge lengths change from $x_0$ to $x_0 + dx$

**39.** The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from $x_0$ to $x_0 + dx$

**40.** The change in the lateral surface area $S = \pi r\sqrt{r^2 + h^2}$ of a right circular cone when the radius changes from $r_0$ to $r_0 + dr$ and the height does not change

**41.** The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from $r_0$ to $r_0 + dr$ and the height does not change

**42.** The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from $h_0$ to $h_0 + dh$ and the radius does not change

## Applications

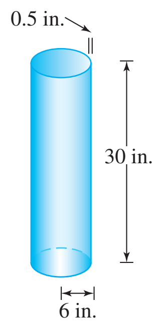
**43.** The radius of a circle is increased from 2.00 to 2.02 m.

**a.** Estimate the resulting change in area.

**b.** Express the estimate as a percentage of the circle's original area.

**44.** The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? The tree's cross-section area?

**45. Estimating volume** Estimate the volume of material in a cylindrical shell with height 30 in., radius 6 in., and shell thickness 0.5 in.




**46. Estimating height of a building** A surveyor, standing 30 ft from the base of a building, measures the angle of elevation to the top of the building to be 75°. How accurately must the angle be measured for the percentage error in estimating the height of the building to be less than 4%?

**47. Tolerance** The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of $h$, expressed as a percentage of $h$.

**48. Tolerance**

**a.** About how accurately must the interior diameter of a 10-m-high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?

**b.** About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?

**49. Minting coins** A manufacturer contracts to mint coins for the federal government. How much variation $dr$ in the radius of the coins can be tolerated if the coins are to weigh within $1/1000$ of their ideal weight? Assume that the thickness does not vary.

**50. Sketching the change in a cube's volume** The volume $V = x^3$ of a cube with edges of length $x$ increases by an amount $\Delta V$ when $x$ increases by an amount $\Delta x$. Show with a sketch how to represent $\Delta V$ geometrically as the sum of the volumes of

**a.** three slabs of dimensions $x$ by $x$ by $\Delta x$

**b.** three bars of dimensions $x$ by $\Delta x$ by $\Delta x$

**c.** one cube of dimensions $\Delta x$ by $\Delta x$ by $\Delta x$.

The differential formula $dV = 3x^2\,dx$ estimates the change in $V$ with the three slabs.

**51. The effect of flight maneuvers on the heart** The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where $W$ is the work per unit time, $P$ is the average blood pressure, $V$ is the volume of blood pumped out during the unit of time, $\delta$ ("delta") is the weight density of the blood, $v$ is the average velocity of the exiting blood, and $g$ is the acceleration of gravity.

When $P$, $V$, $\delta$, and $v$ remain constant, $W$ becomes a function of $g$, and the equation takes the simplified form

$$W = a + \frac{b}{g} \quad (a, b \text{ constant}).$$

As a member of NASA's medical team, you want to know how sensitive $W$ is to apparent changes in $g$ caused by flight maneuvers, and this depends on the initial value of $g$. As part of your investigation, you decide to compare the effect on $W$ of a given change $dg$ on the moon, where $g = 5.2\ \text{ft/sec}^2$, with the effect the same change $dg$ would have on Earth, where $g = 32\ \text{ft/sec}^2$. Use the simplified equation above to find the ratio of $dW_{\text{moon}}$ to $dW_{\text{Earth}}$.

**52. Measuring acceleration of gravity** When the length $L$ of a clock pendulum is held constant by controlling its temperature, the pendulum's period $T$ depends on the acceleration of gravity $g$. The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in $g$. By keeping track of $\Delta T$, we can estimate the variation in $g$ from the equation $T = 2\pi(L/g)^{1/2}$ that relates $T$, $g$, and $L$.

**a.** With $L$ held constant and $g$ as the independent variable, calculate $dT$ and use it to answer parts (b) and (c).

**b.** If $g$ increases, will $T$ increase or decrease? Will a pendulum clock speed up or slow down? Explain.

**c.** A clock with a 100-cm pendulum is moved from a location where $g = 980\ \text{cm/sec}^2$ to a new location. This increases the period by $dT = 0.001$ sec. Find $dg$ and estimate the value of $g$ at the new location.

**53.** The edge of a cube is measured as 10 cm with an error of 1%. The cube's volume is to be calculated from this measurement. Estimate the percentage error in the volume calculation.

**54.** About how accurately should you measure the side of a square to be sure of calculating the area within 2% of its true value?

**55.** The diameter of a sphere is measured as $100 \pm 1$ cm and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.

**56.** Estimate the allowable percentage error in measuring the diameter $D$ of a sphere if the volume is to be calculated correctly to within 3%.

**57.** (*Continuation of Example 7.*) Show that a 5% error in measuring $t$ will cause about a 10% error in calculating $s$ from the equation $s = 16t^2$.

**58.** (*Continuation of Example 8.*) By what percentage should $r$ be increased to increase $V$ by 50%?

## Theory and Examples

**59.** Show that the approximation of $\sqrt{1+x}$ by its linearization at the origin must improve as $x \to 0$ by showing that

$$\lim_{x \to 0} \frac{\sqrt{1+x}}{1 + (x/2)} = 1.$$

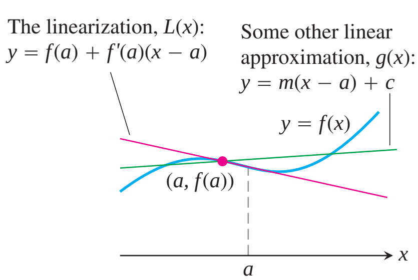
**60.** Show that the approximation of tan $x$ by its linearization at the origin must improve as $x \to 0$ by showing that

$$\lim_{x\to 0} \frac{\tan x}{x} = 1.$$

**61. The linearization is the best linear approximation** (This is why we use the linearization.) Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ is a linear function in which $m$ and $c$ are constants. If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using $g$ as a linear approximation of $f$ instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on $g$ the conditions

1. $E(a) = 0$ — The approximation error is zero at $x = a$.
2. $\lim_{x\to a} \frac{E(x)}{x - a} = 0$ — The error is negligible when compared with $x - a$.

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization $L(x)$ gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $x - a$.




**62. Quadratic approximations**

a. Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties:

i. $Q(a) = f(a)$

ii. $Q'(a) = f'(a)$

iii. $Q''(a) = f''(a)$

Determine the coefficients $b_0$, $b_1$, and $b_2$.

b. Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.

T c. Graph $f(x) = 1/(1 - x)$ and its quadratic approximation at $x = 0$. Then zoom in on the two graphs at the point (0, 1). Comment on what you see.

T d. Find the quadratic approximation to $g(x) = 1/x$ at $x = 1$. Graph $g$ and its quadratic approximation together. Comment on what you see.

T e. Find the quadratic approximation to $h(x) = \sqrt{1 + x}$ at $x = 0$. Graph $h$ and its quadratic approximation together. Comment on what you see.

f. What are the linearizations of $f$, $g$, and $h$ at the respective points in parts (b), (d), and (e)?

T **63. Reading derivatives from graphs** The idea that differentiable curves flatten out when magnified can be used to estimate the values of the derivatives of functions at particular points. We magnify the curve until the portion we see looks like a straight line through the point in question, and then we use the screen's coordinate grid to read the slope of the curve as the slope of the line it resembles.

a. To see how the process works, try it first with the function $y = x^2$ at $x = 1$. The slope you read should be 2.

b. Then try it with the curve $y = e^x$ at $x = 1$, $x = 0$, and $x = -1$. In each case, compare your estimate of the derivative with the value of $e^x$ at the point. What pattern do you see? Test it with other values of $x$. Chapter 7 will explain what is going on.

**64.** Suppose that the graph of a differentiable function $f(x)$ has a horizontal tangent at $x = a$. Can anything be said about the linearization of $f$ at $x = a$? Give reasons for your answer.

**65.** To what relative speed should a body at rest be accelerated to increase its mass by 1%?

T **66. Repeated root-taking**

a. Enter 2 in your calculator and take successive square roots by pressing the square root key repeatedly (or raising the displayed number repeatedly to the 0.5 power). What pattern do you see emerging? Explain what is going on. What happens if you take successive tenth roots instead?

b. Repeat the procedure with 0.5 in place of 2 as the original entry. What happens now? Can you use any positive number $x$ in place of 2? Explain what is going on.

### COMPUTER EXPLORATIONS

## Comparing Functions with Their Linearizations

In Exercises 67–70, use a CAS to estimate the magnitude of the error in using the linearization in place of the function over a specified interval $I$. Perform the following steps:

a. Plot the function $f$ over $I$.

b. Find the linearization $L$ of the function at the point $a$.

c. Plot $f$ and $L$ together on a single graph.

d. Plot the absolute error $|f(x) - L(x)|$ over $I$ and find its maximum value.

e. From your graph in part (d), estimate as large a $\delta > 0$ as you can, satisfying

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - L(x)| < \epsilon$$

for $\epsilon = 0.5, 0.1$, and $0.01$. Then check graphically to see if your $\delta$-estimate holds true.

**67.** $f(x) = x^3 + x^2 - 2x, \quad [-1, 2], \quad a = 1$

**68.** $f(x) = \dfrac{x - 1}{4x^2 + 1}, \quad \left[-\dfrac{3}{4}, 1\right], \quad a = \dfrac{1}{2}$

**69.** $f(x) = x^{2/3}(x - 2), \quad [-2, 3], \quad a = 2$

**70.** $f(x) = \sqrt{x} - \sin x, \quad [0, 2\pi], \quad a = 2$

# Chapter 3 Questions to Guide Your Review

1. What is the derivative of a function $f$? How is its domain related to the domain of $f$? Give examples.
2. What role does the derivative play in defining slopes, tangents, and rates of change?
3. How can you sometimes graph the derivative of a function when all you have is a table of the function's values?
4. What does it mean for a function to be differentiable on an open interval? On a closed interval?
5. How are derivatives and one-sided derivatives related?
6. Describe geometrically when a function typically does *not* have a derivative at a point.
7. How is a function's differentiability at a point related to its continuity there, if at all?
8. Could the unit step function

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

possibly be the derivative of some other function on $[-1, 1]$? Explain.

9. What rules do you know for calculating derivatives? Give some examples.
10. Explain how the three formulas

**a.** $\dfrac{d}{dx}(x^n) = nx^{n-1}$

**b.** $\dfrac{d}{dx}(cu) = c\dfrac{du}{dx}$

**c.** $\dfrac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \dfrac{du_1}{dx} + \dfrac{du_2}{dx} + \cdots + \dfrac{du_n}{dx}$

enable us to differentiate any polynomial.

11. What formula do we need, in addition to the three listed in Question 10, to differentiate rational functions?
12. What is a second derivative? A third derivative? How many derivatives do the functions you know have? Give examples.
13. What is the relationship between a function's average and instantaneous rates of change? Give an example.
14. How do derivatives arise in the study of motion? What can you learn about a body's motion along a line by examining the derivatives of the body's position function? Give examples.
15. How can derivatives arise in economics?
16. Give examples of still other applications of derivatives.
17. What do the limits $\lim_{h\to 0}((\sin h)/h)$ and $\lim_{h\to 0}((\cos h - 1)/h)$ have to do with the derivatives of the sine and cosine functions? What *are* the derivatives of these functions?
18. Once you know the derivatives of $\sin x$ and $\cos x$, how can you find the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$? What *are* the derivatives of these functions?
19. At what points are the six basic trigonometric functions continuous? How do you know?
20. What is the rule for calculating the derivative of a composite of two differentiable functions? How is such a derivative evaluated? Give examples.
21. What is the formula for the slope $dy/dx$ of a parametrized curve $x = f(t), y = g(t)$? When does the formula apply? When can you expect to be able to find $d^2y/dx^2$ as well? Give examples.
22. If $u$ is a differentiable function of $x$, how do you find $(d/dx)(u^n)$ if $n$ is an integer? If $n$ is a rational number? Give examples.
23. What is implicit differentiation? When do you need it? Give examples.
24. How do related rates problems arise? Give examples.
25. Outline a strategy for solving related rates problems. Illustrate with an example.
26. What is the linearization $L(x)$ of a function $f(x)$ at a point $x = a$? What is required of $f$ at $a$ for the linearization to exist? How are linearizations used? Give examples.
27. If $x$ moves from $a$ to a nearby value $a + dx$, how do you estimate the corresponding change in the value of a differentiable function $f(x)$? How do you estimate the relative change? The percentage change? Give an example.

# Chapter 3 Practice Exercises

## Derivatives of Functions

Find the derivatives of the functions in Exercises 1-40.

**1.** $y = x^5 - 0.125x^2 + 0.25x$

**2.** $y = 3 - 0.7x^3 + 0.3x^7$

**3.** $y = x^3 - 3(x^2 + \pi^2)$

**4.** $y = x^7 + \sqrt{7}x - \dfrac{1}{\pi + 1}$

**5.** $y = (x + 1)^2(x^2 + 2x)$

**6.** $y = (2x - 5)(4 - x)^{-1}$

**7.** $y = (\theta^2 + \sec\theta + 1)^3$

**8.** $y = \left(-1 - \dfrac{\csc\theta}{2} - \dfrac{\theta^2}{4}\right)^2$

**9.** $s = \dfrac{\sqrt{t}}{1 + \sqrt{t}}$

**10.** $s = \dfrac{1}{\sqrt{t} - 1}$

**11.** $y = 2\tan^2 x - \sec^2 x$

**12.** $y = \frac{1}{\sin^2 x} - \frac{2}{\sin x}$

**13.** $s = \cos^4(1 - 2t)$

**14.** $s = \cot^3\left(\frac{2}{t}\right)$

**15.** $s = (\sec t + \tan t)^5$

**16.** $s = \csc^5(1 - t + 3t^2)$

**17.** $r = \sqrt{2\theta \sin\theta}$

**18.** $r = 2\theta\sqrt{\cos\theta}$

**19.** $r = \sin\sqrt{2\theta}$

**20.** $r = \sin\left(\theta + \sqrt{\theta + 1}\right)$

**21.** $y = \frac{1}{2}x^2 \csc\frac{2}{x}$

**22.** $y = 2\sqrt{x}\sin\sqrt{x}$

**23.** $y = x^{-1/2}\sec(2x)^2$

**24.** $y = \sqrt{x}\csc(x + 1)^3$

**25.** $y = 5\cot x^2$

**26.** $y = x^2\cot 5x$

**27.** $y = x^2\sin^2(2x^2)$

**28.** $y = x^{-2}\sin^2(x^3)$

**29.** $s = \left(\frac{4t}{t + 1}\right)^{-2}$

**30.** $s = \frac{-1}{15(15t - 1)^3}$

**31.** $y = \left(\frac{\sqrt{x}}{1 + x}\right)^2$

**32.** $y = \left(\frac{2\sqrt{x}}{2\sqrt{x} + 1}\right)^2$

**33.** $y = \sqrt{\frac{x^2 + x}{x^2}}$

**34.** $y = 4x\sqrt{x + \sqrt{x}}$

**35.** $r = \left(\frac{\sin\theta}{\cos\theta - 1}\right)^2$

**36.** $r = \left(\frac{1 + \sin\theta}{1 - \cos\theta}\right)^2$

**37.** $y = (2x + 1)\sqrt{2x + 1}$

**38.** $y = 20(3x - 4)^{1/4}(3x - 4)^{-1/5}$

**39.** $y = \frac{3}{(5x^2 + \sin 2x)^{3/2}}$

**40.** $y = (3 + \cos^3 3x)^{-1/3}$

## Implicit Differentiation

In Exercises 41–48, find $dy/dx$.

**41.** $xy + 2x + 3y = 1$

**42.** $x^2 + xy + y^2 - 5x = 2$

**43.** $x^3 + 4xy - 3y^{4/3} = 2x$

**44.** $5x^{4/5} + 10y^{6/5} = 15$

**45.** $\sqrt{xy} = 1$

**46.** $x^2y^2 = 1$

**47.** $y^2 = \frac{x}{x + 1}$

**48.** $y^2 = \sqrt{\frac{1 + x}{1 - x}}$

In Exercises 49 and 50, find $dp/dq$.

**49.** $p^3 + 4pq - 3q^2 = 2$

**50.** $q = (5p^2 + 2p)^{-3/2}$

In Exercises 51 and 52, find $dr/ds$.

**51.** $r\cos 2s + \sin^2 s = \pi$

**52.** $2rs - r - s + s^2 = -3$

**53.** Find $d^2y/dx^2$ by implicit differentiation:

**a.** $x^3 + y^3 = 1$ **b.** $y^2 = 1 - \frac{2}{x}$

**54. a.** By differentiating $x^2 - y^2 = 1$ implicitly, show that $dy/dx = x/y$.

**b.** Then show that $d^2y/dx^2 = -1/y^3$.

## Numerical Values of Derivatives

**55.** Suppose that functions $f(x)$ and $g(x)$ and their first derivatives have the following values at $x = 0$ and $x = 1$.

| $x$ | $f(x)$ | $g(x)$ | $f'(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 0 | 1 | 1 | −3 | 1/2 |
| 1 | 3 | 5 | 1/2 | −4 |

Find the first derivatives of the following combinations at the given value of $x$.

**a.** $6f(x) - g(x)$, $x = 1$ **b.** $f(x)g^2(x)$, $x = 0$

**c.** $\frac{f(x)}{g(x) + 1}$, $x = 1$ **d.** $f(g(x))$, $x = 0$

**e.** $g(f(x))$, $x = 0$ **f.** $(x + f(x))^{3/2}$, $x = 1$

**g.** $f(x + g(x))$, $x = 0$

**56.** Suppose that the function $f(x)$ and its first derivative have the following values at $x = 0$ and $x = 1$.

| $x$ | $f(x)$ | $f'(x)$ |
|---|---|---|
| 0 | 9 | −2 |
| 1 | −3 | 1/5 |

Find the first derivatives of the following combinations at the given value of $x$.

**a.** $\sqrt{x}\,f(x)$, $x = 1$ **b.** $\sqrt{f(x)}$, $x = 0$

**c.** $f(\sqrt{x})$, $x = 1$ **d.** $f(1 - 5\tan x)$, $x = 0$

**e.** $\frac{f(x)}{2 + \cos x}$, $x = 0$ **f.** $10\sin\left(\frac{\pi x}{2}\right)f^2(x)$, $x = 1$

**57.** Find the value of $dy/dt$ at $t = 0$ if $y = 3\sin 2x$ and $x = t^2 + \pi$.

**58.** Find the value of $ds/du$ at $u = 2$ if $s = t^2 + 5t$ and $t = (u^2 + 2u)^{1/3}$.

**59.** Find the value of $dw/ds$ at $s = 0$ if $w = \sin\left(\sqrt{r} - 2\right)$ and $r = 8\sin(s + \pi/6)$.

**60.** Find the value of $dr/dt$ at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.

**61.** If $y^3 + y = 2\cos x$, find the value of $d^2y/dx^2$ at the point (0, 1).

**62.** If $x^{1/3} + y^{1/3} = 4$, find $d^2y/dx^2$ at the point (8, 8).

## Derivative Definition

In Exercises 63 and 64, find the derivative using the definition.

**63.** $f(t) = \frac{1}{2t + 1}$

**64.** $g(x) = 2x^2 + 1$

**65. a.** Graph the function

$$f(x) = \begin{cases} x^2, & -1 \le x < 0 \\ -x^2, & 0 \le x \le 1. \end{cases}$$

**b.** Is $f$ continuous at $x = 0$?

**c.** Is $f$ differentiable at $x = 0$?

Give reasons for your answers.

**66. a.** Graph the function

$$f(x) = \begin{cases} x, & -1 \le x < 0 \\ \tan x, & 0 \le x \le \pi/4. \end{cases}$$

**b.** Is $f$ continuous at $x = 0$?

**c.** Is $f$ differentiable at $x = 0$?

Give reasons for your answers.

**67. a.** Graph the function

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$$

**b.** Is $f$ continuous at $x = 1$?

**c.** Is $f$ differentiable at $x = 1$?

Give reasons for your answers.

**68.** For what value or values of the constant $m$, if any, is

$$f(x) = \begin{cases} \sin 2x, & x \le 0 \\ mx, & x > 0 \end{cases}$$

**a.** continuous at $x = 0$?

**b.** differentiable at $x = 0$?

Give reasons for your answers.

## Slopes, Tangents, and Normals

**69. Tangents with specified slope** Are there any points on the curve $y = (x/2) + 1/(2x - 4)$ where the slope is $-3/2$? If so, find them.

**70. Tangents with specified slope** Are there any points on the curve $y = x - 1/(2x)$ where the slope is 3? If so, find them.

**71. Horizontal tangents** Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the $x$-axis.

**72. Tangent intercepts** Find the $x$- and $y$-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

**73. Tangents perpendicular or parallel to lines** Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is

**a.** perpendicular to the line $y = 1 - (x/24)$.

**b.** parallel to the line $y = \sqrt{2} - 12x$.

**74. Intersecting tangents** Show that the tangents to the curve $y = (\pi \sin x)/x$ at $x = \pi$ and $x = -\pi$ intersect at right angles.

**75. Normals parallel to a line** Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the normal is parallel to the line $y = -x/2$. Sketch the curve and normals together, labeling each with its equation.

**76. Tangent and normal lines** Find equations for the tangent and normal to the curve $y = 1 + \cos x$ at the point $(\pi/2, 1)$. Sketch the curve, tangent, and normal together, labeling each with its equation.

**77. Tangent parabola** The parabola $y = x^2 + C$ is to be tangent to the line $y = x$. Find $C$.

**78. Slope of tangent** Show that the tangent to the curve $y = x^3$ at any point $(a, a^3)$ meets the curve again at a point where the slope is four times the slope at $(a, a^3)$.

**79. Tangent curve** For what value of $c$ is the curve $y = c/(x + 1)$ tangent to the line through the points $(0, 3)$ and $(5, -2)$?

**80. Normal to a circle** Show that the normal line at any point of the circle $x^2 + y^2 = a^2$ passes through the origin.

## Tangents and Normals to Implicitly Defined Curves

In Exercises 81–86, find equations for the lines that are tangent and normal to the curve at the given point.

**81.** $x^2 + 2y^2 = 9, \quad (1, 2)$

**82.** $x^3 + y^2 = 2, \quad (1, 1)$

**83.** $xy + 2x - 5y = 2, \quad (3, 2)$

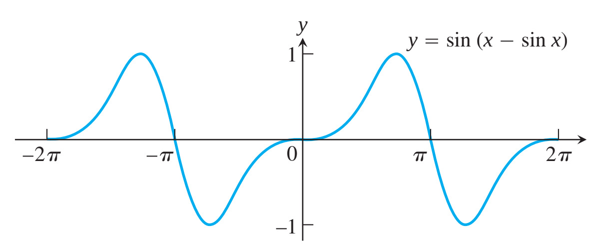
**84.** $(y - x)^2 = 2x + 4, \quad (6, 2)$

**85.** $x + \sqrt{xy} = 6, \quad (4, 1)$

**86.** $x^{3/2} + 2y^{3/2} = 17, \quad (1, 4)$

**87.** Find the slope of the curve $x^3y^3 + y^2 = x + y$ at the points $(1, 1)$ and $(1, -1)$.

**88.** The graph shown suggests that the curve $y = \sin(x - \sin x)$ might have horizontal tangents at the $x$-axis. Does it? Give reasons for your answer.




## Tangents to Parametrized Curves

In Exercises 89 and 90, find an equation for the line in the $xy$-plane that is tangent to the curve at the point corresponding to the given value of $t$. Also, find the value of $d^2y/dx^2$ at this point.

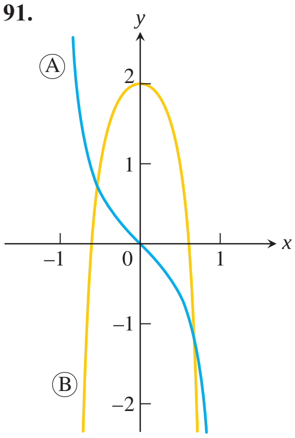
**89.** $x = (1/2)\tan t, \quad y = (1/2)\sec t, \quad t = \pi/3$

**90.** $x = 1 + 1/t^2, \quad y = 1 - 3/t, \quad t = 2$

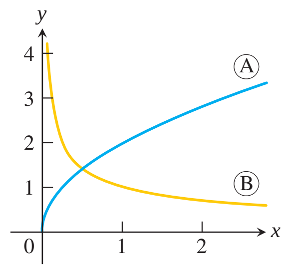
## Analyzing Graphs

Each of the figures in Exercises 91 and 92 shows two graphs, the graph of a function $y = f(x)$ together with the graph of its derivative $f'(x)$. Which graph is which? How do you know?

**91.**




**92.**

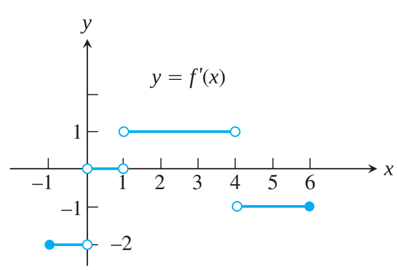



**93.** Use the following information to graph the function $y = f(x)$ for $-1 \le x \le 6$.

**i.** The graph of $f$ is made of line segments joined end to end.

**ii.** The graph starts at the point $(-1, 2)$.

**iii.** The derivative of $f$, where defined, agrees with the step function shown here.




**94.** Repeat Exercise 93, supposing that the graph starts at $(-1, 0)$ instead of $(-1, 2)$.

Exercises 95 and 96 are about the graphs in Figure 3.53 (right-hand column). The graphs in part (a) show the numbers of rabbits and foxes in a small arctic population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Figure 3.53b shows the graph of the derivative of the rabbit population. We made it by plotting slopes.

**95. a.** What is the value of the derivative of the rabbit population in Figure 3.53 when the number of rabbits is largest? Smallest?

**b.** What is the size of the rabbit population in Figure 3.53 when its derivative is largest? Smallest (negative value)?

**96.** In what units should the slopes of the rabbit and fox population curves be measured?

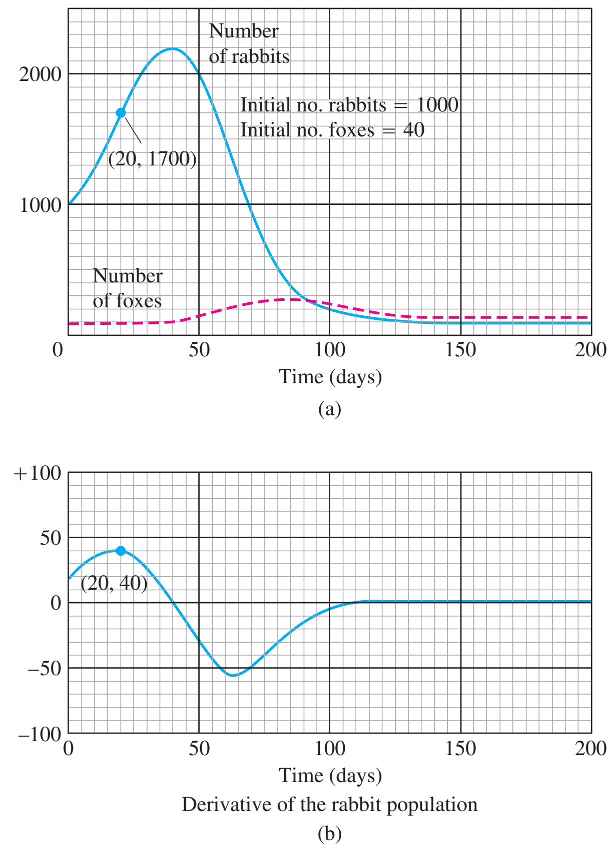



**FIGURE 3.53** Rabbits and foxes in an arctic predator-prey food chain.

## Trigonometric Limits

**97.** $\displaystyle\lim_{x\to 0} \frac{\sin x}{2x^2 - x}$

**98.** $\displaystyle\lim_{x\to 0} \frac{3x - \tan 7x}{2x}$

**99.** $\displaystyle\lim_{r\to 0} \frac{\sin r}{\tan 2r}$

**100.** $\displaystyle\lim_{\theta\to 0} \frac{\sin(\sin\theta)}{\theta}$

**101.** $\displaystyle\lim_{\theta\to(\pi/2)^-} \frac{4\tan^2\theta + \tan\theta + 1}{\tan^2\theta + 5}$

**102.** $\displaystyle\lim_{\theta\to 0^+} \frac{1 - 2\cot^2\theta}{5\cot^2\theta - 7\cot\theta - 8}$

**103.** $\displaystyle\lim_{x\to 0} \frac{x\sin x}{2 - 2\cos x}$

**104.** $\displaystyle\lim_{\theta\to 0} \frac{1 - \cos\theta}{\theta^2}$

Show how to extend the functions in Exercises 105 and 106 to be continuous at the origin.

**105.** $g(x) = \dfrac{\tan(\tan x)}{\tan x}$

**106.** $f(x) = \dfrac{\tan(\tan x)}{\sin(\sin x)}$

## Related Rates

**107. Right circular cylinder** The total surface area $S$ of a right circular cylinder is related to the base radius $r$ and height $h$ by the equation $S = 2\pi r^2 + 2\pi rh$.

**a.** How is $dS/dt$ related to $dr/dt$ if $h$ is constant?

**b.** How is $dS/dt$ related to $dh/dt$ if $r$ is constant?

**c.** How is $dS/dt$ related to $dr/dt$ and $dh/dt$ if neither $r$ nor $h$ is constant?

**d.** How is $dr/dt$ related to $dh/dt$ if $S$ is constant?

**108. Right circular cone** The lateral surface area $S$ of a right circular cone is related to the base radius $r$ and height $h$ by the equation $S = \pi r\sqrt{r^2 + h^2}$.

**a.** How is $dS/dt$ related to $dr/dt$ if $h$ is constant?

**b.** How is $dS/dt$ related to $dh/dt$ if $r$ is constant?

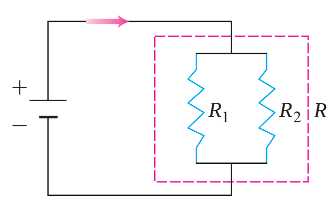
**c.** How is $dS/dt$ related to $dr/dt$ and $dh/dt$ if neither $r$ nor $h$ is constant?

**109. Circle's changing area** The radius of a circle is changing at the rate of $-2/\pi$ m/sec. At what rate is the circle's area changing when $r = 10$ m?

**110. Cube's changing edges** The volume of a cube is increasing at the rate of 1200 cm$^3$/min at the instant its edges are 20 cm long. At what rate are the lengths of the edges changing at that instant?

**111. Resistors connected in parallel** If two resistors of $R_1$ and $R_2$ ohms are connected in parallel in an electric circuit to make an $R$-ohm resistor, the value of $R$ can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$




If $R_1$ is decreasing at the rate of 1 ohm/sec and $R_2$ is increasing at the rate of 0.5 ohm/sec, at what rate is $R$ changing when $R_1 = 75$ ohms and $R_2 = 50$ ohms?

**112. Impedance in a series circuit** The impedance $Z$ (ohms) in a series circuit is related to the resistance $R$ (ohms) and reactance $X$ (ohms) by the equation $Z = \sqrt{R^2 + X^2}$. If $R$ is increasing at 3 ohms/sec and $X$ is decreasing at 2 ohms/sec, at what rate is $Z$ changing when $R = 10$ ohms and $X = 20$ ohms?

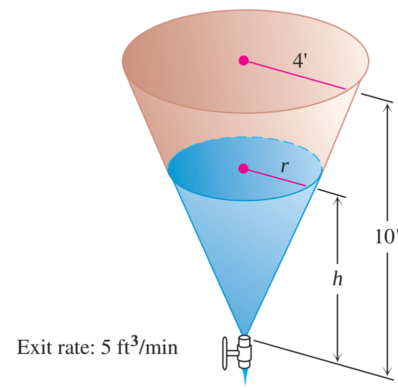
**113. Speed of moving particle** The coordinates of a particle moving in the metric $xy$-plane are differentiable functions of time $t$ with $dx/dt = 10$ m/sec and $dy/dt = 5$ m/sec. How fast is the particle moving away from the origin as it passes through the point $(3, -4)$?

**114. Motion of a particle** A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. Find $dx/dt$ when $x = 3$.

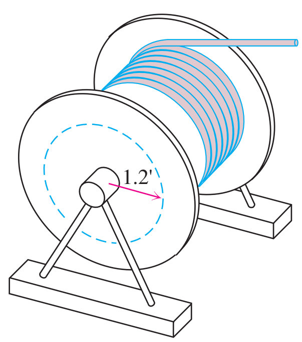
**115. Draining a tank** Water drains from the conical tank shown in the accompanying figure at the rate of 5 ft$^3$/min.

**a.** What is the relation between the variables $h$ and $r$ in the figure?

**b.** How fast is the water level dropping when $h = 6$ ft?



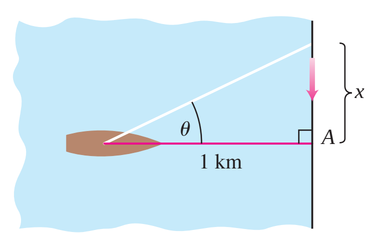

**116. Rotating spool** As television cable is pulled from a large spool to be strung from the telephone poles along a street, it unwinds from the spool in layers of constant radius (see accompanying figure). If the truck pulling the cable moves at a steady 6 ft/sec (a touch over 4 mph), use the equation $s = r\theta$ to find how fast (radians per second) the spool is turning when the layer of radius 1.2 ft is being unwound.




**117. Moving searchlight beam** The figure shows a boat 1 km offshore, sweeping the shore with a searchlight. The light turns at a constant rate, $d\theta/dt = -0.6$ rad/sec.

**a.** How fast is the light moving along the shore when it reaches point $A$?

**b.** How many revolutions per minute is 0.6 rad/sec?




**118. Points moving on coordinate axes** Points $A$ and $B$ move along the $x$- and $y$-axes, respectively, in such a way that the distance $r$ (meters) along the perpendicular from the origin to the line $AB$ remains constant. How fast is $OA$ changing, and is it increasing, or decreasing, when $OB = 2r$ and $B$ is moving toward $O$ at the rate of $0.3r$ m/sec?

## Linearization

**119.** Find the linearizations of

**a.** $\tan x$ at $x = -\pi/4$ **b.** $\sec x$ at $x = -\pi/4$.

Graph the curves and linearizations together.

**120.** We can obtain a useful linear approximation of the function $f(x) = 1/(1 + \tan x)$ at $x = 0$ by combining the approximations

$$\frac{1}{1+x} \approx 1 - x \qquad \text{and} \qquad \tan x \approx x$$

to get

$$\frac{1}{1 + \tan x} \approx 1 - x.$$

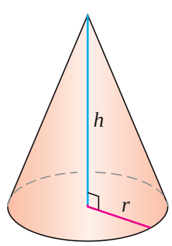
Show that this result is the standard linear approximation of $1/(1 + \tan x)$ at $x = 0$.

**121.** Find the linearization of $f(x) = \sqrt{1 + x} + \sin x - 0.5$ at $x = 0$.

**122.** Find the linearization of $f(x) = 2/(1 - x) + \sqrt{1 + x} - 3.1$ at $x = 0$.

## Differential Estimates of Change

**123. Surface area of a cone** Write a formula that estimates the change that occurs in the lateral surface area of a right circular cone when the height changes from $h_0$ to $h_0 + dh$ and the radius does not change.




$V = \frac{1}{3}\pi r^2 h$

$S = \pi r\sqrt{r^2 + h^2}$

(Lateral surface area)

**124. Controlling error**

**a.** How accurately should you measure the edge of a cube to be reasonably sure of calculating the cube's surface area with an error of no more than 2%?

**b.** Suppose that the edge is measured with the accuracy required in part (a). About how accurately can the cube's volume be calculated from the edge measurement? To find out, estimate the percentage error in the volume calculation that might result from using the edge measurement.

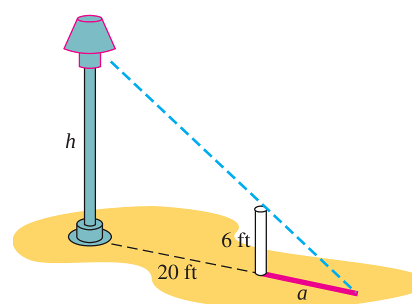
**125. Compounding error** The circumference of the equator of a sphere is measured as 10 cm with a possible error of 0.4 cm. This measurement is then used to calculate the radius. The radius is then used to calculate the surface area and volume of the sphere. Estimate the percentage errors in the calculated values of

**a.** the radius.

**b.** the surface area.

**c.** the volume.

**126. Finding height** To find the height of a lamppost (see accompanying figure), you stand a 6 ft pole 20 ft from the lamp and measure the length $a$ of its shadow, finding it to be 15 ft, give or take an inch. Calculate the height of the lamppost using the value $a = 15$ and estimate the possible error in the result.



# Chapter 3 Additional and Advanced Exercises

1. An equation like $\sin^2\theta + \cos^2\theta = 1$ is called an **identity** because it holds for all values of $\theta$. An equation like $\sin\theta = 0.5$ is not an identity because it holds only for selected values of $\theta$, not all. If you differentiate both sides of a trigonometric identity in $\theta$ with respect to $\theta$, the resulting new equation will also be an identity.

   Differentiate the following to show that the resulting equations hold for all $\theta$.

   **a.** $\sin 2\theta = 2\sin\theta\cos\theta$

   **b.** $\cos 2\theta = \cos^2\theta - \sin^2\theta$

2. If the identity $\sin(x + a) = \sin x \cos a + \cos x \sin a$ is differentiated with respect to $x$, is the resulting equation also an identity? Does this principle apply to the equation $x^2 - 2x - 8 = 0$? Explain.

3. **a.** Find values for the constants $a$, $b$, and $c$ that will make

$$f(x) = \cos x \quad \text{and} \quad g(x) = a + bx + cx^2$$

   satisfy the conditions

$$f(0) = g(0), \quad f'(0) = g'(0), \quad \text{and} \quad f''(0) = g''(0).$$

**b.** Find values for $b$ and $c$ that will make

$$f(x) = \sin(x + a) \quad \text{and} \quad g(x) = b\sin x + c\cos x$$

satisfy the conditions

$$f(0) = g(0) \quad \text{and} \quad f'(0) = g'(0).$$

**c.** For the determined values of $a$, $b$, and $c$, what happens for the third and fourth derivatives of $f$ and $g$ in each of parts (a) and (b)?

**4. Solutions to differential equations**

**a.** Show that $y = \sin x$, $y = \cos x$, and $y = a\cos x + b\sin x$ ($a$ and $b$ constants) all satisfy the equation

$$y'' + y = 0.$$

**b.** How would you modify the functions in part (a) to satisfy the equation

$$y'' + 4y = 0?$$

Generalize this result.

**5. An osculating circle** Find the values of $h$, $k$, and $a$ that make the circle $(x - h)^2 + (y - k)^2 = a^2$ tangent to the parabola $y = x^2 + 1$ at the point (1, 2) and that also make the second derivatives $d^2y/dx^2$ have the same value on both curves there. Circles like this one that are tangent to a curve and have the same second derivative as the curve at the point of tangency are called *osculating circles* (from the Latin *osculari*, meaning "to kiss"). We encounter them again in Chapter 13.

**6. Marginal revenue** A bus will hold 60 people. The number $x$ of people per trip who use the bus is related to the fare charged ($p$ dollars) by the law $p = [3 - (x/40)]^2$. Write an expression for the total revenue $r(x)$ per trip received by the bus company. What number of people per trip will make the marginal revenue $dr/dx$ equal to zero? What is the corresponding fare? (This fare is the one that maximizes the revenue, so the bus company should probably rethink its fare policy.)

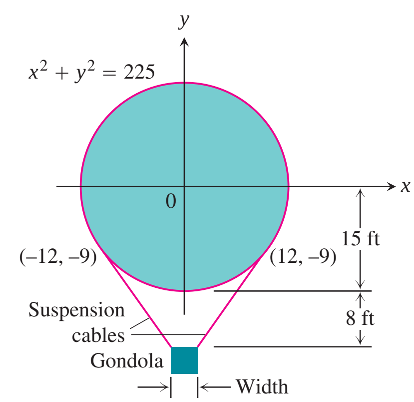
**7. Industrial production**

**a.** Economists often use the expression "rate of growth" in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time $t$ in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.)

Let $v = g(t)$ be the average production per person in the labor force at time $t$. The total production is then $y = uv$. If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, $y$.

**b.** Suppose that the labor force in part (a) is decreasing at the rate of 2% per year while the production per person is increasing at the rate of 3% per year. Is the total production increasing, or is it decreasing, and at what rate?

**8. Designing a gondola** The designer of a 30-ft-diameter spherical hot air balloon wants to suspend the gondola 8 ft below the bottom of the balloon with cables tangent to the surface of the balloon, as shown. Two of the cables are shown running from the top edges of the gondola to their points of tangency, $(-12, -9)$ and $(12, -9)$. How wide should the gondola be?




NOT TO SCALE

**9. Pisa by parachute** The photograph shows Mike McCarthy parachuting from the top of the Tower of Pisa on August 5, 1988. Make a rough sketch to show the shape of the graph of his speed during the jump.

Photograph is not available.

Mike McCarthy of London jumped from the Tower of Pisa and then opened his parachute in what he said was a world record low-level parachute jump of 179 ft. (*Source: Boston Globe*, Aug. 6, 1988.)

**10. Motion of a particle** The position at time $t \geq 0$ of a particle moving along a coordinate line is

$$s = 10\cos(t + \pi/4).$$

**a.** What is the particle's starting position $(t = 0)$?

**b.** What are the points farthest to the left and right of the origin reached by the particle?

**c.** Find the particle's velocity and acceleration at the points in part (b).

**d.** When does the particle first reach the origin? What are its velocity, speed, and acceleration then?

**11. Shooting a paper clip** On Earth, you can easily shoot a paper clip 64 ft straight up into the air with a rubber band. In $t$ sec after firing, the paper clip is $s = 64t - 16t^2$ ft above your hand.

**a.** How long does it take the paper clip to reach its maximum height? With what velocity does it leave your hand?

**b.** On the moon, the same acceleration will send the paper clip to a height of $s = 64t - 2.6t^2$ ft in $t$ sec. About how long will it take the paper clip to reach its maximum height, and how high will it go?

**12. Velocities of two particles** At time $t$ sec, the positions of two particles on a coordinate line are $s_1 = 3t^3 - 12t^2 + 18t + 5$ m and $s_2 = -t^3 + 9t^2 - 12t$ m. When do the particles have the same velocities?

**13. Velocity of a particle** A particle of constant mass $m$ moves along the $x$-axis. Its velocity $v$ and position $x$ satisfy the equation

$$\frac{1}{2}m(v^2 - v_0{}^2) = \frac{1}{2}k(x_0{}^2 - x^2),$$

where $k$, $v_0$, and $x_0$ are constants. Show that whenever $v \neq 0$,

$$m\frac{dv}{dt} = -kx.$$

**14. Average and instantaneous velocity**

**a.** Show that if the position $x$ of a moving point is given by a quadratic function of $t$, $x = At^2 + Bt + C$, then the average velocity over any time interval $[t_1, t_2]$ is equal to the instantaneous velocity at the midpoint of the time interval.

**b.** What is the geometric significance of the result in part (a)?

**15.** Find all values of the constants $m$ and $b$ for which the function

$$y = \begin{cases} \sin x, & x < \pi \\ mx + b, & x \geq \pi \end{cases}$$

is

**a.** continuous at $x = \pi$.

**b.** differentiable at $x = \pi$.

**16.** Does the function

$$f(x) = \begin{cases} \dfrac{1 - \cos x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a derivative at $x = 0$? Explain.

**17. a.** For what values of $a$ and $b$ will

$$f(x) = \begin{cases} ax, & x < 2 \\ ax^2 - bx + 3, & x \geq 2 \end{cases}$$

be differentiable for all values of $x$?

**b.** Discuss the geometry of the resulting graph of $f$.

**18. a.** For what values of $a$ and $b$ will

$$g(x) = \begin{cases} ax + b, & x \leq -1 \\ ax^3 + x + 2b, & x > -1 \end{cases}$$

be differentiable for all values of $x$?

**b.** Discuss the geometry of the resulting graph of $g$.

**19. Odd differentiable functions** Is there anything special about the derivative of an odd differentiable function of $x$? Give reasons for your answer.

**20. Even differentiable functions** Is there anything special about the derivative of an even differentiable function of $x$? Give reasons for your answer.

**21.** Suppose that the functions $f$ and $g$ are defined throughout an open interval containing the point $x_0$, that $f$ is differentiable at $x_0$, that $f(x_0) = 0$, and that $g$ is continuous at $x_0$. Show that the product $fg$ is differentiable at $x_0$. This process shows, for example, that although $|x|$ is not differentiable at $x = 0$, the product $x|x|$ *is* differentiable at $x = 0$.

**22.** *(Continuation of Exercise 21.)* Use the result of Exercise 21 to show that the following functions are differentiable at $x = 0$.

**a.** $|x|\sin x$ **b.** $x^{2/3}\sin x$ **c.** $\sqrt[3]{x}(1 - \cos x)$

**d.** $h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

**23.** Is the derivative of

$$h(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

continuous at $x = 0$? How about the derivative of $k(x) = xh(x)$? Give reasons for your answers.

**24.** Suppose that a function $f$ satisfies the following conditions for all real values of $x$ and $y$:

**i.** $f(x + y) = f(x) \cdot f(y)$.

**ii.** $f(x) = 1 + xg(x)$, where $\lim_{x \to 0} g(x) = 1$.

Show that the derivative $f'(x)$ exists at every value of $x$ and that $f'(x) = f(x)$.

**25. The generalized product rule** Use mathematical induction to prove that if $y = u_1 u_2 \cdots u_n$ is a finite product of differentiable functions, then $y$ is differentiable on their common domain and

$$\frac{dy}{dx} = \frac{du_1}{dx}u_2 \cdots u_n + u_1\frac{du_2}{dx} \cdots u_n + \cdots + u_1 u_2 \cdots u_{n-1}\frac{du_n}{dx}.$$

**26. Leibniz's rule for higher-order derivatives of products** Leibniz's rule for higher-order derivatives of products of differentiable functions says that

**a.** $\dfrac{d^2(uv)}{dx^2} = \dfrac{d^2u}{dx^2}v + 2\dfrac{du}{dx}\dfrac{dv}{dx} + u\dfrac{d^2v}{dx^2}$

**b.** $\dfrac{d^3(uv)}{dx^3} = \dfrac{d^3u}{dx^3}v + 3\dfrac{d^2u}{dx^2}\dfrac{dv}{dx} + 3\dfrac{du}{dx}\dfrac{d^2v}{dx^2} + u\dfrac{d^3v}{dx^3}$

**c.** 
$$\frac{d^n(uv)}{dx^n} = \frac{d^nu}{dx^n}v + n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{d^{n-k}u}{dx^{n-k}}\frac{d^kv}{dx^k} + \cdots + u\frac{d^nv}{dx^n}.$$

The equations in parts (a) and (b) are special cases of the equation in part (c). Derive the equation in part (c) by mathematical induction, using

$$\binom{m}{k} + \binom{m}{k+1} = \frac{m!}{k!(m-k)!} + \frac{m!}{(k+1)!(m-k-1)!}.$$

**27. The period of a clock pendulum** The period $T$ of a clock pendulum (time for one full swing and back) is given by the formula $T^2 = 4\pi^2 L/g$, where $T$ is measured in seconds, $g = 32.2$ ft/sec$^2$, and $L$, the length of the pendulum, is measured in feet. Find approximately

**a.** the length of a clock pendulum whose period is $T = 1$ sec.

**b.** the change $dT$ in $T$ if the pendulum in part (a) is lengthened 0.01 ft.

**c.** the amount the clock gains or loses in a day as a result of the period's changing by the amount $dT$ found in part (b).

**28. The melting ice cube** Assume an ice cube retains its cubical shape as it melts. If we call its edge length $s$, its volume is $V = s^3$ and its surface area is $6s^2$. We assume that $V$ and $s$ are differentiable functions of time $t$. We assume also that the cube's volume decreases at a rate that is proportional to its surface area. (This latter assumption seems reasonable enough when we think that the melting takes place at the surface: Changing the amount of surface changes the amount of ice exposed to melt.) In mathematical terms,

$$\frac{dV}{dt} = -k(6s^2), \qquad k > 0.$$

The minus sign indicates that the volume is decreasing. We assume that the proportionality factor $k$ is constant. (It probably depends on many things, such as the relative humidity of the surrounding air, the air temperature, and the incidence or absence of sunlight, to name only a few.) Assume a particular set of conditions in which the cube lost $1/4$ of its volume during the first hour, and that the volume is $V_0$ when $t = 0$. How long will it take the ice cube to melt?