APEX CALCULUS

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12: Functions of Several Variables

A function of the form $y = f(x)$ is a function of a single variable; given a value of x, we can find a value y. Even the vector–valued functions of Chapter 11 are single–variable functions; the input is a single variable though the output is a vector.

There are many situations where a desired quantity is a function of two or more variables. For instance, wind chill is measured by knowing the temperature and wind speed; the volume of a gas can be computed knowing the pressure and temperature of the gas; to compute a baseball player's batting average, one needs to know the number of hits and the number of at–bats.

This chapter studies **multivariable** functions, that is, functions with more than one input.

12.1 Introduction to Multivariable Functions

Definition 12.1.1 Function of Two Variables

Let D be a subset of $\mathbb{R}^2$. A **function *f* of two variables** is a rule that assigns each pair (x, y) in D a value $z = f(x, y)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Example 12.1.1 Understanding a function of two variables

Let $z = f(x, y) = x^2 - y$. Evaluate $f(1, 2)$, $f(2, 1)$, and $f(-2, 4)$; find the domain and range of f.

Solution Using the definition $f(x, y) = x^2 - y$, we have:

$$\begin{aligned} f(1,2) &= 1^2 - 2 = -1 \\ f(2,1) &= 2^2 - 1 = 3 \\ f(-2,4) &= (-2)^2 - 4 = 0 \end{aligned}$$

The domain is not specified, so we take it to be all possible pairs in $\mathbb{R}^2$ for which f is defined. In this example, f is defined for *all* pairs (x, y), so the domain D of f is $\mathbb{R}^2$.

The output of f can be made as large or small as possible; any real number r can be the output. (In fact, given any real number r, $f(0, -r) = r$.) So the range R of f is $\mathbb{R}$.

Example 12.1.2 Understanding a function of two variables

Let $f(x,y) = \sqrt{1 - \dfrac{x^2}{9} - \dfrac{y^2}{4}}$. Find the domain and range of f.

Solution The domain is all pairs (x,y) allowable as input in f. Because of the square–root, we need (x,y) such that $0 \le 1 - \frac{x^2}{9} - \frac{y^2}{4}$:

$$0 \le 1 - \frac{x^2}{9} - \frac{y^2}{4}$$
$$\frac{x^2}{9} + \frac{y^2}{4} \le 1$$

The above equation describes an ellipse and its interior as shown in Figure 12.1.1. We can represent the domain D graphically with the figure; in set notation, we can write $D = \left\{(x,y) \mid \frac{x^2}{9} + \frac{y^2}{4} \le 1\right\}$.

The range is the set of all possible output values. The square–root ensures that all output is ≥ 0. Since the x and y terms are squared, then subtracted, inside the square–root, the largest output value comes at $x = 0$, $y = 0$: $f(0,0) = 1$. Thus the range R is the interval $[0,1]$.

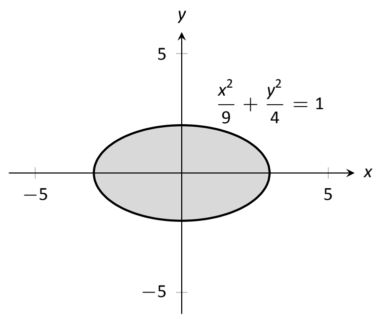

Figure 12.1.1: Illustrating the domain of $f(x,y)$ in Example 12.1.2.

Graphing Functions of Two Variables

The **graph** of a function f of two variables is the set of all points $\big(x, y, f(x,y)\big)$ where (x,y) is in the domain of f. This creates a **surface** in space.

One can begin sketching a graph by plotting points, but this has limitations. Consider Figure 12.1.2(a) where 25 points have been plotted of $f(x,y) = \dfrac{1}{x^2+y^2+1}$. More points have been plotted than one would reasonably want to do by hand, yet it is not clear at all what the graph of the function looks like. Technology allows us to plot lots of points, connect adjacent points with lines and add shading to create a graph like Figure 12.1.2b which does a far better job of illustrating the behavior of f.

While technology is readily available to help us graph functions of two variables, there is still a paper–and–pencil approach that is useful to understand and master as it, combined with high–quality graphics, gives one great insight into the behavior of a function. This technique is known as sketching **level curves**.

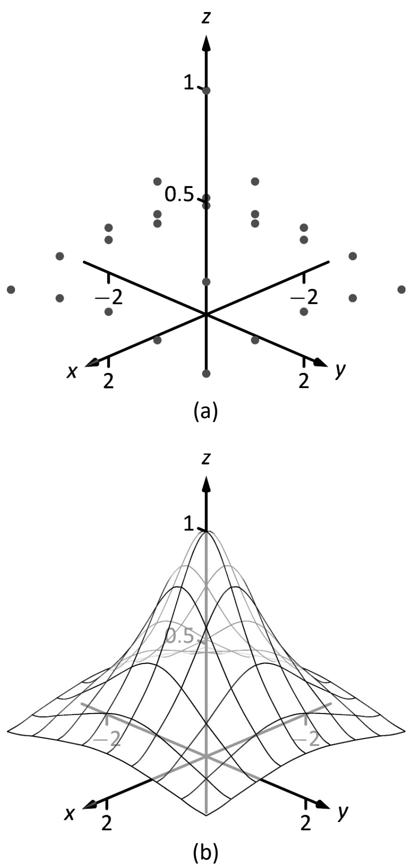

Figure 12.1.2: Graphing a function of two variables.

Level Curves

It may be surprising to find that the problem of representing a three dimensional surface on paper is familiar to most people (they just don't realize it). Topographical maps, like the one shown in Figure 12.1.3, represent the surface of Earth by indicating points with the same elevation with **contour lines**. The

Notes:

elevations marked are equally spaced; in this example, each thin line indicates an elevation change in 50ft increments and each thick line indicates a change of 200ft. When lines are drawn close together, elevation changes rapidly (as one does not have to travel far to rise 50ft). When lines are far apart, such as near "Aspen Campground," elevation changes more gradually as one has to walk farther to rise 50ft.

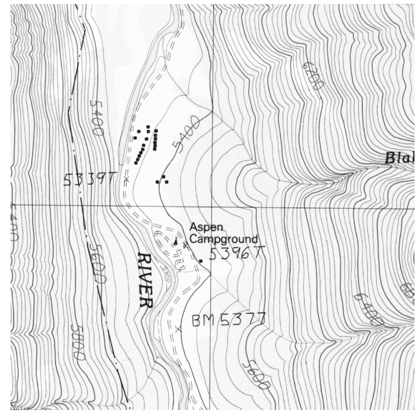

Figure 12.1.3: A topographical map displays elevation by drawing contour lines, along with the elevation is constant.
Sample taken from the public domain USGS Digital Raster Graphics, http://topmaps.usgs.gove/drg/.

Given a function $z = f(x, y)$, we can draw a "topographical map" of f by drawing **level curves** (or, contour lines). A level curve at $z = c$ is a curve in the x-y plane such that for all points (x, y) on the curve, $f(x, y) = c$.

When drawing level curves, it is important that the c values are spaced equally apart as that gives the best insight to how quickly the "elevation" is changing. Examples will help one understand this concept.

Example 12.1.3 Drawing Level Curves

Let $f(x, y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$. Find the level curves of f for $c = 0$, 0.2, 0.4, 0.6, 0.8 and 1.

Solution Consider first $c = 0$. The level curve for $c = 0$ is the set of all points (x, y) such that $0 = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$. Squaring both sides gives us

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

an ellipse centered at $(0, 0)$ with horizontal major axis of length 6 and minor axis of length 4. Thus for any point (x, y) on this curve, $f(x, y) = 0$.

Now consider the level curve for $c = 0.2$

$$\begin{aligned} 0.2 &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ 0.04 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 0.96 \\ \frac{x^2}{8.64} + \frac{y^2}{3.84} &= 1. \end{aligned}$$

This is also an ellipse, where $a = \sqrt{8.64} \approx 2.94$ and $b = \sqrt{3.84} \approx 1.96$.

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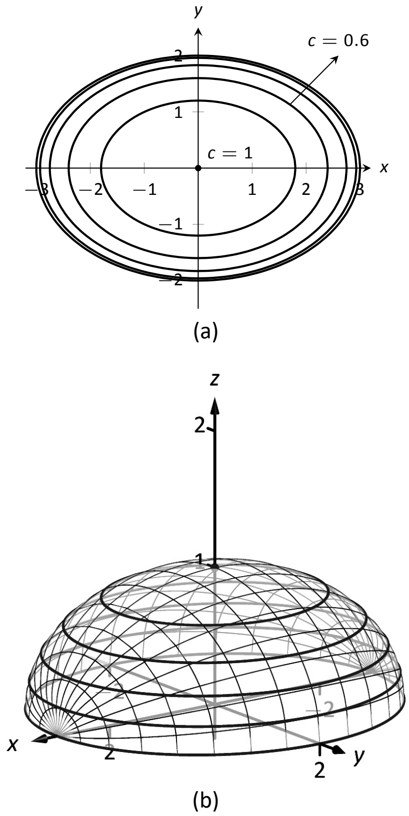

Figure 12.1.4: Graphing the level curves in Example 12.1.3.

In general, for $z = c$, the level curve is:

$$\begin{aligned} c &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ c^2 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 1 - c^2 \\ \frac{x^2}{9(1-c^2)} + \frac{y^2}{4(1-c^2)} &= 1, \end{aligned}$$

ellipses that are decreasing in size as c increases. A special case is when $c = 1$; there the ellipse is just the point $(0,0)$.

The level curves are shown in Figure 12.1.4(a). Note how the level curves for $c = 0$ and $c = 0.2$ are very, very close together: this indicates that f is growing rapidly along those curves.

In Figure 12.1.4(b), the curves are drawn on a graph of f in space. Note how the elevations are evenly spaced. Near the level curves of $c = 0$ and $c = 0.2$ we can see that f indeed is growing quickly.

Example 12.1.4 Analyzing Level Curves

Let $f(x,y) = \dfrac{x+y}{x^2+y^2+1}$. Find the level curves for $z = c$.

Solution We begin by setting $f(x,y) = c$ for an arbitrary c and seeing if algebraic manipulation of the equation reveals anything significant.

$$\begin{aligned} \frac{x+y}{x^2+y^2+1} &= c \\ x + y &= c(x^2+y^2+1). \end{aligned}$$

We recognize this as a circle, though the center and radius are not yet clear. By completing the square, we can obtain:

$$\left(x - \frac{1}{2c}\right)^2 + \left(y - \frac{1}{2c}\right)^2 = \frac{1}{2c^2} - 1,$$

a circle centered at $\big(1/(2c), 1/(2c)\big)$ with radius $\sqrt{1/(2c^2) - 1}$, where $|c| < 1/\sqrt{2}$. The level curves for $c = \pm 0.2$, ± 0.4 and ± 0.6 are sketched in Figure 12.1.5(a). To help illustrate "elevation," we use thicker lines for c values near 0, and dashed lines indicate where $c < 0$.

There is one special level curve, when $c = 0$. The level curve in this situation is $x + y = 0$, the line $y = -x$.

Notes:

In Figure 12.1.5(b) we see a graph of the surface. Note how the y-axis is pointing away from the viewer to more closely resemble the orientation of the level curves in (a).

Seeing the level curves helps us understand the graph. For instance, the graph does not make it clear that one can "walk" along the line $y = -x$ without elevation change, though the level curve does.

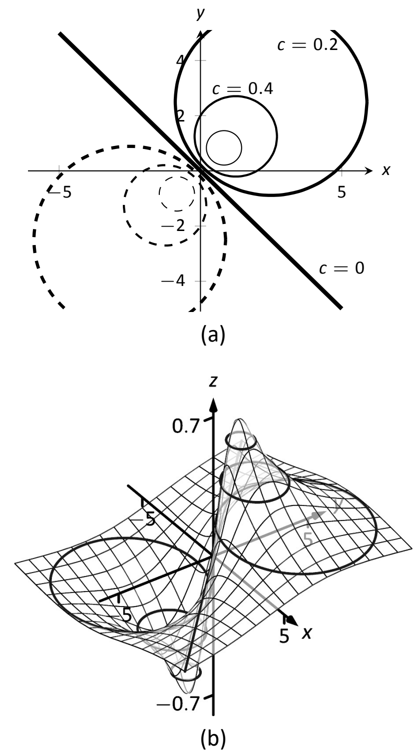

Figure 12.1.5: Graphing the level curves in Example 12.1.4.

Functions of Three Variables

We extend our study of multivariable functions to functions of three variables. (One can make a function of as many variables as one likes; we limit our study to three variables.)

> **Definition 12.1.2 Function of Three Variables**
>
> Let D be a subset of $\mathbb{R}^3$. A **function f of three variables** is a rule that assigns each triple (x, y, z) in D a value $w = f(x, y, z)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Note how this definition closely resembles that of Definition 12.1.1.

Example 12.1.5 Understanding a function of three variables

Let $f(x, y, z) = \dfrac{x^2 + z + 3\sin y}{x + 2y - z}$. Evaluate f at the point $(3, 0, 2)$ and find the domain and range of f.

Solution $$f(3, 0, 2) = \frac{3^2 + 2 + 3\sin 0}{3 + 2(0) - 2} = 11.$$

As the domain of f is not specified, we take it to be the set of all triples (x, y, z) for which $f(x, y, z)$ is defined. As we cannot divide by 0, we find the domain D is

$$D = \{(x, y, z) \mid x + 2y - z \neq 0\}.$$

We recognize that the set of all points in $\mathbb{R}^3$ that *are not* in D form a plane in space that passes through the origin (with normal vector $\langle 1, 2, -1\rangle$).

We determine the range R is $\mathbb{R}$; that is, all real numbers are possible outputs of f. There is no set way of establishing this. Rather, to get numbers near 0 we can let $y = 0$ and choose $z \approx -x^2$. To get numbers of arbitrarily large magnitude, we can let $z \approx x + 2y$.

Notes:

Level Surfaces

It is very difficult to produce a meaningful graph of a function of three variables. A function of *one* variable is a *curve* drawn in *2* dimensions; a function of *two* variables is a *surface* drawn in *3* dimensions; a function of *three* variables is a *hypersurface* drawn in *4* dimensions.

There are a few techniques one can employ to try to "picture" a graph of three variables. One is an analogue of level curves: **level surfaces**. Given $w = f(x, y, z)$, the level surface at $w = c$ is the surface in space formed by all points (x, y, z) where $f(x, y, z) = c$.

Example 12.1.6 Finding level surfaces

If a point source S is radiating energy, the intensity I at a given point P in space is inversely proportional to the square of the distance between S and P. That is, when $S = (0, 0, 0)$, $I(x, y, z) = \dfrac{k}{x^2 + y^2 + z^2}$ for some constant k.

Let $k = 1$; find the level surfaces of I.

Solution We can (mostly) answer this question using "common sense." If energy (say, in the form of light) is emanating from the origin, its intensity will be the same at all points equidistant from the origin. That is, at any point on the surface of a sphere centered at the origin, the intensity should be the same. Therefore, the level surfaces are spheres.

We now find this mathematically. The level surface at $I = c$ is defined by

$$c = \frac{1}{x^2 + y^2 + z^2}.$$

A small amount of algebra reveals

$$x^2 + y^2 + z^2 = \frac{1}{c}.$$

Given an intensity c, the level surface $I = c$ is a sphere of radius $1/\sqrt{c}$, centered at the origin.

Figure 12.1.6 gives a table of the radii of the spheres for given c values. Normally one would use equally spaced c values, but these values have been chosen purposefully. At a distance of 0.25 from the point source, the intensity is 16; to move to a point of half that intensity, one just moves out 0.1 to 0.35 – not much at all. To again halve the intensity, one moves 0.15, a little more than before.

Note how each time the intensity if halved, the distance required to move away grows. We conclude that the closer one is to the source, the more rapidly the intensity changes.

c	r
16.	0.25
8.	0.35
4.	0.5
2.	0.71
1.	1.
0.5	1.41
0.25	2.
0.125	2.83
0.0625	4.

Figure 12.1.6: A table of c values and the corresponding radius r of the spheres of constant value in Example 12.1.6.

In the next section we apply the concepts of limits to functions of two or more variables.

Notes:

Exercises 12.1

Terms and Concepts

1. Give two examples (other than those given in the text) of "real world" functions that require more than one input.

2. The graph of a function of two variables is a ________.

3. Most people are familiar with the concept of level curves in the context of ________ maps.

4. T/F: Along a level curve, the output of a function does not change.

5. The analogue of a level curve for functions of three variables is a level ________.

6. What does it mean when level curves are close together? Far apart?

Problems

In Exercises 7 – 14, give the domain and range of the multivariable function.

7. $f(x, y) = x^2 + y^2 + 2$

8. $f(x, y) = x + 2y$

9. $f(x, y) = x - 2y$

10. $f(x, y) = \dfrac{1}{x + 2y}$

11. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

12. $f(x, y) = \sin x \cos y$

13. $f(x, y) = \sqrt{9 - x^2 - y^2}$

14. $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2 - 9}}$

In Exercises 15 – 22, describe in words and sketch the level curves for the function and given *c* values.

15. $f(x, y) = 3x - 2y$; $c = -2, 0, 2$

16. $f(x, y) = x^2 - y^2$; $c = -1, 0, 1$

17. $f(x, y) = x - y^2$; $c = -2, 0, 2$

18. $f(x, y) = \dfrac{1 - x^2 - y^2}{2y - 2x}$; $c = -2, 0, 2$

19. $f(x, y) = \dfrac{2x - 2y}{x^2 + y^2 + 1}$; $c = -1, 0, 1$

20. $f(x, y) = \dfrac{y - x^3 - 1}{x}$; $c = -3, -1, 0, 1, 3$

21. $f(x, y) = \sqrt{x^2 + 4y^2}$; $c = 1, 2, 3, 4$

22. $f(x, y) = x^2 + 4y^2$; $c = 1, 2, 3, 4$

In Exercises 23 – 26, give the domain and range of the functions of three variables.

23. $f(x, y, z) = \dfrac{x}{x + 2y - 4z}$

24. $f(x, y, z) = \dfrac{1}{1 - x^2 - y^2 - z^2}$

25. $f(x, y, z) = \sqrt{z - x^2 + y^2}$

26. $f(x, y, z) = z^2 \sin x \cos y$

In Exercises 27 – 30, describe the level surfaces of the given functions of three variables.

27. $f(x, y, z) = x^2 + y^2 + z^2$

28. $f(x, y, z) = z - x^2 + y^2$

29. $f(x, y, z) = \dfrac{x^2 + y^2}{z}$

30. $f(x, y, z) = \dfrac{z}{x - y}$

31. Compare the level curves of Exercises 21 and 22. How are they similar, and how are they different? Each surface is a quadric surface; describe how the level curves are consistent with what we know about each surface.

12.2 Limits and Continuity of Multivariable Functions

We continue with the pattern we have established in this text: after defining a new kind of function, we apply calculus ideas to it. The previous section defined functions of two and three variables; this section investigates what it means for these functions to be "continuous."

We begin with a series of definitions. We are used to "open intervals" such as $(1,3)$, which represents the set of all x such that $1 < x < 3$, and "closed intervals" such as $[1,3]$, which represents the set of all x such that $1 \leq x \leq 3$. We need analogous definitions for open and closed sets in the x-y plane.

Definition 12.2.1 Open Disk, Boundary and Interior Points, Open and Closed Sets, Bounded Sets

An **open disk** B in $\mathbb{R}^2$ centered at (x_0, y_0) with radius r is the set of all points (x,y) such that $\sqrt{(x-x_0)^2+(y-y_0)^2} < r$.

Let S be a set of points in $\mathbb{R}^2$. A point P in $\mathbb{R}^2$ is a **boundary point** of S if all open disks centered at P contain both points in S and points not in S.

A point P in S is an **interior point** of S if there is an open disk centered at P that contains only points in S.

A set S is **open** if every point in S is an interior point.

A set S is **closed** if it contains all of its boundary points.

A set S is **bounded** if there is an $M > 0$ such that the open disk, centered at the origin with radius M, contains S. A set that is not bounded is **unbounded**.

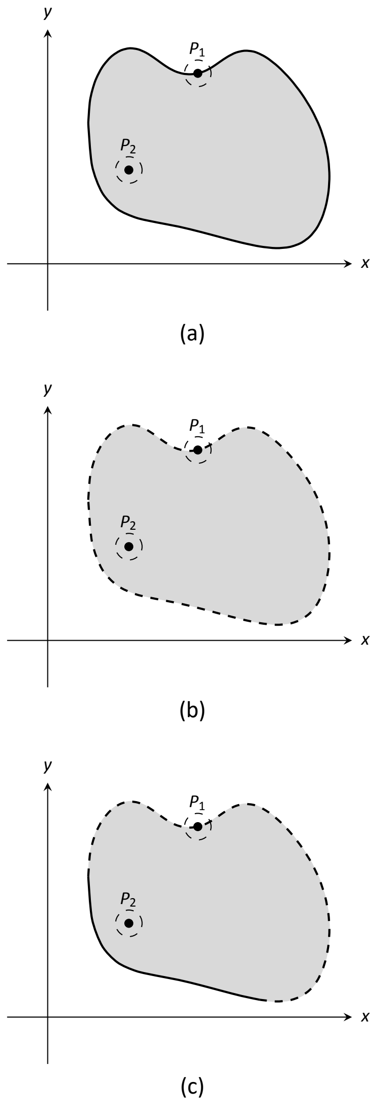

Figure 12.2.1: Illustrating open and closed sets in the x-y plane.

Figure 12.2.1 shows several sets in the x-y plane. In each set, point P_1 lies on the boundary of the set as all open disks centered there contain both points in, and not in, the set. In contrast, point P_2 is an interior point for there is an open disk centered there that lies entirely within the set.

The set depicted in Figure 12.2.1(a) is a closed set as it contains all of its boundary points. The set in (b) is open, for all of its points are interior points (or, equivalently, it does not contain any of its boundary points). The set in (c) is neither open nor closed as it contains some of its boundary points.

Notes:

Example 12.2.1 Determining open/closed, bounded/unbounded

Determine if the domain of the function $f(x,y) = \sqrt{1-x^2/9-y^2/4}$ is open, closed, or neither, and if it is bounded.

SOLUTION This domain of this function was found in Example 12.1.2 to be $D = \{(x,y) \mid \frac{x^2}{9}+\frac{y^2}{4} \leq 1\}$, the region *bounded* by the ellipse $\frac{x^2}{9}+\frac{y^2}{4} = 1$. Since the region includes the boundary (indicated by the use of "$\leq$"), the set contains all of its boundary points and hence is closed. The region is bounded as a disk of radius 4, centered at the origin, contains D.

Example 12.2.2 Determining open/closed, bounded/unbounded

Determine if the domain of $f(x,y) = \frac{1}{x-y}$ is open, closed, or neither.

SOLUTION As we cannot divide by 0, we find the domain to be $D = \{(x,y) \mid x-y \neq 0\}$. In other words, the domain is the set of all points (x,y) *not* on the line $y = x$.

The domain is sketched in Figure 12.2.2. Note how we can draw an open disk around any point in the domain that lies entirely inside the domain, and also note how the only boundary points of the domain are the points on the line $y = x$. We conclude the domain is an open set. The set is unbounded.

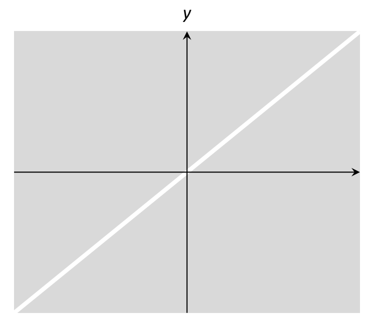

Figure 12.2.2: Sketching the domain of the function in Example 12.2.2.

Limits

Recall a pseudo–definition of the limit of a function of one variable: "$\lim\limits_{x\to c} f(x) = L$" means that if x is "really close" to c, then $f(x)$ is "really close" to L. A similar pseudo–definition holds for functions of two variables. We'll say that

$$\text{“}\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L\text{”}$$

means "if the point (x,y) is really close to the point (x_0,y_0), then $f(x,y)$ is really close to L." The formal definition is given below.

Definition 12.2.2 Limit of a Function of Two Variables

Let S be a set containing $P = (x_0,y_0)$ where every open disk centered at P contains points in S other than P, let f be a function of two variables defined on S, except possibly at P, and let L be a real number. The **limit of** $f(x,y)$ **as** (x,y) **approaches** (x_0,y_0) **is** L, denoted

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all (x,y) in S, where $(x,y) \neq (x_0,y_0)$, if (x,y) is in the open disk centered at (x_0,y_0) with radius δ, then $|f(x,y) - L| < \varepsilon$.

Note: While our first limit definition was defined over an open interval, we now define limits over a set S in the plane (where S does not have to be open). As planar sets can be far more complicated than intervals, our definition adds the restriction ". . . where every open disk centered at P contains points in S other than P." In this text, all sets we'll consider will satisfy this condition and we won't bother to check; it is included in the definition for completeness.

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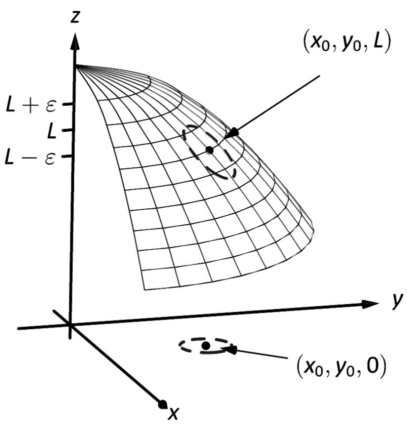


Figure 12.2.3: **Illustrating the definition of a limit.** The open disk in the x-y plane has radius δ. Let (x, y) be any point in this disk; $f(x, y)$ is within ε of L.

The concept behind Definition 12.2.2 is sketched in Figure 12.2.3. Given $\varepsilon > 0$, find $\delta > 0$ such that if (x, y) is any point in the open disk centered at (x_0, y_0) in the x-y plane with radius δ, then $f(x, y)$ should be within ε of L.

Computing limits using this definition is rather cumbersome. The following theorem allows us to evaluate limits much more easily.

Theorem 12.2.1 Basic Limit Properties of Functions of Two Variables

Let b, x_0, y_0, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x, y) = K.$$

The following limits hold.

1. Constants: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b = b$
2. Identity $\displaystyle\lim_{(x,y)\to(x_0,y_0)} x = x_0; \qquad \lim_{(x,y)\to(x_0,y_0)} y = y_0$
3. Sums/Differences: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \big(f(x, y) \pm g(x, y)\big) = L \pm K$
4. Scalar Multiples: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b \cdot f(x, y) = bL$
5. Products: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y) \cdot g(x, y) = LK$
6. Quotients: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)/g(x, y) = L/K$, $(K \neq 0)$
7. Powers: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)^n = L^n$

This theorem, combined with Theorems 1.3.2 and 1.3.3 of Section 1.3, allows us to evaluate many limits.

Example 12.2.3 Evaluating a limit

Evaluate the following limits:

$$1. \lim_{(x,y)\to(1,\pi)} \left(\frac{y}{x} + \cos(xy)\right) \qquad\qquad 2. \lim_{(x,y)\to(0,0)} \frac{3xy}{x^2 + y^2}$$

Notes:

Solution

1. The aforementioned theorems allow us to simply evaluate $y/x + \cos(xy)$ when $x = 1$ and $y = \pi$. If an indeterminate form is returned, we must do more work to evaluate the limit; otherwise, the result is the limit. Therefore
$$\begin{aligned}\lim_{(x,y)\to(1,\pi)} \left(\frac{y}{x} + \cos(xy)\right) &= \frac{\pi}{1} + \cos\pi \\ &= \pi - 1.\end{aligned}$$

2. We attempt to evaluate the limit by substituting 0 in for x and y, but the result is the indeterminate form "0/0." To evaluate this limit, we must "do more work," but we have not yet learned what "kind" of work to do. Therefore we cannot yet evaluate this limit.

When dealing with functions of a single variable we also considered one–sided limits and stated
$$\lim_{x\to c} f(x) = L \quad \text{if, and only if,} \quad \lim_{x\to c^+} f(x) = L \quad \textbf{and} \quad \lim_{x\to c^-} f(x) = L.$$

That is, the limit is L if and only if $f(x)$ approaches L when x approaches c from **either** direction, the left or the right.

In the plane, there are infinitely many directions from which (x, y) might approach (x_0, y_0). In fact, we do not have to restrict ourselves to approaching (x_0, y_0) from a particular direction, but rather we can approach that point along a path that is not a straight line. It is possible to arrive at different limiting values by approaching (x_0, y_0) along different paths. If this happens, we say that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist (this is analogous to the left and right hand limits of single variable functions not being equal).

Our theorems tell us that we can evaluate most limits quite simply, without worrying about paths. When indeterminate forms arise, the limit may or may not exist. If it does exist, it can be difficult to prove this as we need to show the same limiting value is obtained regardless of the path chosen. The case where the limit does not exist is often easier to deal with, for we can often pick two paths along which the limit is different.

Example 12.2.4 Showing limits do not exist

1. Show $\lim\limits_{(x,y)\to(0,0)} \dfrac{3xy}{x^2 + y^2}$ does not exist by finding the limits along the lines $y = mx$.

Notes:

2. Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist by finding the limit along the path $y = -\sin x$.

Solution

1. Evaluating $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$ along the lines $y = mx$ means replace all y's with mx and evaluating the resulting limit:

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{3x(mx)}{x^2+(mx)^2} &= \lim_{x\to 0} \frac{3mx^2}{x^2(m^2+1)}\\ &= \lim_{x\to 0} \frac{3m}{m^2+1}\\ &= \frac{3m}{m^2+1}.\end{aligned}$$

While the limit exists for each choice of m, we get a *different* limit for each choice of m. That is, along different lines we get differing limiting values, meaning *the* limit does not exist.

2. Let $f(x,y) = \frac{\sin(xy)}{x+y}$. We are to show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist by finding the limit along the path $y = -\sin x$. First, however, consider the limits found along the lines $y = mx$ as done above.

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{\sin\big(x(mx)\big)}{x+mx} &= \lim_{x\to 0} \frac{\sin(mx^2)}{x(m+1)}\\ &= \lim_{x\to 0} \frac{\sin(mx^2)}{x}\cdot\frac{1}{m+1}.\end{aligned}$$

By applying L'Hôpital's Rule, we can show this limit is 0 *except* when $m = -1$, that is, along the line $y = -x$. This line is not in the domain of f, so we have found the following fact: along every line $y = mx$ in the domain of f, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Now consider the limit along the path $y = -\sin x$:

$$\lim_{(x,-\sin x)\to(0,0)} \frac{\sin\big(-x\sin x\big)}{x-\sin x} = \lim_{x\to 0} \frac{\sin\big(-x\sin x\big)}{x-\sin x}$$

Now apply L'Hôpital's Rule twice:

$$\begin{aligned}&= \lim_{x\to 0} \frac{\cos\big(-x\sin x\big)(-\sin x - x\cos x)}{1-\cos x} \qquad (\text{“} = 0/0\text{”})\\ &= \lim_{x\to 0} \frac{-\sin\big(-x\sin x\big)(-\sin x - x\cos x)^2 + \cos\big(-x\sin x\big)(-2\cos x + x\sin x)}{\sin x}\\ &= \text{“}{-2}/0\text{”} \Rightarrow \text{the limit does not exist.}\end{aligned}$$

Notes:

Step back and consider what we have just discovered. Along any line $y = mx$ in the domain of the $f(x, y)$, the limit is 0. However, along the path $y = -\sin x$, which lies in the domain of $f(x, y)$ for all $x \neq 0$, the limit does not exist. Since the limit is not the same along every path to $(0, 0)$, we say $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist.

Example 12.2.5 Finding a limit

Let $f(x, y) = \dfrac{5x^2y^2}{x^2+y^2}$. Find $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$.

Solution It is relatively easy to show that along any line $y = mx$, the limit is 0. This is not enough to prove that the limit exists, as demonstrated in the previous example, but it tells us that if the limit does exist then it must be 0.

To prove the limit is 0, we apply Definition 12.2.2. Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that if $\sqrt{(x-0)^2+(y-0)^2} < \delta$, then $|f(x, y) - 0| < \varepsilon$.

Set $\delta < \sqrt{\varepsilon/5}$. Note that $\left|\dfrac{5y^2}{x^2+y^2}\right| < 5$ for all $(x, y) \neq (0, 0)$, and that if $\sqrt{x^2+y^2} < \delta$, then $x^2 < \delta^2$.

Let $\sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta$. Consider $|f(x, y) - 0|$:

$$\begin{aligned}
|f(x, y) - 0| &= \left|\frac{5x^2y^2}{x^2+y^2} - 0\right| \\
&= \left|x^2 \cdot \frac{5y^2}{x^2+y^2}\right| \\
&< \delta^2 \cdot 5 \\
&< \frac{\varepsilon}{5} \cdot 5 \\
&= \varepsilon.
\end{aligned}$$

Thus if $\sqrt{(x-0)^2+(y-0)^2} < \delta$ then $|f(x, y) - 0| < \varepsilon$, which is what we wanted to show. Thus $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^2+y^2} = 0$.

Continuity

Definition 1.5.1 defines what it means for a function of one variable to be continuous. In brief, it meant that the graph of the function did not have breaks, holes, jumps, etc. We define continuity for functions of two variables in a similar way as we did for functions of one variable.

Notes:

Definition 12.2.3 Continuous

Let a function $f(x,y)$ be defined on a set S containing the point (x_0,y_0).

1. f is **continuous** at (x_0,y_0) if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.
2. f is **continuous on** S if f is continuous at all points in S. If f is continuous at all points in $\mathbb{R}^2$, we say that f is **continuous everywhere**.

Example 12.2.6 Continuity of a function of two variables

Let $f(x,y) = \begin{cases} \frac{\cos y \sin x}{x} & x \neq 0 \\ \cos y & x = 0 \end{cases}$. Is f continuous at $(0,0)$? Is f continuous everywhere?

Solution To determine if f is continuous at $(0,0)$, we need to compare $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ to $f(0,0)$.

Applying the definition of f, we see that $f(0,0) = \cos 0 = 1$.

We now consider the limit $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$. Substituting 0 for x and y in $(\cos y \sin x)/x$ returns the indeterminate form "0/0", so we need to do more work to evaluate this limit.

Consider two related limits: $\displaystyle\lim_{(x,y)\to(0,0)} \cos y$ and $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin x}{x}$. The first limit does not contain x, and since $\cos y$ is continuous,

$$\lim_{(x,y)\to(0,0)} \cos y = \lim_{y\to 0} \cos y = \cos 0 = 1.$$

The second limit does not contain y. By Theorem 1.3.5 we can say

$$\lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Finally, Theorem 12.2.1 of this section states that we can combine these two limits as follows:

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x} &= \lim_{(x,y)\to(0,0)} (\cos y)\left(\frac{\sin x}{x}\right) \\ &= \left(\lim_{(x,y)\to(0,0)} \cos y\right)\left(\lim_{(x,y)\to(0,0)} \frac{\sin x}{x}\right) \\ &= (1)(1) \\ &= 1. \end{aligned}$$

Notes:

We have found that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x} = f(0,0)$, so f is continuous at $(0,0)$.

A similar analysis shows that f is continuous at all points in $\mathbb{R}^2$. As long as $x \neq 0$, we can evaluate the limit directly; when $x = 0$, a similar analysis shows that the limit is $\cos y$. Thus we can say that f is continuous everywhere. A graph of f is given in Figure 12.2.4. Notice how it has no breaks, jumps, etc.

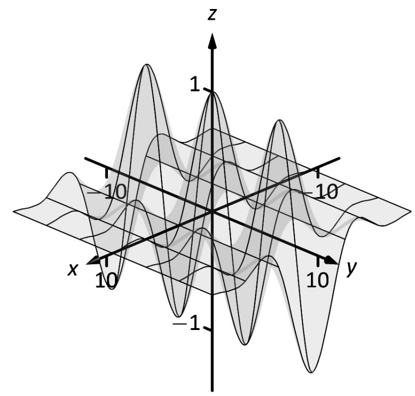


Figure 12.2.4: A graph of $f(x,y)$ in Example 12.2.6.

The following theorem is very similar to Theorem 1.5.1, giving us ways to combine continuous functions to create other continuous functions.

Theorem 12.2.2 Properties of Continuous Functions

Let f and g be continuous on a set S, let c be a real number, and let n be a positive integer. The following functions are continuous on S.

1. Sums/Differences: $f \pm g$
2. Constant Multiples: $c \cdot f$
3. Products: $f \cdot g$
4. Quotients: f/g (as longs as $g \neq 0$ on S)
5. Powers: f^n
6. Roots: $\sqrt[n]{f}$ (if n is even then $f \geq 0$ on S; if n is odd, then true for all values of f on S.)
7. Compositions: Adjust the definitions of f and g to: Let f be continuous on S, where the range of f on S is J, and let g be a single variable function that is continuous on J. Then $g \circ f$, i.e., $g(f(x,y))$, is continuous on S.

Example 12.2.7 Establishing continuity of a function

Let $f(x,y) = \sin(x^2 \cos y)$. Show f is continuous everywhere.

Solution We will apply both Theorems 1.5.1 and 12.2.2. Let $f_1(x,y) = x^2$. Since y is not actually used in the function, and polynomials are continuous (by Theorem 1.5.1), we conclude f_1 is continuous everywhere. A similar statement can be made about $f_2(x,y) = \cos y$. Part 3 of Theorem 12.2.2 states that $f_3 = f_1 \cdot f_2$ is continuous everywhere, and Part 7 of the theorem states the composition of sine with f_3 is continuous: that is, $\sin(f_3) = \sin(x^2 \cos y)$ is continuous everywhere.

Notes:

Functions of Three Variables

The definitions and theorems given in this section can be extended in a natural way to definitions and theorems about functions of three (or more) variables. We cover the key concepts here; some terms from Definitions 12.2.1 and 12.2.3 are not redefined but their analogous meanings should be clear to the reader.

Definition 12.2.4 Open Balls, Limit, Continuous

1. An **open ball** in $\mathbb{R}^3$ centered at (x_0, y_0, z_0) with radius r is the set of all points (x, y, z) such that $\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = r$.

2. Let D be a set in $\mathbb{R}^3$ containing (x_0, y_0, z_0) where every open ball centered at (x_0, y_0, z_0) contains points of D other than (x_0, y_0, z_0), and let $f(x, y, z)$ be a function of three variables defined on D, except possibly at (x_0, y_0, z_0). The **limit** of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0) is L, denoted
$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = L,$$
means that given any $\varepsilon > 0$, there is a $\delta > 0$ such that for all (x, y, z) in D, $(x, y, z) \neq (x_0, y_0, z_0)$, if (x, y, z) is in the open ball centered at (x_0, y_0, z_0) with radius δ, then $|f(x, y, z) - L| < \varepsilon$.

3. Let $f(x, y, z)$ be defined on a set D containing (x_0, y_0, z_0). f is **continuous** at (x_0, y_0, z_0) if $\displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x, y, z) = f(x_0, y_0, z_0)$; if f is continuous at all points in D, we say f is **continuous on** D.

These definitions can also be extended naturally to apply to functions of four or more variables. Theorem 12.2.2 also applies to function of three or more variables, allowing us to say that the function
$$f(x, y, z) = \frac{e^{x^2+y}\sqrt{y^2+z^2+3}}{\sin(xyz)+5}$$
is continuous everywhere.

When considering single variable functions, we studied limits, then continuity, then the derivative. In our current study of multivariable functions, we have studied limits and continuity. In the next section we study derivation, which takes on a slight twist as we are in a multivarible context.

Notes:

Exercises 12.2

Terms and Concepts

1. Describe in your own words the difference between boundary and interior points of a set.

2. Use your own words to describe (informally) what $\lim_{(x,y)\to(1,2)} f(x,y) = 17$ means.

3. Give an example of a closed, bounded set.

4. Give an example of a closed, unbounded set.

5. Give an example of a open, bounded set.

6. Give an example of a open, unbounded set.

Problems

In Exercises 7 – 10, a set S is given.

(a) Give one boundary point and one interior point, when possible, of S.

(b) State whether S is open, closed, or neither.

(c) State whether S is bounded or unbounded.

7. $S = \left\{(x,y) \;\middle|\; \dfrac{(x-1)^2}{4} + \dfrac{(y-3)^2}{9} \leq 1\right\}$

8. $S = \left\{(x,y) \mid y \neq x^2\right\}$

9. $S = \left\{(x,y) \mid x^2 + y^2 = 1\right\}$

10. $S = \{(x,y) | y > \sin x\}$

In Exercises 11 – 14:

(a) Find the domain D of the given function.

(b) State whether D is an open or closed set.

(c) State whether D is bounded or unbounded.

11. $f(x,y) = \sqrt{9 - x^2 - y^2}$

12. $f(x,y) = \sqrt{y - x^2}$

13. $f(x,y) = \dfrac{1}{\sqrt{y - x^2}}$

14. $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

In Exercises 15 – 20, a limit is given. Evaluate the limit along the paths given, then state why these results show the given limit does not exist.

15. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$

    (a) Along the path $y = 0$.

    (b) Along the path $x = 0$.

16. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x + y}{x - y}$

    (a) Along the path $y = mx$.

17. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy - y^2}{y^2 + x}$

    (a) Along the path $y = mx$.

    (b) Along the path $x = 0$.

18. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2)}{y}$

    (a) Along the path $y = mx$.

    (b) Along the path $y = x^2$.

19. $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x + y - 3}{x^2 - 1}$

    (a) Along the path $y = 2$.

    (b) Along the path $y = x + 1$.

20. $\displaystyle\lim_{(x,y)\to(\pi,\pi/2)} \frac{\sin x}{\cos y}$

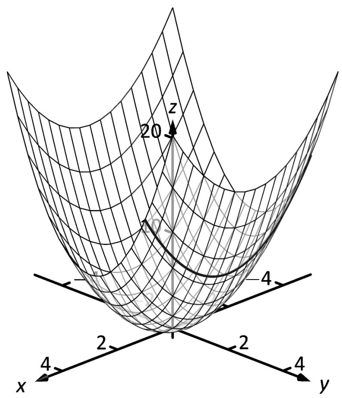
    (a) Along the path $x = \pi$.

    (b) Along the path $y = x - \pi/2$.

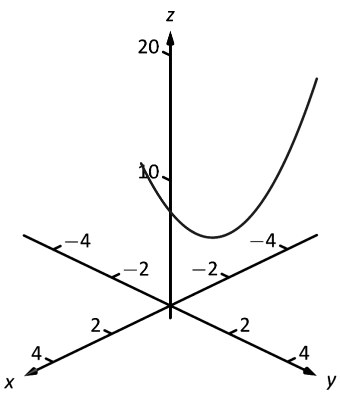
12.3 Partial Derivatives

Let y be a function of x. We have studied in great detail the derivative of y with respect to x, that is, $\frac{dy}{dx}$, which measures the rate at which y changes with respect to x. Consider now $z = f(x, y)$. It makes sense to want to know how z changes with respect to x and/or y. This section begins our investigation into these rates of change.

Consider the function $z = f(x, y) = x^2 + 2y^2$, as graphed in Figure 12.3.1(a). By fixing $y = 2$, we focus our attention to all points on the surface where the y-value is 2, shown in both parts (a) and (b) of the figure. These points form a curve in space: $z = f(x, 2) = x^2 + 8$ which is a function of just one variable. We can take the derivative of z with respect to x along this curve and find equations of tangent lines, etc.



(a)



(b)

Figure 12.3.1: By fixing $y = 2$, the surface $f(x, y) = x^2 + 2y^2$ is a curve in space.

The key notion to extract from this example is: by treating y as constant (it does not vary) we can consider how z changes with respect to x. In a similar fashion, we can hold x constant and consider how z changes with respect to y. This is the underlying principle of **partial derivatives**. We state the formal, limit–based definition first, then show how to compute these partial derivatives without directly taking limits.

Definition 12.3.1 Partial Derivative

Let $z = f(x, y)$ be a continuous function on a set S in $\mathbb{R}^2$.

1. The **partial derivative of f with respect to x** is:

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

2. The **partial derivative of f with respect to y** is:

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$

Alternate notations for $f_x(x, y)$ include:

$$\frac{\partial}{\partial x}f(x, y), \quad \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x}, \quad \text{and } z_x,$$

with similar notations for $f_y(x, y)$. For ease of notation, $f_x(x, y)$ is often abbreviated f_x.

Example 12.3.1 Computing partial derivatives with the limit definition

Let $f(x, y) = x^2y + 2x + y^3$. Find $f_x(x, y)$ using the limit definition.

Notes:

Solution Using Definition 12.3.1, we have:

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\\ &= \lim_{h\to 0}\frac{(x+h)^2y+2(x+h)+y^3-(x^2y+2x+y^3)}{h}\\ &= \lim_{h\to 0}\frac{x^2y+2xhy+h^2y+2x+2h+y^3-(x^2y+2x+y^3)}{h}\\ &= \lim_{h\to 0}\frac{2xhy+h^2y+2h}{h}\\ &= \lim_{h\to 0} 2xy+hy+2\\ &= 2xy+2. \end{aligned}$$

We have found $f_x(x,y) = 2xy+2$.

Example 12.3.1 found a partial derivative using the formal, limit–based definition. Using limits is not necessary, though, as we can rely on our previous knowledge of derivatives to compute partial derivatives easily. When computing $f_x(x,y)$, we hold y fixed – it does not vary. Therefore we can compute the derivative with respect to x by treating y as a constant or coefficient.

Just as $\frac{d}{dx}\big(5x^2\big) = 10x$, we compute $\frac{\partial}{\partial x}\big(x^2y\big) = 2xy$. Here we are treating y as a coefficient.

Just as $\frac{d}{dx}\big(5^3\big) = 0$, we compute $\frac{\partial}{\partial x}\big(y^3\big) = 0$. Here we are treating y as a constant. More examples will help make this clear.

Example 12.3.2 Finding partial derivatives

Find $f_x(x,y)$ and $f_y(x,y)$ in each of the following.

1. $f(x,y) = x^3y^2+5y^2-x+7$
2. $f(x,y) = \cos(xy^2)+\sin x$
3. $f(x,y) = e^{x^2y^3}\sqrt{x^2+1}$

Solution

1. We have $f(x,y) = x^3y^2+5y^2-x+7$.
   Begin with $f_x(x,y)$. Keep y fixed, treating it as a constant or coefficient, as appropriate:
   $$f_x(x,y) = 3x^2y^2-1.$$
   Note how the $5y^2$ and 7 terms go to zero.

Notes:

To compute $f_y(x,y)$, we hold x fixed:

$$f_y(x,y) = 2x^3y + 10y.$$

Note how the $-x$ and 7 terms go to zero.

2. We have $f(x,y) = \cos(xy^2) + \sin x$.
Begin with $f_x(x,y)$. We need to apply the Chain Rule with the cosine term; y^2 is the coefficient of the x-term inside the cosine function.

$$f_x(x,y) = -\sin(xy^2)(y^2) + \cos x = -y^2\sin(xy^2) + \cos x.$$

To find $f_y(x,y)$, note that x is the coefficient of the y^2 term inside of the cosine term; also note that since x is fixed, $\sin x$ is also fixed, and we treat it as a constant.

$$f_y(x,y) = -\sin(xy^2)(2xy) = -2xy\sin(xy^2).$$

3. We have $f(x,y) = e^{x^2y^3}\sqrt{x^2+1}$.
Beginning with $f_x(x,y)$, note how we need to apply the Product Rule.

$$\begin{aligned} f_x(x,y) &= e^{x^2y^3}(2xy^3)\sqrt{x^2+1} + e^{x^2y^3}\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x) \\ &= 2xy^3e^{x^2y^3}\sqrt{x^2+1} + \frac{xe^{x^2y^3}}{\sqrt{x^2+1}}. \end{aligned}$$

Note that when finding $f_y(x,y)$ we do not have to apply the Product Rule; since $\sqrt{x^2+1}$ does not contain y, we treat it as fixed and hence becomes a coefficient of the $e^{x^2y^3}$ term.

$$f_y(x,y) = e^{x^2y^3}(3x^2y^2)\sqrt{x^2+1} = 3x^2y^2e^{x^2y^3}\sqrt{x^2+1}.$$

We have shown *how* to compute a partial derivative, but it may still not be clear what a partial derivative *means*. Given $z = f(x,y)$, $f_x(x,y)$ measures the rate at which z changes as only x varies: y is held constant.

Imagine standing in a rolling meadow, then beginning to walk due east. Depending on your location, you might walk up, sharply down, or perhaps not change elevation at all. This is similar to measuring z_x: you are moving only east (in the "x"-direction) and not north/south at all. Going back to your original location, imagine now walking due north (in the "y"-direction). Perhaps walking due north does not change your elevation at all. This is analogous to $z_y = 0$: z does not change with respect to y. We can see that z_x and z_y do not have to be the same, or even similar, as it is easy to imagine circumstances where walking east means you walk downhill, though walking north makes you walk uphill.

Notes:

The following example helps us visualize this more.

Example 12.3.3 Evaluating partial derivatives
Let $z = f(x, y) = -x^2 - \frac{1}{2}y^2 + xy + 10$. Find $f_x(2, 1)$ and $f_y(2, 1)$ and interpret their meaning.

Solution We begin by computing $f_x(x, y) = -2x + y$ and $f_y(x, y) = -y + x$. Thus

$$f_x(2, 1) = -3 \quad \text{and} \quad f_y(2, 1) = 1.$$

It is also useful to note that $f(2, 1) = 7.5$. What does each of these numbers mean?

Consider $f_x(2, 1) = -3$, along with Figure 12.3.2(a). If one "stands" on the surface at the point $(2, 1, 7.5)$ and moves parallel to the x-axis (i.e., only the x-value changes, not the y-value), then the instantaneous rate of change is -3. Increasing the x-value will decrease the z-value; decreasing the x-value will increase the z-value.

Now consider $f_y(2, 1) = 1$, illustrated in Figure 12.3.2(b). Moving along the curve drawn on the surface, i.e., parallel to the y-axis and not changing the x-values, increases the z-value instantaneously at a rate of 1. Increasing the y-value by 1 would increase the z-value by approximately 1.

Since the magnitude of f_x is greater than the magnitude of f_y at $(2, 1)$, it is "steeper" in the x-direction than in the y-direction.

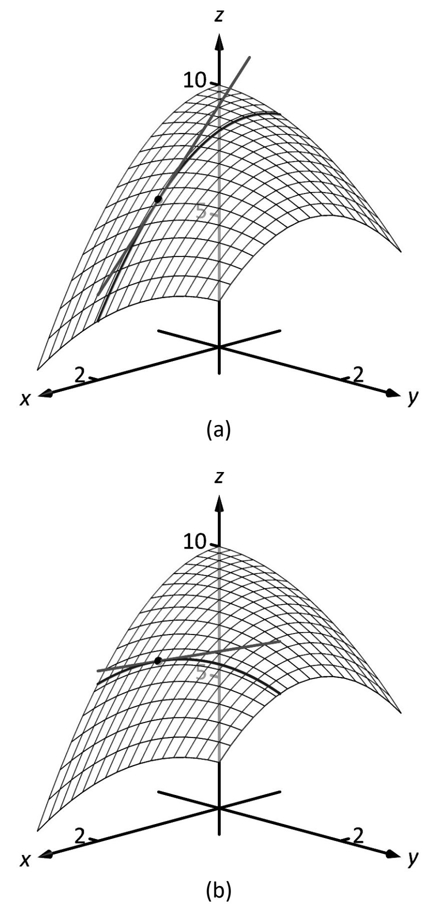


Figure 12.3.2: Illustrating the meaning of partial derivatives.

Second Partial Derivatives

Let $z = f(x, y)$. We have learned to find the partial derivatives $f_x(x, y)$ and $f_y(x, y)$, which are each functions of x and y. Therefore we can take partial derivatives of them, each with respect to x and y. We define these "second partials" along with the notation, give examples, then discuss their meaning.

Notes:

Definition 12.3.2 Second Partial Derivative, Mixed Partial Derivative

Let $z = f(x, y)$ be continuous on a set S.

1. The **second partial derivative of** f **with respect to** x **then** x is

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \left(f_x\right)_x = f_{xx}$$

2. The **second partial derivative of** f **with respect to** x **then** y is

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \left(f_x\right)_y = f_{xy}$$

Similar definitions hold for $\dfrac{\partial^2 f}{\partial y^2} = f_{yy}$ and $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$.

The second partial derivatives f_{xy} and f_{yx} are **mixed partial derivatives**.

Note: The terms in Definition 12.3.2 all depend on limits, so each definition comes with the caveat "where the limit exists."

The notation of second partial derivatives gives some insight into the notation of the second derivative of a function of a single variable. If $y = f(x)$, then $f''(x) = \dfrac{d^2y}{dx^2}$. The "$d^2y$" portion means "take the derivative of y twice," while "dx^2" means "with respect to x both times." When we only know of functions of a single variable, this latter phrase seems silly: there is only one variable to take the derivative with respect to. Now that we understand functions of multiple variables, we see the importance of specifying which variables we are referring to.

Example 12.3.4 Second partial derivatives

For each of the following, find all six first and second partial derivatives. That is, find

$$f_x, \quad f_y, \quad f_{xx}, \quad f_{yy}, \quad f_{xy} \quad \text{and} \quad f_{yx}\,.$$

1. $f(x, y) = x^3y^2 + 2xy^3 + \cos x$
2. $f(x, y) = \dfrac{x^3}{y^2}$
3. $f(x, y) = e^x \sin(x^2y)$

SOLUTION In each, we give f_x and f_y immediately and then spend time de-

Notes:

riving the second partial derivatives.

1. $f(x,y) = x^3y^2 + 2xy^3 + \cos x$

   $f_x(x,y) = 3x^2y^2 + 2y^3 - \sin x$

   $f_y(x,y) = 2x^3y + 6xy^2$

   $f_{xx}(x,y) = \dfrac{\partial}{\partial x}\big(f_x\big) = \dfrac{\partial}{\partial x}\big(3x^2y^2 + 2y^3 - \sin x\big) = 6xy^2 - \cos x$

   $f_{yy}(x,y) = \dfrac{\partial}{\partial y}\big(f_y\big) = \dfrac{\partial}{\partial y}\big(2x^3y + 6xy^2\big) = 2x^3 + 12xy$

   $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\big(f_x\big) = \dfrac{\partial}{\partial y}\big(3x^2y^2 + 2y^3 - \sin x\big) = 6x^2y + 6y^2$

   $f_{yx}(x,y) = \dfrac{\partial}{\partial x}\big(f_y\big) = \dfrac{\partial}{\partial x}\big(2x^3y + 6xy^2\big) = 6x^2y + 6y^2$

2. $f(x,y) = \dfrac{x^3}{y^2} = x^3y^{-2}$

   $f_x(x,y) = \dfrac{3x^2}{y^2}$

   $f_y(x,y) = -\dfrac{2x^3}{y^3}$

   $f_{xx}(x,y) = \dfrac{\partial}{\partial x}\big(f_x\big) = \dfrac{\partial}{\partial x}\Big(\dfrac{3x^2}{y^2}\Big) = \dfrac{6x}{y^2}$

   $f_{yy}(x,y) = \dfrac{\partial}{\partial y}\big(f_y\big) = \dfrac{\partial}{\partial y}\Big(-\dfrac{2x^3}{y^3}\Big) = \dfrac{6x^3}{y^4}$

   $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\big(f_x\big) = \dfrac{\partial}{\partial y}\Big(\dfrac{3x^2}{y^2}\Big) = -\dfrac{6x^2}{y^3}$

   $f_{yx}(x,y) = \dfrac{\partial}{\partial x}\big(f_y\big) = \dfrac{\partial}{\partial x}\Big(-\dfrac{2x^3}{y^3}\Big) = -\dfrac{6x^2}{y^3}$

3. $f(x,y) = e^x\sin(x^2y)$

   Because the following partial derivatives get rather long, we omit the extra notation and just give the results. In several cases, multiple applications of the Product and Chain Rules will be necessary, followed by some basic combination of like terms.

   $f_x(x,y) = e^x\sin(x^2y) + 2xye^x\cos(x^2y)$

   $f_y(x,y) = x^2e^x\cos(x^2y)$

   $f_{xx}(x,y) = e^x\sin(x^2y) + 4xye^x\cos(x^2y) + 2ye^x\cos(x^2y) - 4x^2y^2e^x\sin(x^2y)$

   $f_{yy}(x,y) = -x^4e^x\sin(x^2y)$

   $f_{xy}(x,y) = x^2e^x\cos(x^2y) + 2xe^x\cos(x^2y) - 2x^3ye^x\sin(x^2y)$

   $f_{yx}(x,y) = x^2e^x\cos(x^2y) + 2xe^x\cos(x^2y) - 2x^3ye^x\sin(x^2y)$

Notes:

Notice how in each of the three functions in Example 12.3.4, $f_{xy} = f_{yx}$. Due to the complexity of the examples, this likely is not a coincidence. The following theorem states that it is not.

Theorem 12.3.1 Mixed Partial Derivatives

Let f be defined such that f_{xy} and f_{yx} are continuous on a set S. Then for each point (x, y) in S, $f_{xy}(x, y) = f_{yx}(x, y)$.

Finding f_{xy} and f_{yx} independently and comparing the results provides a convenient way of checking our work.

Understanding Second Partial Derivatives

Now that we know *how* to find second partials, we investigate *what* they tell us.

Again we refer back to a function $y = f(x)$ of a single variable. The second derivative of f is "the derivative of the derivative," or "the rate of change of the rate of change." The second derivative measures how much the derivative is changing. If $f''(x) < 0$, then the derivative is getting smaller (so the graph of f is concave down); if $f''(x) > 0$, then the derivative is growing, making the graph of f concave up.

Now consider $z = f(x, y)$. Similar statements can be made about f_{xx} and f_{yy} as could be made about $f''(x)$ above. When taking derivatives with respect to x twice, we measure how much f_x changes with respect to x. If $f_{xx}(x, y) < 0$, it means that as x increases, f_x decreases, and the graph of f will be concave down *in the x-direction*. Using the analogy of standing in the rolling meadow used earlier in this section, f_{xx} measures whether one's path is concave up/down when walking due east.

Similarly, f_{yy} measures the concavity in the y-direction. If $f_{yy}(x, y) > 0$, then f_y is increasing with respect to y and the graph of f will be concave up in the y-direction. Appealing to the rolling meadow analogy again, f_{yy} measures whether one's path is concave up/down when walking due north.

We now consider the mixed partials f_{xy} and f_{yx}. The mixed partial f_{xy} measures how much f_x changes with respect to y. Once again using the rolling meadow analogy, f_x measures the slope if one walks due east. Looking east, begin walking *north* (side–stepping). Is the path towards the east getting steeper? If so, $f_{xy} > 0$. Is the path towards the east not changing in steepness? If so, then $f_{xy} = 0$. A similar thing can be said about f_{yx}: consider the steepness of paths heading north while side–stepping to the east.

The following example examines these ideas with concrete numbers and

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Notes:

graphs.

Example 12.3.5 Understanding second partial derivatives
Let $z = x^2 - y^2 + xy$. Evaluate the 6 first and second partial derivatives at $(-1/2, 1/2)$ and interpret what each of these numbers mean.

Solution We find that:
$f_x(x,y) = 2x + y$, $f_y(x,y) = -2y + x$, $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = -2$ and $f_{xy}(x,y) = f_{yx}(x,y) = 1$. Thus at $(-1/2, 1/2)$ we have

$$f_x(-1/2, 1/2) = -1/2, \qquad f_y(-1/2, 1/2) = -3/2.$$

The slope of the tangent line at $(-1/2, 1/2, -1/4)$ in the direction of x is $-1/2$: if one moves from that point parallel to the x-axis, the instantaneous rate of change will be $-1/2$. The slope of the tangent line at this point in the direction of y is $-3/2$: if one moves from this point parallel to the y-axis, the instantaneous rate of change will be $-3/2$. These tangents lines are graphed in Figure 12.3.3(a) and (b), respectively, where the tangent lines are drawn in a solid line.

Now consider only Figure 12.3.3(a). Three directed tangent lines are drawn (two are dashed), each in the direction of x; that is, each has a slope determined by f_x. Note how as y increases, the slope of these lines get closer to 0. Since the slopes are all negative, getting closer to 0 means the *slopes are increasing.* The slopes given by f_x are increasing as y increases, meaning f_{xy} must be positive.

Since $f_{xy} = f_{yx}$, we also expect f_y to increase as x increases. Consider Figure 12.3.3(b) where again three directed tangent lines are drawn, this time each in the direction of y with slopes determined by f_y. As x increases, the slopes become less steep (closer to 0). Since these are negative slopes, this means the slopes are increasing.

Thus far we have a visual understanding of f_x, f_y, and $f_{xy} = f_{yx}$. We now interpret f_{xx} and f_{yy}. In Figure 12.3.3(a), we see a curve drawn where x is held constant at $x = -1/2$: only y varies. This curve is clearly concave down, corresponding to the fact that $f_{yy} < 0$. In part (b) of the figure, we see a similar curve where y is constant and only x varies. This curve is concave up, corresponding to the fact that $f_{xx} > 0$.

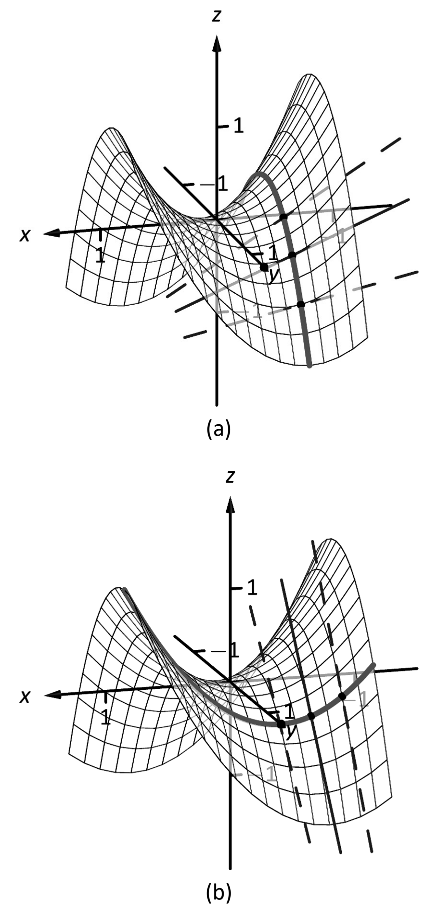


Figure 12.3.3: Understanding the second partial derivatives in Example 12.3.5.

Partial Derivatives and Functions of Three Variables

The concepts underlying partial derivatives can be easily extend to more than two variables. We give some definitions and examples in the case of three variables and trust the reader can extend these definitions to more variables if needed.

Notes:

Definition 12.3.3 Partial Derivatives with Three Variables

Let $w = f(x, y, z)$ be a continuous function on a set D in $\mathbb{R}^3$.
The **partial derivative of** f **with respect to** x is:

$$f_x(x, y, z) = \lim_{h\to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}.$$

Similar definitions hold for $f_y(x, y, z)$ and $f_z(x, y, z)$.

By taking partial derivatives of partial derivatives, we can find second partial derivatives of f with respect to z then y, for instance, just as before.

Example 12.3.6 Partial derivatives of functions of three variables

For each of the following, find f_x, f_y, f_z, f_{xz}, f_{yz}, and f_{zz}.

1. $f(x, y, z) = x^2y^3z^4 + x^2y^2 + x^3z^3 + y^4z^4$
2. $f(x, y, z) = x\sin(yz)$

Solution

1. $f_x = 2xy^3z^4 + 2xy^2 + 3x^2z^3$; $\quad f_y = 3x^2y^2z^4 + 2x^2y + 4y^3z^4$;
   $f_z = 4x^2y^3z^3 + 3x^3z^2 + 4y^4z^3$; $\quad f_{xz} = 8xy^3z^3 + 9x^2z^2$;
   $f_{yz} = 12x^2y^2z^3 + 16y^3z^3$; $\quad f_{zz} = 12x^2y^3z^2 + 6x^3z + 12y^4z^2$
2. $f_x = \sin(yz)$; $\quad f_y = xz\cos(yz)$; $\quad f_z = xy\cos(yz)$;
   $f_{xz} = y\cos(yz)$; $\quad f_{yz} = x\cos(yz) - xyz\sin(yz)$; $\quad f_{zz} = -xy^2\sin(xy)$

Higher Order Partial Derivatives

We can continue taking partial derivatives of partial derivatives of partial derivatives of ...; we do not have to stop with second partial derivatives. These higher order partial derivatives do not have a tidy graphical interpretation; nevertheless they are not hard to compute and worthy of some practice.

We do not formally define each higher order derivative, but rather give just a few examples of the notation.

$$f_{xyx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) \quad \text{and}$$

$$f_{xyz}(x, y, z) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right).$$

Notes:

Example 12.3.7 Higher order partial derivatives

1. Let $f(x,y) = x^2y^2 + \sin(xy)$. Find f_{xxy} and f_{yxx}.
2. Let $f(x,y,z) = x^3e^{xy} + \cos(z)$. Find f_{xyz}.

Solution

1. To find f_{xxy}, we first find f_x, then f_{xx}, then f_{xxy}:

$$\begin{aligned} f_x &= 2xy^2 + y\cos(xy) \qquad f_{xx} = 2y^2 - y^2\sin(xy) \\ f_{xxy} &= 4y - 2y\sin(xy) - xy^2\cos(xy). \end{aligned}$$

   To find f_{yxx}, we first find f_y, then f_{yx}, then f_{yxx}:

$$\begin{aligned} f_y &= 2x^2y + x\cos(xy) \qquad f_{yx} = 4xy + \cos(xy) - xy\sin(xy) \\ f_{yxx} &= 4y - y\sin(xy) - \big(y\sin(xy) + xy^2\cos(xy)\big) \\ &= 4y - 2y\sin(xy) - xy^2\cos(xy). \end{aligned}$$

   Note how $f_{xxy} = f_{yxx}$.

2. To find f_{xyz}, we find f_x, then f_{xy}, then f_{xyz}:

$$\begin{aligned} f_x &= 3x^2e^{xy} + x^3ye^{xy} \qquad f_{xy} = 3x^3e^{xy} + x^3e^{xy} + x^4ye^{xy} = 4x^3e^{xy} + x^4ye^{xy} \\ f_{xyz} &= 0. \end{aligned}$$

In the previous example we saw that $f_{xxy} = f_{yxx}$; this is not a coincidence. While we do not state this as a formal theorem, as long as each partial derivative is continuous, it does not matter the order in which the partial derivatives are taken. For instance, $f_{xxy} = f_{xyx} = f_{yxx}$.

This can be useful at times. Had we known this, the second part of Example 12.3.7 would have been much simpler to compute. Instead of computing f_{xyz} in the x, y then z orders, we could have applied the z, then x then y order (as $f_{xyz} = f_{zxy}$). It is easy to see that $f_z = -\sin z$; then f_{zx} and f_{zxy} are clearly 0 as f_z does not contain an x or y.

Notes:

A brief review of this section: partial derivatives measure the instantaneous rate of change of a multivariable function with respect to one variable. With $z = f(x, y)$, the partial derivatives f_x and f_y measure the instantaneous rate of change of z when moving parallel to the x- and y-axes, respectively. How do we measure the rate of change at a point when we do not move parallel to one of these axes? What if we move in the direction given by the vector $\langle 2, 1\rangle$? Can we measure that rate of change? The answer is, of course, yes, we can. This is the topic of Section 12.6. First, we need to define what it means for a function of two variables to be *differentiable.*

Notes:

Exercises 12.3

Terms and Concepts

1. What is the difference between a constant and a coefficient?

2. Given a function $z = f(x, y)$, explain in your own words how to compute f_x.

3. In the mixed partial fraction f_{xy}, which is computed first, f_x or f_y?

4. In the mixed partial fraction $\dfrac{\partial^2 f}{\partial x \partial y}$, which is computed first, f_x or f_y?

Problems

In Exercises 5 – 8, evaluate $f_x(x, y)$ and $f_y(x, y)$ at the indicated point.

5. $f(x, y) = x^2y - x + 2y + 3$ at $(1, 2)$

6. $f(x, y) = x^3 - 3x + y^2 - 6y$ at $(-1, 3)$

7. $f(x, y) = \sin y \cos x$ at $(\pi/3, \pi/3)$

8. $f(x, y) = \ln(xy)$ at $(-2, -3)$

In Exercises 9 – 26, find f_x, f_y, f_{xx}, f_{yy}, f_{xy} and f_{yx}.

9. $f(x, y) = x^2y + 3x^2 + 4y - 5$

10. $f(x, y) = y^3 + 3xy^2 + 3x^2y + x^3$

11. $f(x, y) = \dfrac{x}{y}$

12. $f(x, y) = \dfrac{4}{xy}$

13. $f(x, y) = e^{x^2+y^2}$

14. $f(x, y) = e^{x+2y}$

15. $f(x, y) = \sin x \cos y$

16. $f(x, y) = (x + y)^3$

17. $f(x, y) = \cos(5xy^3)$

18. $f(x, y) = \sin(5x^2 + 2y^3)$

19. $f(x, y) = \sqrt{4xy^2 + 1}$

20. $f(x, y) = (2x + 5y)\sqrt{y}$

21. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

22. $f(x, y) = 5x - 17y$

23. $f(x, y) = 3x^2 + 1$

24. $f(x, y) = \ln(x^2 + y)$

25. $f(x, y) = \dfrac{\ln x}{4y}$

26. $f(x, y) = 5e^x \sin y + 9$

In Exercises 27 – 30, form a function $z = f(x, y)$ such that f_x and f_y match those given.

27. $f_x = \sin y + 1, \quad f_y = x \cos y$

28. $f_x = x + y, \quad f_y = x + y$

29. $f_x = 6xy - 4y^2, \quad f_y = 3x^2 - 8xy + 2$

30. $f_x = \dfrac{2x}{x^2 + y^2}, \quad f_y = \dfrac{2y}{x^2 + y^2}$

In Exercises 31 – 34, find f_x, f_y, f_z, f_{yz} and f_{zy}.

31. $f(x, y, z) = x^2e^{2y-3z}$

32. $f(x, y, z) = x^3y^2 + x^3z + y^2z$

33. $f(x, y, z) = \dfrac{3x}{7y^2z}$

34. $f(x, y, z) = \ln(xyz)$

12.4 Differentiability and the Total Differential

We studied **differentials** in Section 4.4, where Definition 4.4.1 states that if $y = f(x)$ and f is differentiable, then $dy = f'(x)dx$. One important use of this differential is in Integration by Substitution. Another important application is approximation. Let $\Delta x = dx$ represent a change in x. When dx is small, $dy \approx \Delta y$, the change in y resulting from the change in x. Fundamental in this understanding is this: as dx gets small, the difference between Δy and dy goes to 0. Another way of stating this: as dx goes to 0, the *error* in approximating Δy with dy goes to 0.

We extend this idea to functions of two variables. Let $z = f(x, y)$, and let $\Delta x = dx$ and $\Delta y = dy$ represent changes in x and y, respectively. Let $\Delta z = f(x + dx, y + dy) - f(x, y)$ be the change in z over the change in x and y. Recalling that f_x and f_y give the instantaneous rates of z-change in the x- and y-directions, respectively, we can approximate Δz with $dz = f_x dx + f_y dy$; in words, the total change in z is approximately the change caused by changing x plus the change caused by changing y. In a moment we give an indication of whether or not this approximation is any good. First we give a name to dz.

Note: From Definition 12.4.1, we can write

$$dz = \langle f_x, f_y \rangle \cdot \langle dx, dy \rangle.$$

While not explored in this section, the vector $\langle f_x, f_y \rangle$ is seen again in the next section and fully defined in Section 12.6.

Definition 12.4.1 Total Differential

Let $z = f(x, y)$ be continuous on a set S. Let dx and dy represent changes in x and y, respectively. Where the partial derivatives f_x and f_y exist, the **total differential of** z is

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

Example 12.4.1 Finding the total differential

Let $z = x^4 e^{3y}$. Find dz.

Solution We compute the partial derivatives: $f_x = 4x^3 e^{3y}$ and $f_y = 3x^4 e^{3y}$. Following Definition 12.4.1, we have

$$dz = 4x^3 e^{3y} dx + 3x^4 e^{3y} dy.$$

We *can* approximate Δz with dz, but as with all approximations, there is error involved. A good approximation is one in which the error is small. At a given point (x_0, y_0), let E_x and E_y be functions of dx and dy such that $E_x dx + E_y dy$ describes this error. Then

$$\begin{aligned}\Delta z &= dz + E_x dx + E_y dy \\ &= f_x(x_0, y_0)dx + f_y(x_0, y_0)dy + E_x dx + E_y dy.\end{aligned}$$

Notes:

If the approximation of Δz by dz is good, then as dx and dy get small, so does $E_x dx + E_y dy$. The approximation of Δz by dz is even better if, as dx and dy go to 0, so do E_x and E_y. This leads us to our definition of differentiability.

Definition 12.4.2 Multivariable Differentiability

Let $z = f(x, y)$ be defined on a set S containing (x_0, y_0) where $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist. Let dz be the total differential of z at (x_0, y_0), let $\Delta z = f(x_0 + dx, y_0 + dy) - f(x_0, y_0)$, and let E_x and E_y be functions of dx and dy such that

$$\Delta z = dz + E_x dx + E_y dy.$$

1. We say f is **differentiable at** (x_0, y_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|| \langle dx, dy \rangle || < \delta$, then $|| \langle E_x, E_y \rangle || < \varepsilon$. That is, as dx and dy go to 0, so do E_x and E_y.
2. We say f is **differentiable on** S if f is differentiable at every point in S. If f is differentiable on $\mathbb{R}^2$, we say that f is **differentiable everywhere**.

Example 12.4.2 Showing a function is differentiable

Show $f(x, y) = xy + 3y^2$ is differentiable using Definition 12.4.2.

Solution We begin by finding $f(x + dx, y + dy)$, Δz, f_x and f_y.

$$\begin{aligned} f(x + dx, y + dy) &= (x + dx)(y + dy) + 3(y + dy)^2 \\ &= xy + xdy + ydx + dxdy + 3y^2 + 6ydy + 3dy^2. \end{aligned}$$

$\Delta z = f(x + dx, y + dy) - f(x, y)$, so

$$\Delta z = xdy + ydx + dxdy + 6ydy + 3dy^2.$$

It is straightforward to compute $f_x = y$ and $f_y = x + 6y$. Consider once more Δz:

$$\begin{aligned} \Delta z &= xdy + ydx + dxdy + 6ydy + 3dy^2 \qquad \text{(now reorder)} \\ &= ydx + xdy + 6ydy + dxdy + 3dy^2 \\ &= \underbrace{(y)}_{f_x} dx + \underbrace{(x + 6y)}_{f_y} dy + \underbrace{(dy)}_{E_x} dx + \underbrace{(3dy)}_{E_y} dy \\ &= f_x dx + f_y dy + E_x dx + E_y dy. \end{aligned}$$

With $E_x = dy$ and $E_y = 3dy$, it is clear that as dx and dy go to 0, E_x and E_y also go to 0. Since this did not depend on a specific point (x_0, y_0), we can say that $f(x, y)$

Notes:

is differentiable for all pairs (x, y) in $\mathbb{R}^2$, or, equivalently, that f is differentiable everywhere.

Our intuitive understanding of differentiability of functions $y = f(x)$ of one variable was that the graph of f was "smooth." A similar intuitive understanding of functions $z = f(x, y)$ of two variables is that the surface defined by f is also "smooth," not containing cusps, edges, breaks, etc. The following theorem states that differentiable functions are continuous, followed by another theorem that provides a more tangible way of determining whether a great number of functions are differentiable or not.

Theorem 12.4.1 Continuity and Differentiability of Multivariable Functions

Let $z = f(x, y)$ be defined on a set S containing (x_0, y_0). If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Theorem 12.4.2 Differentiability of Multivariable Functions

Let $z = f(x, y)$ be defined on a set S. If f_x and f_y are both continuous on S, then f is differentiable on S.

The theorems assure us that essentially all functions that we see in the course of our studies here are differentiable (and hence continuous) on their natural domains. There is a difference between Definition 12.4.2 and Theorem 12.4.2, though: it is possible for a function f to be differentiable yet f_x and/or f_y is *not* continuous. Such strange behavior of functions is a source of delight for many mathematicians.

When f_x and f_y exist at a point but are not continuous at that point, we need to use other methods to determine whether or not f is differentiable at that point.

For instance, consider the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Notes:

We can find $f_x(0,0)$ and $f_y(0,0)$ using Definition 12.3.1:

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0}\frac{0}{h^2}=0; \\ f_y(0,0) &= \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} \\ &= \lim_{h\to 0}\frac{0}{h^2}=0. \end{aligned}$$

Both f_x and f_y *exist* at $(0,0)$, but they are not continuous at $(0,0)$, as

$$f_x(x,y)=\frac{y(y^2-x^2)}{(x^2+y^2)^2} \qquad \text{and} \qquad f_y(x,y)=\frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

are not continuous at $(0,0)$. (Take the limit of f_x as $(x,y)\to(0,0)$ along the x- and y-axes; they give different results.) So even though f_x and f_y *exist* at every point in the x-y plane, they are not continuous. Therefore it is possible, by Theorem 12.4.2, for f to not be differentiable.

Indeed, it is not. One can show that f is not continuous at $(0,0)$ (see Example 12.2.4), and by Theorem 12.4.1, this means f is not differentiable at $(0,0)$.

Approximating with the Total Differential

By the definition, when f is differentiable dz is a good approximation for Δz when dx and dy are small. We give some simple examples of how this is used here.

Example 12.4.3 Approximating with the total differential

Let $z=\sqrt{x}\sin y$. Approximate $f(4.1,0.8)$.

Solution Recognizing that $\pi/4\approx 0.785\approx 0.8$, we can approximate $f(4.1,0.8)$ using $f(4,\pi/4)$. We can easily compute $f(4,\pi/4)=\sqrt{4}\sin(\pi/4)=2\left(\frac{\sqrt{2}}{2}\right)=\sqrt{2}\approx 1.414$. Without calculus, this is the best approximation we could reasonably come up with. The total differential gives us a way of adjusting this initial approximation to hopefully get a more accurate answer.

We let $\Delta z=f(4.1,0.8)-f(4,\pi/4)$. The total differential dz is approximately equal to Δz, so

$$f(4.1,0.8)-f(4,\pi/4)\approx dz \quad\Rightarrow\quad f(4.1,0.8)\approx dz+f(4,\pi/4). \qquad (12.1)$$

To find dz, we need f_x and f_y.

Notes:

$$f_x(x,y) = \frac{\sin y}{2\sqrt{x}} \quad \Rightarrow \quad f_x(4,\pi/4) = \frac{\sin \pi/4}{2\sqrt{4}} = \frac{\sqrt{2}/2}{4} = \sqrt{2}/8.$$

$$f_y(x,y) = \sqrt{x}\cos y \quad \Rightarrow \quad f_y(4,\pi/4) = \sqrt{4}\frac{\sqrt{2}}{2} = \sqrt{2}.$$

Approximating 4.1 with 4 gives $dx = 0.1$; approximating 0.8 with $\pi/4$ gives $dy \approx 0.015$. Thus

$$\begin{aligned} dz(4,\pi/4) &= f_x(4,\pi/4)(0.1) + f_y(4,\pi/4)(0.015) \\ &= \frac{\sqrt{2}}{8}(0.1) + \sqrt{2}(0.015) \\ &\approx 0.039. \end{aligned}$$

Returning to Equation (12.1), we have

$$f(4.1, 0.8) \approx 0.039 + 1.414 = 1.4531.$$

We, of course, can compute the actual value of $f(4.1, 0.8)$ with a calculator; the actual value, accurate to 5 places after the decimal, is 1.45254. Obviously our approximation is quite good.

The point of the previous example was *not* to develop an approximation method for known functions. After all, we can very easily compute $f(4.1, 0.8)$ using readily available technology. Rather, it serves to illustrate how well this method of approximation works, and to reinforce the following concept:

“New position = old position $+$ amount of change,” so
“New position $\approx$ old position + approximate amount of change.”

In the previous example, we could easily compute $f(4,\pi/4)$ and could approximate the amount of z-change when computing $f(4.1, 0.8)$, letting us approximate the new z-value.

It may be surprising to learn that it is not uncommon to know the values of f, f_x and f_y at a particular point without actually knowing the function f. The total differential gives a good method of approximating f at nearby points.

Example 12.4.4 Approximating an unknown function

Given that $f(2,-3) = 6$, $f_x(2,-3) = 1.3$ and $f_y(2,-3) = -0.6$, approximate $f(2.1, -3.03)$.

Notes:

Solution The total differential approximates how much f changes from the point $(2,-3)$ to the point $(2.1,-3.03)$. With $dx=0.1$ and $dy=-0.03$, we have

$$\begin{aligned} dz &= f_x(2,-3)dx + f_y(2,-3)dy \\ &= 1.3(0.1) + (-0.6)(-0.03) \\ &= 0.148. \end{aligned}$$

The change in z is approximately 0.148, so we approximate $f(2.1,-3.03)\approx 6.148$.

Error/Sensitivity Analysis

The total differential gives an approximation of the change in z given small changes in x and y. We can use this to approximate error propagation; that is, if the input is a little off from what it should be, how far from correct will the output be? We demonstrate this in an example.

Example 12.4.5 Sensitivity analysis

A cylindrical steel storage tank is to be built that is 10ft tall and 4ft across in diameter. It is known that the steel will expand/contract with temperature changes; is the overall volume of the tank more sensitive to changes in the diameter or in the height of the tank?

Solution A cylindrical solid with height h and radius r has volume $V=\pi r^2h$. We can view V as a function of two variables, r and h. We can compute partial derivatives of V:

$$\frac{\partial V}{\partial r} = V_r(r,h) = 2\pi rh \qquad \text{and} \qquad \frac{\partial V}{\partial h} = V_h(r,h) = \pi r^2.$$

The total differential is $dV=(2\pi rh)dr+(\pi r^2)dh$. When $h=10$ and $r=2$, we have $dV=40\pi dr+4\pi dh$. Note that the coefficient of dr is $40\pi\approx 125.7$; the coefficient of dh is a tenth of that, approximately 12.57. A small change in radius will be multiplied by 125.7, whereas a small change in height will be multiplied by 12.57. Thus the volume of the tank is more sensitive to changes in radius than in height.

The previous example showed that the volume of a particular tank was more sensitive to changes in radius than in height. Keep in mind that this analysis only applies to a tank of those dimensions. A tank with a height of 1ft and radius of 5ft would be more sensitive to changes in height than in radius.

Notes:

One could make a chart of small changes in radius and height and find exact changes in volume given specific changes. While this provides exact numbers, it does not give as much insight as the error analysis using the total differential.

Differentiability of Functions of Three Variables

The definition of differentiability for functions of three variables is very similar to that of functions of two variables. We again start with the total differential.

Definition 12.4.3 Total Differential

Let $w = f(x, y, z)$ be continuous on a set D. Let dx, dy and dz represent changes in x, y and z, respectively. Where the partial derivatives f_x, f_y and f_z exist, the **total differential of** w is

$$dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz.$$

This differential can be a good approximation of the change in w when $w = f(x, y, z)$ is **differentiable**.

Definition 12.4.4 Multivariable Differentiability

Let $w = f(x, y, z)$ be defined on a set D containing (x_0, y_0, z_0) where $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$ and $f_z(x_0, y_0, z_0)$ exist. Let dw be the total differential of w at (x_0, y_0, z_0), let $\Delta w = f(x_0 + dx, y_0 + dy, z_0 + dz) - f(x_0, y_0, z_0)$, and let E_x, E_y and E_z be functions of dx, dy and dz such that

$$\Delta w = dw + E_x dx + E_y dy + E_z dz.$$

1. We say f is **differentiable at** (x_0, y_0, z_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|| \langle dx, dy, dz \rangle || < \delta$, then $|| \langle E_x, E_y, E_z \rangle || < \varepsilon$.

2. We say f is **differentiable on** B if f is differentiable at every point in B. If f is differentiable on $\mathbb{R}^3$, we say that f is **differentiable everywhere**.

Just as before, this definition gives a rigorous statement about what it means to be differentiable that is not very intuitive. We follow it with a theorem similar to Theorem 12.4.2.

Notes:

Theorem 12.4.3 Continuity and Differentiability of Functions of Three Variables

Let $w = f(x, y, z)$ be defined on a set D containing (x_0, y_0, z_0).

1. If f is differentiable at (x_0, y_0, z_0), then f is continuous at (x_0, y_0, z_0).
2. If f_x, f_y and f_z are continuous on B, then f is differentiable on B.

This set of definition and theorem extends to functions of any number of variables. The theorem again gives us a simple way of verifying that most functions that we encounter are differentiable on their natural domains.

This section has given us a formal definition of what it means for a functions to be "differentiable," along with a theorem that gives a more accessible understanding. The following sections return to notions prompted by our study of partial derivatives that make use of the fact that most functions we encounter are differentiable.

Notes:

Exercises 12.4

Terms and Concepts

1. T/F: If $f(x, y)$ is differentiable on S, the f is continuous on S.

2. T/F: If f_x and f_y are continuous on S, then f is differentiable on S.

3. T/F: If $z = f(x, y)$ is differentiable, then the change in z over small changes dx and dy in x and y is approximately dz.

4. Finish the sentence: "The new z-value is approximately the old z-value plus the approximate ________."

Problems

In Exercises 5 – 8, find the total differential dz.

5. $z = x \sin y + x^2$

6. $z = (2x^2 + 3y)^2$

7. $z = 5x - 7y$

8. $z = xe^{x+y}$

In Exercises 9 – 12, a function $z = f(x, y)$ is given. Give the indicated approximation using the total differential.

9. $f(x, y) = \sqrt{x^2 + y}$. Approximate $f(2.95, 7.1)$ knowing $f(3, 7) = 4$.

10. $f(x, y) = \sin x \cos y$. Approximate $f(0.1, -0.1)$ knowing $f(0, 0) = 0$.

11. $f(x, y) = x^2y - xy^2$. Approximate $f(2.04, 3.06)$ knowing $f(2, 3) = -6$.

12. $f(x, y) = \ln(x - y)$. Approximate $f(5.1, 3.98)$ knowing $f(5, 4) = 0$.

Exercises 13 – 16 ask a variety of questions dealing with approximating error and sensitivity analysis.

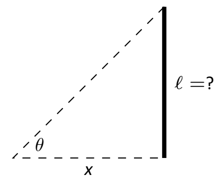
13. A cylindrical storage tank is to be 2ft tall with a radius of 1ft. Is the volume of the tank more sensitive to changes in the radius or the height?

14. **Projectile Motion:** The x-value of an object moving under the principles of projectile motion is $x(\theta, v_0, t) = (v_0 \cos\theta)t$. A particular projectile is fired with an initial velocity of $v_0 = 250$ft/s and an angle of elevation of $\theta = 60^\circ$. It travels a distance of 375ft in 3 seconds.

    Is the projectile more sensitive to errors in initial speed or angle of elevation?

15. The length ℓ of a long wall is to be approximated. The angle θ, as shown in the diagram (not to scale), is measured to be 85°, and the distance x is measured to be 30'. Assume that the triangle formed is a right triangle.

    Is the measurement of the length of ℓ more sensitive to errors in the measurement of x or in θ?



16. It is "common sense" that it is far better to measure a long distance with a long measuring tape rather than a short one. A measured distance D can be viewed as the product of the length ℓ of a measuring tape times the number n of times it was used. For instance, using a 3' tape 10 times gives a length of 30'. To measure the same distance with a 12' tape, we would use the tape 2.5 times. (I.e., $30 = 12 \times 2.5$.) Thus $D = n\ell$.

    Suppose each time a measurement is taken with the tape, the recorded distance is within 1/16" of the actual distance. (I.e., $d\ell = 1/16'' \approx 0.005$ft). Using differentials, show why common sense proves correct in that it is better to use a long tape to measure long distances.

In Exercises 17 – 18, find the total differential dw.

17. $w = x^2yz^3$

18. $w = e^x \sin y \ln z$

In Exercises 19 – 22, use the information provided and the total differential to make the given approximation.

19. $f(3, 1) = 7$, $f_x(3, 1) = 9$, $f_y(3, 1) = -2$. Approximate $f(3.05, 0.9)$.

20. $f(-4, 2) = 13$, $f_x(-4, 2) = 2.6$, $f_y(-4, 2) = 5.1$. Approximate $f(-4.12, 2.07)$.

21. $f(2, 4, 5) = -1$, $f_x(2, 4, 5) = 2$, $f_y(2, 4, 5) = -3$, $f_z(2, 4, 5) = 3.7$. Approximate $f(2.5, 4.1, 4.8)$.

22. $f(3, 3, 3) = 5$, $f_x(3, 3, 3) = 2$, $f_y(3, 3, 3) = 0$, $f_z(3, 3, 3) = -2$. Approximate $f(3.1, 3.1, 3.1)$.

12.5 The Multivariable Chain Rule

Consider driving an off-road vehicle along a dirt road. As you drive, your elevation likely changes. What factors determine how quickly your elevation rises and falls? After some thought, generally one recognizes that one's *velocity* (speed and direction) and the *terrain* influence your rise and fall.

One can represent the terrain as the surface defined by a multivariable function $z = f(x, y)$; one can represent the path of the off-road vehicle, as seen from above, with a vector–valued function $\vec{r}(t) = \langle x(t), y(t)\rangle$; the velocity of the vehicle is thus $\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle$.

Consider Figure 12.5.1 in which a surface $z = f(x, y)$ is drawn, along with a dashed curve in the x-y plane. Restricting f to just the points on this circle gives the curve shown on the surface (i.e., "the path of the off-road vehicle.") The derivative $\frac{df}{dt}$ gives the instantaneous rate of change of f with respect to t. If we consider an object traveling along this path, $\frac{df}{dt} = \frac{dz}{dt}$ gives the rate at which the object rises/falls (i.e., "the rate of elevation change" of the vehicle.) Conceptually, the Multivariable Chain Rule combines terrain and velocity information properly to compute this rate of elevation change.

Abstractly, let z be a function of x and y; that is, $z = f(x, y)$ for some function f, and let x and y each be functions of t. By choosing a t-value, x- and y-values are determined, which in turn determine z: this defines z as a function of t. The Multivariable Chain Rule gives a method of computing $\frac{dz}{dt}$.

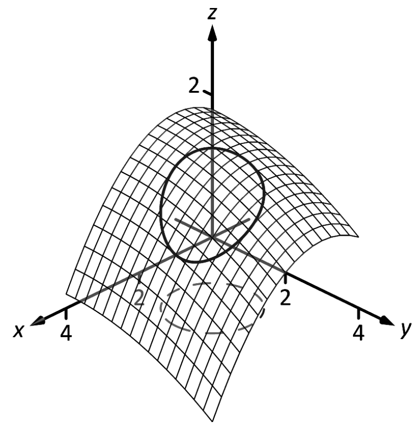


Figure 12.5.1: Understanding the application of the Multivariable Chain Rule.

Theorem 12.5.1 Multivariable Chain Rule, Part I

Let $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, where f, g and h are differentiable functions. Then $z = f(x, y) = f\big(g(t), h(t)\big)$ is a function of t, and

$$\begin{aligned}\frac{dz}{dt} = \frac{df}{dt} &= f_x(x, y)\frac{dx}{dt} + f_y(x, y)\frac{dy}{dt}\\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\\ &= \langle f_x, f_y\rangle \cdot \langle x', y'\rangle.\end{aligned}$$

The Chain Rule of Section 2.5 states that $\dfrac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)g'(x)$. If $t = g(x)$, we can express the Chain Rule as

$$\frac{df}{dx} = \frac{df}{dt}\frac{dt}{dx};$$

recall that the derivative notation is deliberately chosen to reflect their fraction–

Notes:

like properties. A similar effect is seen in Theorem 12.5.1. In the second line of equations, one can think of the dx and ∂x as "sort of" canceling out, and likewise with dy and ∂y.

Notice, too, the third line of equations in Theorem 12.5.1. The vector $\langle f_x, f_y\rangle$ contains information about the surface (terrain); the vector $\langle x', y'\rangle$ can represent velocity. In the context measuring the rate of elevation change of the off-road vehicle, the Multivariable Chain Rule states it can be found through a product of terrain and velocity information.

We now practice applying the Multivariable Chain Rule.

Example 12.5.1 Using the Multivariable Chain Rule

Let $z = x^2y + x$, where $x = \sin t$ and $y = e^{5t}$. Find $\dfrac{dz}{dt}$ using the Chain Rule.

Solution Following Theorem 12.5.1, we find

$$f_x(x, y) = 2xy + 1, \qquad f_y(x, y) = x^2, \qquad \frac{dx}{dt} = \cos t, \qquad \frac{dy}{dt} = 5e^{5t}.$$

Applying the theorem, we have

$$\frac{dz}{dt} = (2xy + 1)\cos t + 5x^2e^{5t}.$$

This may look odd, as it seems that $\frac{dz}{dt}$ is a function of x, y and t. Since x and y are functions of t, $\frac{dz}{dt}$ is really just a function of t, and we can replace x with $\sin t$ and y with e^{5t}:

$$\frac{dz}{dt} = (2xy + 1)\cos t + 5x^2e^{5t} = (2\sin(t)e^{5t} + 1)\cos t + 5e^{5t}\sin^2 t.$$

The previous example can make us wonder: if we substituted for x and y at the end to show that $\frac{dz}{dt}$ is really just a function of t, why not substitute before differentiating, showing clearly that z is a function of t?

That is, $z = x^2y + x = (\sin t)^2e^{5t} + \sin t$. Applying the Chain and Product Rules, we have

$$\frac{dz}{dt} = 2\sin t\cos t\, e^{5t} + 5\sin^2 t\, e^{5t} + \cos t,$$

which matches the result from the example.

This may now make one wonder "What's the point? If we could already find the derivative, why learn another way of finding it?" In some cases, applying this rule makes deriving simpler, but this is hardly the power of the Chain Rule. Rather, in the case where $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, the Chain Rule is

Notes:

extremely powerful when *we do not know what f, g and/or h are*. It may be hard to believe, but often in "the real world" we know rate–of–change information (i.e., information about derivatives) without explicitly knowing the underlying functions. The Chain Rule allows us to combine several rates of change to find another rate of change. The Chain Rule also has theoretic use, giving us insight into the behavior of certain constructions (as we'll see in the next section).

We demonstrate this in the next example.

Example 12.5.2 Applying the Multivarible Chain Rule

An object travels along a path on a surface. The exact path and surface are not known, but at time $t = t_0$ it is known that :

$$\frac{\partial z}{\partial x} = 5, \qquad \frac{\partial z}{\partial y} = -2, \qquad \frac{dx}{dt} = 3 \qquad \text{and} \qquad \frac{dy}{dt} = 7.$$

Find $\frac{dz}{dt}$ at time t_0.

Solution The Multivariable Chain Rule states that

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= 5(3) + (-2)(7)\\ &= 1.\end{aligned}$$

By knowing certain rates–of–change information about the surface and about the path of the particle in the x-y plane, we can determine how quickly the object is rising/falling.

We next apply the Chain Rule to solve a max/min problem.

Example 12.5.3 Applying the Multivariable Chain Rule

Consider the surface $z = x^2 + y^2 - xy$, a paraboloid, on which a particle moves with x and y coordinates given by $x = \cos t$ and $y = \sin t$. Find $\frac{dz}{dt}$ when $t = 0$, and find where the particle reaches its maximum/minimum z-values.

Solution It is straightforward to compute

$$f_x(x, y) = 2x - y, \qquad f_y(x, y) = 2y - x, \qquad \frac{dx}{dt} = -\sin t, \qquad \frac{dy}{dt} = \cos t.$$

Combining these according to the Chain Rule gives:

$$\frac{dz}{dt} = -(2x - y)\sin t + (2y - x)\cos t.$$

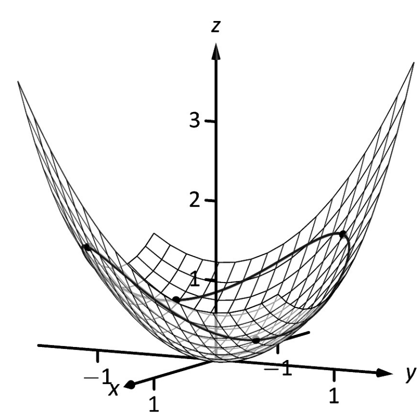


Figure 12.5.2: Plotting the path of a particle on a surface in Example 12.5.3.

Notes:

When $t=0$, $x=1$ and $y=0$. Thus $\dfrac{dz}{dt}=-(2)(0)+(-1)(1)=-1$. When $t=0$, the particle is moving down, as shown in Figure 12.5.2.

To find where z-value is maximized/minimized on the particle's path, we set $\frac{dz}{dt}=0$ and solve for t:

$$\begin{aligned}\frac{dz}{dt}=0&=-(2x-y)\sin t+(2y-x)\cos t\\0&=-(2\cos t-\sin t)\sin t+(2\sin t-\cos t)\cos t\\0&=\sin^2 t-\cos^2 t\\\cos^2 t&=\sin^2 t\\t&=n\frac{\pi}{4}\quad\text{(for odd }n)\end{aligned}$$

We can use the First Derivative Test to find that on $[0,2\pi]$, z has reaches its absolute minimum at $t=\pi/4$ and $5\pi/4$; it reaches its absolute maximum at $t=3\pi/4$ and $7\pi/4$, as shown in Figure 12.5.2.

We can extend the Chain Rule to include the situation where z is a function of more than one variable, and each of these variables is also a function of more than one variable. The basic case of this is where $z=f(x,y)$, and x and y are functions of two variables, say s and t.

Theorem 12.5.2 Multivariable Chain Rule, Part II

1. Let $z=f(x,y)$, $x=g(s,t)$ and $y=h(s,t)$, where f, g and h are differentiable functions. Then z is a function of s and t, and
   - $\dfrac{\partial z}{\partial s}=\dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s}+\dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$, and
   - $\dfrac{\partial z}{\partial t}=\dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t}+\dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}$.
2. Let $z=f(x_1,x_2,\ldots,x_m)$ be a differentiable function of m variables, where each of the x_i is a differentiable function of the variables $t_1,t_2,\ldots,t_n$. Then z is a function of the t_i, and

$$\frac{\partial z}{\partial t_i}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i}+\frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i}+\cdots+\frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_i}.$$

Notes:

Example 12.5.4 Using the Multivarible Chain Rule, Part II

Let $z = x^2y + x$, $x = s^2 + 3t$ and $y = 2s - t$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, and evaluate each when $s = 1$ and $t = 2$.

Solution Following Theorem 12.5.2, we compute the following partial derivatives:

$$\frac{\partial f}{\partial x} = 2xy + 1 \qquad \frac{\partial f}{\partial y} = x^2,$$

$$\frac{\partial x}{\partial s} = 2s \qquad \frac{\partial x}{\partial t} = 3 \qquad \frac{\partial y}{\partial s} = 2 \qquad \frac{\partial y}{\partial t} = -1.$$

Thus

$$\frac{\partial z}{\partial s} = (2xy + 1)(2s) + (x^2)(2) = 4xys + 2s + 2x^2, \quad \text{and}$$

$$\frac{\partial z}{\partial t} = (2xy + 1)(3) + (x^2)(-1) = 6xy - x^2 + 3.$$

When $s = 1$ and $t = 2$, $x = 7$ and $y = 0$, so

$$\frac{\partial z}{\partial s} = 100 \qquad \text{and} \qquad \frac{\partial z}{\partial t} = -46.$$

Example 12.5.5 Using the Multivarible Chain Rule, Part II

Let $w = xy + z^2$, where $x = t^2e^s$, $y = t\cos s$, and $z = s\sin t$. Find $\frac{\partial w}{\partial t}$ when $s = 0$ and $t = \pi$.

Solution Following Theorem 12.5.2, we compute the following partial derivatives:

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x \qquad \frac{\partial f}{\partial z} = 2z,$$

$$\frac{\partial x}{\partial t} = 2te^s \qquad \frac{\partial y}{\partial t} = \cos s \qquad \frac{\partial z}{\partial t} = s\cos t.$$

Thus

$$\frac{\partial w}{\partial t} = y(2te^s) + x(\cos s) + 2z(s\cos t).$$

When $s = 0$ and $t = \pi$, we have $x = \pi^2$, $y = \pi$ and $z = 0$. Thus

$$\frac{\partial w}{\partial t} = \pi(2\pi) + \pi^2 = 3\pi^2.$$

Implicit Differentiation

We studied finding $\frac{dy}{dx}$ when y is given as an implicit function of x in detail in Section 2.6. We find here that the Multivariable Chain Rule gives a simpler method of finding $\frac{dy}{dx}$.

Notes:

For instance, consider the implicit function $x^2y - xy^3 = 3$. We learned to use the following steps to find $\frac{dy}{dx}$:

$$\frac{d}{dx}\Big(x^2y - xy^3\Big) = \frac{d}{dx}\Big(3\Big)$$
$$2xy + x^2\frac{dy}{dx} - y^3 - 3xy^2\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = -\frac{2xy - y^3}{x^2 - 3xy^2}. \tag{12.2}$$

Instead of using this method, consider $z = x^2y - xy^3$. The implicit function above describes the level curve $z = 3$. Considering x and y as functions of x, the Multivariable Chain Rule states that

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx}. \tag{12.3}$$

Since z is constant (in our example, $z = 3$), $\frac{dz}{dx} = 0$. We also know $\frac{dx}{dx} = 1$. Equation (12.3) becomes

$$0 = \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}\frac{dy}{dx} \quad \Rightarrow$$
$$\frac{dy}{dx} = -\frac{\partial z}{\partial x}\Big/\frac{\partial z}{\partial y}$$
$$= -\frac{f_x}{f_y}.$$

Note how our solution for $\frac{dy}{dx}$ in Equation (12.2) is just the partial derivative of z with respect to x, divided by the partial derivative of z with respect to y, all multiplied by (-1).

We state the above as a theorem.

Theorem 12.5.3 Implicit Differentiation

Let f be a differentiable function of x and y, where $f(x, y) = c$ defines y as an implicit function of x, for some constant c. Then

$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}.$$

We practice using Theorem 12.5.3 by applying it to a problem from Section 2.6.

Notes:

Example 12.5.6 Implicit Differentiation

Given the implicitly defined function $\sin(x^2y^2)+y^3=x+y$, find y'. Note: this is the same problem as given in Example 2.6.4 of Section 2.6, where the solution took about a full page to find.

Solution Let $f(x,y)=\sin(x^2y^2)+y^3-x-y$; the implicitly defined function above is equivalent to $f(x,y)=0$. We find $\frac{dy}{dx}$ by applying Theorem 12.5.3. We find

$$f_x(x,y)=2xy^2\cos(x^2y^2)-1 \qquad \text{and} \qquad f_y(x,y)=2x^2y\cos(x^2y^2)+3y^2-1,$$

so

$$\frac{dy}{dx}=-\frac{2xy^2\cos(x^2y^2)-1}{2x^2y\cos(x^2y^2)+3y^2-1},$$

which matches our solution from Example 2.6.4.

In Section 12.3 we learned how partial derivatives give certain instantaneous rate of change information about a function $z=f(x,y)$. In that section, we measured the rate of change of f by holding one variable constant and letting the other vary (such as, holding y constant and letting x vary gives f_x). We can visualize this change by considering the surface defined by f at a point and moving parallel to the x-axis.

What if we want to move in a direction that is not parallel to a coordinate axis? Can we still measure instantaneous rates of change? Yes; we find out how in the next section. In doing so, we'll see how the Multivariable Chain Rule informs our understanding of these *directional derivatives*.

Notes:

Exercises 12.5

Terms and Concepts

1. Let a level curve of $z = f(x, y)$ be described by $x = g(t)$, $y = h(t)$. Explain why $\frac{dz}{dt} = 0$.

2. Fill in the blank: The single variable Chain Rule states $\frac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)\cdot$________.

3. Fill in the blank: The Multivariable Chain Rule states $\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot$ ________ $+$ ________ $\cdot \frac{dy}{dt}$.

4. If $z = f(x, y)$, where $x = g(t)$ and $y = h(t)$, we can substitute and write z as an explicit function of t.
T/F: Using the Multivariable Chain Rule to find $\frac{dz}{dt}$ is sometimes easier than first substituting and then taking the derivative.

5. T/F: The Multivariable Chain Rule is only useful when all the related functions are known explicitly.

6. The Multivariable Chain Rule allows us to compute implicit derivatives easily by just computing two ________ derivatives.

Problems

In Exercises 7 – 12, functions $z = f(x, y)$, $x = g(t)$ and $y = h(t)$ are given.

(a) Use the Multivariable Chain Rule to compute $\frac{dz}{dt}$.

(b) Evaluate $\frac{dz}{dt}$ at the indicated t-value.

7. $z = 3x + 4y, \quad x = t^2, \quad y = 2t; \quad t = 1$

8. $z = x^2 - y^2, \quad x = t, \quad y = t^2 - 1; \quad t = 1$

9. $z = 5x + 2y, \quad x = 2\cos t + 1, \quad y = \sin t - 3; \quad t = \pi/4$

10. $z = \frac{x}{y^2 + 1}, \quad x = \cos t, \quad y = \sin t; \quad t = \pi/2$

11. $z = x^2 + 2y^2, \quad x = \sin t, \quad y = 3\sin t; \quad t = \pi/4$

12. $z = \cos x \sin y, \quad x = \pi t, \quad y = 2\pi t + \pi/2; \quad t = 3$

In Exercises 13 – 18, functions $z = f(x, y)$, $x = g(t)$ and $y = h(t)$ are given. Find the values of t where $\frac{dz}{dt} = 0$. Note: these are the same surfaces/curves as found in Exercises 7 – 12.

13. $z = 3x + 4y, \quad x = t^2, \quad y = 2t$

14. $z = x^2 - y^2, \quad x = t, \quad y = t^2 - 1$

15. $z = 5x + 2y, \quad x = 2\cos t + 1, \quad y = \sin t - 3$

16. $z = \frac{x}{y^2 + 1}, \quad x = \cos t, \quad y = \sin t$

17. $z = x^2 + 2y^2, \quad x = \sin t, \quad y = 3\sin t$

18. $z = \cos x \sin y, \quad x = \pi t, \quad y = 2\pi t + \pi/2$

In Exercises 19 – 22, functions $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$ are given.

(a) Use the Multivariable Chain Rule to compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

(b) Evaluate $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at the indicated s and t values.

19. $z = x^2 y, \quad x = s - t, \quad y = 2s + 4t; \quad s = 1, t = 0$

20. $z = \cos\left(\pi x + \frac{\pi}{2} y\right), \quad x = st^2, \quad y = s^2 t; \quad s = 1, t = 1$

21. $z = x^2 + y^2, \quad x = s\cos t, \quad y = s\sin t; \quad s = 2, t = \pi/4$

22. $z = e^{-(x^2+y^2)}, \quad x = t, \quad y = st^2; \quad s = 1, t = 1$

In Exercises 23 – 26, find $\frac{dy}{dx}$ using Implicit Differentiation and Theorem 12.5.3.

23. $x^2 \tan y = 50$

24. $(3x^2 + 2y^3)^4 = 2$

25. $\frac{x^2 + y}{x + y^2} = 17$

26. $\ln(x^2 + xy + y^2) = 1$

In Exercises 27 – 30, find $\frac{dz}{dt}$, or $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, using the supplied information.

27. $\frac{\partial z}{\partial x} = 2, \quad \frac{\partial z}{\partial y} = 1, \quad \frac{dx}{dt} = 4, \quad \frac{dy}{dt} = -5$

28. $\frac{\partial z}{\partial x} = 1, \quad \frac{\partial z}{\partial y} = -3, \quad \frac{dx}{dt} = 6, \quad \frac{dy}{dt} = 2$

29. $\frac{\partial z}{\partial x} = -4, \quad \frac{\partial z}{\partial y} = 9,$
$\frac{\partial x}{\partial s} = 5, \quad \frac{\partial x}{\partial t} = 7, \quad \frac{\partial y}{\partial s} = -2, \quad \frac{\partial y}{\partial t} = 6$

30. $\frac{\partial z}{\partial x} = 2, \quad \frac{\partial z}{\partial y} = 1,$
$\frac{\partial x}{\partial s} = -2, \quad \frac{\partial x}{\partial t} = 3, \quad \frac{\partial y}{\partial s} = 2, \quad \frac{\partial y}{\partial t} = -1$

12.6 Directional Derivatives

Partial derivatives give us an understanding of how a surface changes when we move in the x and y directions. We made the comparison to standing in a rolling meadow and heading due east: the amount of rise/fall in doing so is comparable to f_x. Likewise, the rise/fall in moving due north is comparable to f_y. The steeper the slope, the greater in magnitude f_y.

But what if we didn't move due north or east? What if we needed to move northeast and wanted to measure the amount of rise/fall? Partial derivatives alone cannot measure this. This section investigates **directional derivatives**, which do measure this rate of change.

We begin with a definition.

Definition 12.6.1 Directional Derivatives

Let $z = f(x, y)$ be continuous on a set S and let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. For all points (x, y), the **directional derivative of f at (x, y) in the direction of $\vec{u}$** is

$$D_{\vec{u}}f(x, y) = \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2) - f(x, y)}{h}.$$

The partial derivatives f_x and f_y are defined with similar limits, but only x or y varies with h, not both. Here both x and y vary with a weighted h, determined by a particular unit vector $\vec{u}$. This may look a bit intimidating but in reality it is not too difficult to deal with; it often just requires extra algebra. However, the following theorem reduces this algebraic load.

Theorem 12.6.1 Directional Derivatives

Let $z = f(x, y)$ be differentiable on a set S containing (x_0, y_0), and let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector. The directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$ is

$$D_{\vec{u}}f(x_0, y_0) = f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2.$$

Example 12.6.1 Computing directional derivatives

Let $z = 14 - x^2 - y^2$ and let $P = (1, 2)$. Find the directional derivative of f, at P, in the following directions:

1. toward the point $Q = (3, 4)$,
2. in the direction of $\langle 2, -1 \rangle$, and

Notes:

3. toward the origin.

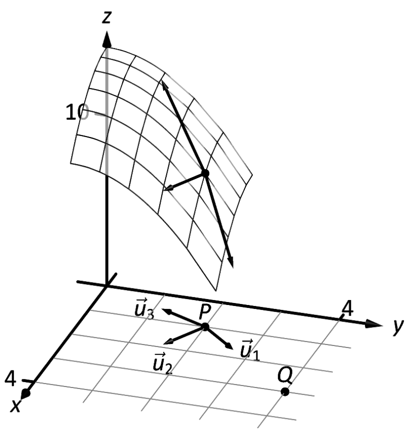


Figure 12.6.1: Understanding the directional derivative in Example 12.6.1.

Solution The surface is plotted in Figure 12.6.1, where the point $P = (1,2)$ is indicated in the x,y-plane as well as the point $(1,2,9)$ which lies on the surface of f. We find that $f_x(x,y) = -2x$ and $f_x(1,2) = -2$; $f_y(x,y) = -2y$ and $f_y(1,2) = -4$.

1. Let $\vec{u}_1$ be the unit vector that points from the point $(1,2)$ to the point $Q = (3,4)$, as shown in the figure. The vector $\overrightarrow{PQ} = \langle 2,2\rangle$; the unit vector in this direction is $\vec{u}_1 = \left\langle 1/\sqrt{2}, 1/\sqrt{2}\right\rangle$. Thus the directional derivative of f at $(1,2)$ in the direction of $\vec{u}_1$ is

$$D_{\vec{u}_1}f(1,2) = -2(1/\sqrt{2}) + (-4)(1/\sqrt{2}) = -6/\sqrt{2} \approx -4.24.$$

   Thus the instantaneous rate of change in moving from the point $(1,2,9)$ on the surface in the direction of $\vec{u}_1$ (which points toward the point Q) is about -4.24. Moving in this direction moves one steeply downward.

2. We seek the directional derivative in the direction of $\langle 2,-1\rangle$. The unit vector in this direction is $\vec{u}_2 = \left\langle 2/\sqrt{5}, -1/\sqrt{5}\right\rangle$. Thus the directional derivative of f at $(1,2)$ in the direction of $\vec{u}_2$ is

$$D_{\vec{u}_2}f(1,2) = -2(2/\sqrt{5}) + (-4)(-1/\sqrt{5}) = 0.$$

   Starting on the surface of f at $(1,2)$ and moving in the direction of $\langle 2,-1\rangle$ (or $\vec{u}_2$) results in no instantaneous change in z-value. This is analogous to standing on the side of a hill and choosing a direction to walk that does not change the elevation. One neither walks up nor down, rather just "along the side" of the hill.

   Finding these directions of "no elevation change" is important.

3. At $P = (1,2)$, the direction towards the origin is given by the vector $\langle -1,-2\rangle$; the unit vector in this direction is $\vec{u}_3 = \left\langle -1/\sqrt{5}, -2/\sqrt{5}\right\rangle$. The directional derivative of f at P in the direction of the origin is

$$D_{\vec{u}_3}f(1,2) = -2(-1/\sqrt{5}) + (-4)(-2/\sqrt{5}) = 10/\sqrt{5} \approx 4.47.$$

   Moving towards the origin means "walking uphill" quite steeply, with an initial slope of about 4.47.

As we study directional derivatives, it will help to make an important connection between the unit vector $\vec{u} = \langle u_1, u_2\rangle$ that describes the direction and the partial derivatives f_x and f_y. We start with a definition and follow this with a Key Idea.

Notes:

Definition 12.6.2 Gradient

Let $z = f(x, y)$ be differentiable on a set S that contains the point (x_0, y_0).

1. The **gradient of** f is $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$.
2. The **gradient of** f **at** (x_0, y_0) **is** $\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$**.**

To simplify notation, we often express the gradient as $\nabla f = \langle f_x, f_y \rangle$. The gradient allows us to compute directional derivatives in terms of a dot product.

Key Idea 12.6.1 The Gradient and Directional Derivatives

The directional derivative of $z = f(x, y)$ in the direction of $\vec{u}$ is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}.$$

Note: The symbol "∇" is named "nabla," derived from the Greek name of a Jewish harp. Oddly enough, in mathematics the expression ∇f is pronounced "del f."

The properties of the dot product previously studied allow us to investigate the properties of the directional derivative. Given that the directional derivative gives the instantaneous rate of change of z when moving in the direction of $\vec{u}$, three questions naturally arise:

1. In what direction(s) is the change in z the greatest (i.e., the "steepest uphill")?
2. In what direction(s) is the change in z the least (i.e., the "steepest downhill")?
3. In what direction(s) is there no change in z?

Using the key property of the dot product, we have

$$\nabla f \cdot \vec{u} = ||\, \nabla f \,||\, ||\, \vec{u} \,|| \cos\theta = ||\, \nabla f \,|| \cos\theta, \tag{12.4}$$

where θ is the angle between the gradient and $\vec{u}$. (Since $\vec{u}$ is a unit vector, $||\, \vec{u} \,|| = 1$.) This equation allows us to answer the three questions stated previously.

1. Equation 12.4 is maximized when $\cos\theta = 1$, i.e., when the gradient and $\vec{u}$ have the same direction. We conclude the gradient points in the direction of greatest z change.

Notes:

2. Equation 12.4 is minimized when $\cos\theta = -1$, i.e., when the gradient and $\vec{u}$ have opposite directions. We conclude the gradient points in the opposite direction of the least z change.

3. Equation 12.4 is 0 when $\cos\theta = 0$, i.e., when the gradient and $\vec{u}$ are orthogonal to each other. We conclude the gradient is orthogonal to directions of no z change.

This result is rather amazing. Once again imagine standing in a rolling meadow and face the direction that leads you steepest uphill. Then the direction that leads steepest downhill is directly behind you, and side–stepping either left or right (i.e., moving perpendicularly to the direction you face) does not change your elevation at all.

Recall that a level curve is defined as a curve in the x-y plane along which the z-values of a function do not change. Let a surface $z = f(x, y)$ be given, and let's represent one such level curve as a vector–valued function, $\vec{r}(t) = \langle x(t), y(t)\rangle$. As the output of f does not change along this curve, $f\big(x(t), y(t)\big) = c$ for all t, for some constant c.

Since f is constant for all t, $\frac{df}{dt} = 0$. By the Multivariable Chain Rule, we also know

$$\begin{aligned}\frac{df}{dt} &= f_x(x,y)x'(t) + f_y(x,y)y'(t)\\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle x'(t), y'(t)\rangle\\ &= \nabla f \cdot \vec{r}\,'(t)\\ &= 0.\end{aligned}$$

This last equality states $\nabla f \cdot \vec{r}\,'(t) = 0$: the gradient is orthogonal to the derivative of $\vec{r}$, meaning the gradient is orthogonal to the graph of $\vec{r}$. Our conclusion: at any point on a surface, the gradient at that point is orthogonal to the level curve that passes through that point.

We restate these ideas in a theorem, then use them in an example.

Theorem 12.6.2 The Gradient and Directional Derivatives

Let $z = f(x, y)$ be differentiable on a set S with gradient ∇f, let $P = (x_0, y_0)$ be a point in S and let $\vec{u}$ be a unit vector.

1. The maximum value of $D_{\vec{u}}\,f(x_0, y_0)$ is $||\ \nabla f(x_0, y_0)\ ||$; the direction of maximal z increase is $\nabla f(x_0, y_0)$.

2. The minimum value of $D_{\vec{u}}\,f(x_0, y_0)$ is $-||\ \nabla f(x_0, y_0)\ ||$; the direction of minimal z increase is $-\nabla f(x_0, y_0)$.

3. At P, $\nabla f(x_0, y_0)$ is orthogonal to the level curve passing through $\big(x_0, y_0, f(x_0, y_0)\big)$.

Notes:

Example 12.6.2 Finding directions of maximal and minimal increase

Let $f(x,y) = \sin x \cos y$ and let $P = (\pi/3, \pi/3)$. Find the directions of maximal/minimal increase, and find a direction where the instantaneous rate of z change is 0.

SOLUTION We begin by finding the gradient. $f_x = \cos x \cos y$ and $f_y = -\sin x \sin y$, thus

$$\nabla f = \langle \cos x \cos y, -\sin x \sin y \rangle \quad \text{and, at } P, \quad \nabla f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \left\langle \frac{1}{4}, -\frac{3}{4} \right\rangle.$$

Thus the direction of maximal increase is $\langle 1/4, -3/4 \rangle$. In this direction, the instantaneous rate of z change is $|| \langle 1/4, -3/4 \rangle || = \sqrt{10}/4 \approx 0.79$.

Figure 12.6.2 shows the surface plotted from two different perspectives. In each, the gradient is drawn at P with a dashed line (because of the nature of this surface, the gradient points "into" the surface). Let $\vec{u} = \langle u_1, u_2 \rangle$ be the unit vector in the direction of ∇f at P. Each graph of the figure also contains the vector $\langle u_1, u_2, ||\nabla f|| \rangle$. This vector has a "run" of 1 (because in the x-y plane it moves 1 unit) and a "rise" of $||\nabla f||$, hence we can think of it as a vector with slope of $||\nabla f||$ in the direction of ∇f, helping us visualize how "steep" the surface is in its steepest direction.

The direction of minimal increase is $\langle -1/4, 3/4 \rangle$; in this direction the instantaneous rate of z change is $-\sqrt{10}/4 \approx -0.79$.

Any direction orthogonal to ∇f is a direction of no z change. We have two choices: the direction of $\langle 3, 1 \rangle$ and the direction of $\langle -3, -1 \rangle$. The unit vector in the direction of $\langle 3, 1 \rangle$ is shown in each graph of the figure as well. The level curve at $z = \sqrt{3}/4$ is drawn: recall that along this curve the z-values do not change. Since $\langle 3, 1 \rangle$ is a direction of no z-change, this vector is tangent to the level curve at P.

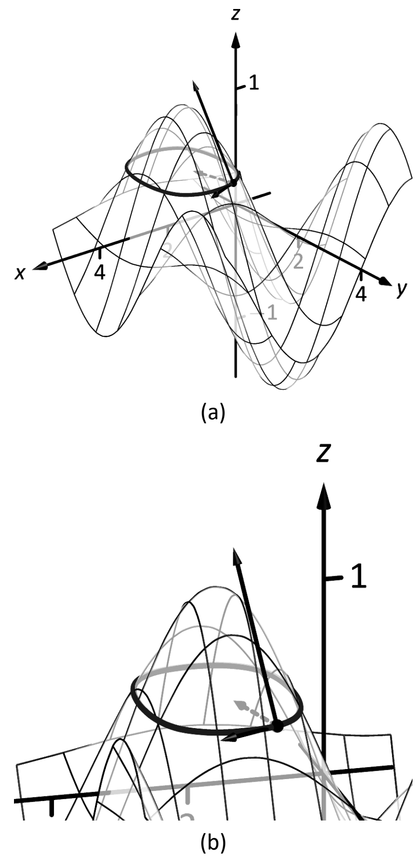


Figure 12.6.2: Graphing the surface and important directions in Example 12.6.2.

Example 12.6.3 Understanding when $\nabla f = \vec{0}$

Let $f(x,y) = -x^2 + 2x - y^2 + 2y + 1$. Find the directional derivative of f in any direction at $P = (1,1)$.

SOLUTION We find $\nabla f = \langle -2x + 2, -2y + 2 \rangle$. At P, we have $\nabla f(1,1) = \langle 0, 0 \rangle$. According to Theorem 12.6.2, this is the direction of maximal increase. However, $\langle 0, 0 \rangle$ is directionless; it has no displacement. And regardless of the unit vector $\vec{u}$ chosen, $D_{\vec{u}} f = 0$.

Figure 12.6.3 helps us understand what this means. We can see that P lies at the top of a paraboloid. In all directions, the instantaneous rate of change is 0.

So what is the direction of maximal increase? It is fine to give an answer of $\vec{0} = \langle 0, 0 \rangle$, as this indicates that all directional derivatives are 0.

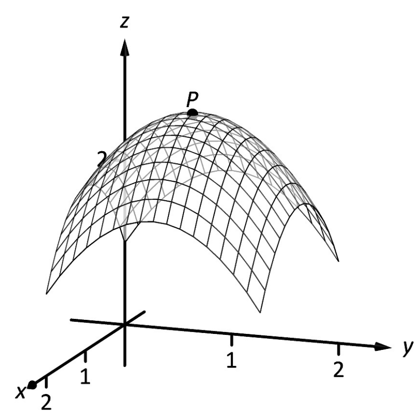


Figure 12.6.3: At the top of a paraboloid, all directional derivatives are 0.

Notes:

The fact that the gradient of a surface always points in the direction of steepest increase/decrease is very useful, as illustrated in the following example.

Example 12.6.4 The flow of water downhill
Consider the surface given by $f(x,y) = 20 - x^2 - 2y^2$. Water is poured on the surface at $(1, 1/4)$. What path does it take as it flows downhill?

SOLUTION Let $\vec{r}(t) = \langle x(t), y(t)\rangle$ be the vector–valued function describing the path of the water in the x-y plane; we seek $x(t)$ and $y(t)$. We know that water will always flow downhill in the steepest direction; therefore, at any point on its path, it will be moving in the direction of $-\nabla f$. (We ignore the physical effects of momentum on the water.) Thus $\vec{r}\,'(t)$ will be parallel to ∇f, and there is some constant c such that $c\nabla f = \vec{r}\,'(t) = \langle x'(t), y'(t)\rangle$.

We find $\nabla f = \langle -2x, -4y\rangle$ and write $x'(t)$ as $\frac{dx}{dt}$ and $y'(t)$ as $\frac{dy}{dt}$. Then

$$\begin{aligned} c\nabla f &= \langle x'(t), y'(t)\rangle \\ \langle -2cx, -4cy\rangle &= \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle. \end{aligned}$$

This implies

$$\begin{aligned} -2cx = \frac{dx}{dt} \quad &\text{and} \quad -4cy = \frac{dy}{dt}, \text{ i.e.,} \\ c = -\frac{1}{2x}\frac{dx}{dt} \quad &\text{and} \quad c = -\frac{1}{4y}\frac{dy}{dt}. \end{aligned}$$

As c equals both expressions, we have

$$\frac{1}{2x}\frac{dx}{dt} = \frac{1}{4y}\frac{dy}{dt}.$$

To find an explicit relationship between x and y, we can integrate both sides with respect to t. Recall from our study of differentials that $\frac{dx}{dt}dt = dx$. Thus:

$$\begin{aligned} \int \frac{1}{2x}\frac{dx}{dt}\,dt &= \int \frac{1}{4y}\frac{dy}{dt}\,dt \\ \int \frac{1}{2x}\,dx &= \int \frac{1}{4y}\,dy \\ \frac{1}{2}\ln|x| &= \frac{1}{4}\ln|y| + C_1 \\ 2\ln|x| &= \ln|y| + C_1 \\ \ln|x^2| &= \ln|y| + C_1 \end{aligned}$$

Notes:

Now raise both sides as a power of e:

$$\begin{aligned} x^2 &= e^{\ln|y|+C_1} \\ x^2 &= e^{\ln|y|}e^{C_1} \qquad \text{(Note that } e^{C_1} \text{ is just a constant.)} \\ x^2 &= yC_2 \\ \frac{1}{C_2}x^2 &= y \qquad \text{(Note that } 1/C_2 \text{ is just a constant.)} \\ Cx^2 &= y. \end{aligned}$$

As the water started at the point $(1, 1/4)$, we can solve for C:

$$C(1)^2 = \frac{1}{4} \quad \Rightarrow \quad C = \frac{1}{4}.$$

Thus the water follows the curve $y = x^2/4$ in the x-y plane. The surface and the path of the water is graphed in Figure 12.6.4(a). In part (b) of the figure, the level curves of the surface are plotted in the x-y plane, along with the curve $y = x^2/4$. Notice how the path intersects the level curves at right angles. As the path follows the gradient downhill, this reinforces the fact that the gradient is orthogonal to level curves.

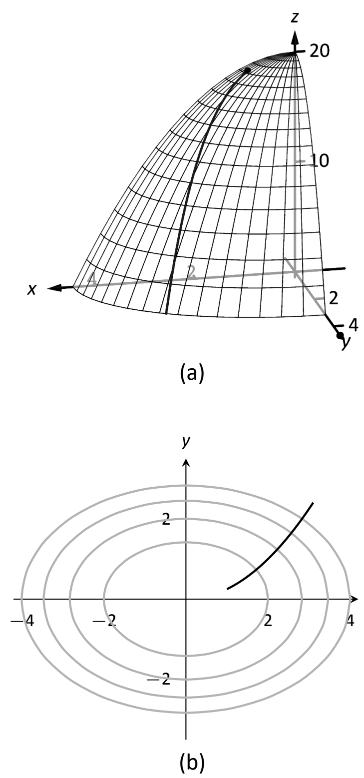


Figure 12.6.4: A graph of the surface described in Example 12.6.4 along with the path in the x-y plane with the level curves.

Functions of Three Variables

The concepts of directional derivatives and the gradient are easily extended to three (and more) variables. We combine the concepts behind Definitions 12.6.1 and 12.6.2 and Theorem 12.6.1 into one set of definitions.

Definition 12.6.3 Directional Derivatives and Gradient with Three Variables

Let $w = F(x, y, z)$ be differentiable on a set D and let $\vec{u}$ be a unit vector in $\mathbb{R}^3$.

1. The **gradient** of F is $\nabla F = \langle F_x, F_y, F_z \rangle$.
2. The **directional derivative of F in the direction of $\vec{u}$** is

$$D_{\vec{u}}\,F = \nabla F \cdot \vec{u}.$$

The same properties of the gradient given in Theorem 12.6.2, when f is a

Notes:

function of two variables, hold for F, a function of three variables.

Theorem 12.6.3 The Gradient and Directional Derivatives with Three Variables

Let $w = F(x, y, z)$ be differentiable on a set D, let ∇F be the gradient of F, and let $\vec{u}$ be a unit vector.

1. The maximum value of $D_{\vec{u}}\,F$ is $||\ \nabla F\ ||$, obtained when the angle between ∇F and $\vec{u}$ is 0, i.e., the direction of maximal increase is ∇F.
2. The minimum value of $D_{\vec{u}}\,F$ is $-||\ \nabla F\ ||$, obtained when the angle between ∇F and $\vec{u}$ is π, i.e., the direction of minimal increase is $-\nabla F$.
3. $D_{\vec{u}}\,F = 0$ when ∇F and $\vec{u}$ are orthogonal.

We interpret the third statement of the theorem as "the gradient is orthogonal to level surfaces," the three–variable analogue to level curves.

Example 12.6.5 Finding directional derivatives with functions of three variables

If a point source S is radiating energy, the intensity I at a given point P in space is inversely proportional to the square of the distance between S and P. That is, when $S = (0, 0, 0)$, $I(x, y, z) = \dfrac{k}{x^2 + y^2 + z^2}$ for some constant k.

Let $k = 1$, let $\vec{u} = \langle 2/3, 2/3, 1/3\rangle$ be a unit vector, and let $P = (2, 5, 3)$. Measure distances in inches. Find the directional derivative of I at P in the direction of $\vec{u}$, and find the direction of greatest intensity increase at P.

Solution We need the gradient ∇I, meaning we need I_x, I_y and I_z. Each partial derivative requires a simple application of the Quotient Rule, giving

$$\begin{aligned}
\nabla I &= \left\langle \frac{-2x}{(x^2+y^2+z^2)^2}, \frac{-2y}{(x^2+y^2+z^2)^2}, \frac{-2z}{(x^2+y^2+z^2)^2} \right\rangle \\
\nabla I(2,5,3) &= \left\langle \frac{-4}{1444}, \frac{-10}{1444}, \frac{-6}{1444} \right\rangle \approx \langle -0.003, -0.007, -0.004\rangle \\
D_{\vec{u}}\,I &= \nabla I(2,5,3)\cdot\vec{u} \\
&= -\frac{17}{2166} \approx -0.0078.
\end{aligned}$$

The directional derivative tells us that moving in the direction of $\vec{u}$ from P results in a decrease in intensity of about -0.008 units per inch. (The intensity is decreasing as $\vec{u}$ moves one farther from the origin than P.)

Notes:

The gradient gives the direction of greatest intensity increase. Notice that

$$\begin{aligned}\nabla I(2,5,3) &= \left\langle \frac{-4}{1444}, \frac{-10}{1444}, \frac{-6}{1444} \right\rangle \\ &= \frac{2}{1444}\left\langle -2,-5,-3\right\rangle.\end{aligned}$$

That is, the gradient at $(2,5,3)$ is pointing in the direction of $\langle -2,-5,-3\rangle$, that is, towards the origin. That should make intuitive sense: the greatest increase in intensity is found by moving towards to source of the energy.

The directional derivative allows us to find the instantaneous rate of z change in any direction at a point. We can use these instantaneous rates of change to define lines and planes that are *tangent* to a surface at a point, which is the topic of the next section.

Notes:

Exercises 12.6

Terms and Concepts

1. What is the difference between a directional derivative and a partial derivative?

2. For what $\vec{u}$ is $D_{\vec{u}}f = f_x$?

3. For what $\vec{u}$ is $D_{\vec{u}}f = f_y$?

4. The gradient is ________ to level curves.

5. The gradient points in the direction of ________ increase.

6. It is generally more informative to view the directional derivative not as the result of a limit, but rather as the result of a ________ product.

Problems

In Exercises 7 – 12, a function $z = f(x, y)$ is given. Find ∇f.

7. $f(x, y) = -x^2y + xy^2 + xy$

8. $f(x, y) = \sin x \cos y$

9. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

10. $f(x, y) = -4x + 3y$

11. $f(x, y) = x^2 + 2y^2 - xy - 7x$

12. $f(x, y) = x^2y^3 - 2x$

In Exercises 13 – 18, a function $z = f(x, y)$ and a point P are given. Find the directional derivative of f in the indicated directions. Note: these are the same functions as in Exercises 7 through 12.

13. $f(x, y) = -x^2y + xy^2 + xy$, $P = (2, 1)$
    (a) In the direction of $\vec{v} = \langle 3, 4\rangle$
    (b) In the direction toward the point $Q = (1, -1)$.

14. $f(x, y) = \sin x \cos y$, $P = \left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$
    (a) In the direction of $\vec{v} = \langle 1, 1\rangle$.
    (b) In the direction toward the point $Q = (0, 0)$.

15. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$, $P = (1, 1)$.
    (a) In the direction of $\vec{v} = \langle 1, -1\rangle$.
    (b) In the direction toward the point $Q = (-2, -2)$.

16. $f(x, y) = -4x + 3y$, $P = (5, 2)$
    (a) In the direction of $\vec{v} = \langle 3, 1\rangle$.
    (b) In the direction toward the point $Q = (2, 7)$.

17. $f(x, y) = x^2 + 2y^2 - xy - 7x$, $P = (4, 1)$
    (a) In the direction of $\vec{v} = \langle -2, 5\rangle$
    (b) In the direction toward the point $Q = (4, 0)$.

18. $f(x, y) = x^2y^3 - 2x$, $P = (1, 1)$
    (a) In the direction of $\vec{v} = \langle 3, 3\rangle$
    (b) In the direction toward the point $Q = (1, 2)$.

In Exercises 19 – 24, a function $z = f(x, y)$ and a point P are given.

(a) Find the direction of maximal increase of f at P.

(b) What is the maximal value of $D_{\vec{u}}f$ at P?

(c) Find the direction of minimal increase of f at P.

(d) Give a direction $\vec{u}$ such that $D_{\vec{u}}f = 0$ at P.

Note: these are the same functions and points as in Exercises 13 through 18.

19. $f(x, y) = -x^2y + xy^2 + xy$, $P = (2, 1)$

20. $f(x, y) = \sin x \cos y$, $P = \left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$

21. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$, $P = (1, 1)$.

22. $f(x, y) = -4x + 3y$, $P = (5, 4)$.

23. $f(x, y) = x^2 + 2y^2 - xy - 7x$, $P = (4, 1)$

24. $f(x, y) = x^2y^3 - 2x$, $P = (1, 1)$

In Exercises 25 – 28, a function $w = F(x, y, z)$, a vector $\vec{v}$ and a point P are given.

(a) Find $\nabla F(x, y, z)$.

(b) Find $D_{\vec{u}}F$ at P, where $\vec{u}$ is the unit vector in the direction of $\vec{v}$.

25. $F(x, y, z) = 3x^2z^3 + 4xy - 3z^2$, $\vec{v} = \langle 1, 1, 1\rangle$, $P = (3, 2, 1)$

26. $F(x, y, z) = \sin(x)\cos(y)e^z$, $\vec{v} = \langle 2, 2, 1\rangle$, $P = (0, 0, 0)$

27. $F(x, y, z) = x^2y^2 - y^2z^2$, $\vec{v} = \langle -1, 7, 3\rangle$, $P = (1, 0, -1)$

28. $F(x, y, z) = \dfrac{2}{x^2 + y^2 + z^2}$, $\vec{v} = \langle 1, 1, -2\rangle$, $P = (1, 1, 1)$

12.7 Tangent Lines, Normal Lines, and Tangent Planes

Derivatives and tangent lines go hand–in–hand. Given $y = f(x)$, the line tangent to the graph of f at $x = x_0$ is the line through $\big(x_0, f(x_0)\big)$ with slope $f'(x_0)$; that is, the slope of the tangent line is the instantaneous rate of change of f at x_0.

When dealing with functions of two variables, the graph is no longer a curve but a surface. At a given point on the surface, it seems there are many lines that fit our intuition of being "tangent" to the surface.

In Figure 12.7.1 we see lines that are tangent to curves in space. Since each curve lies on a surface, it makes sense to say that the lines are also tangent to the surface. The next definition formally defines what it means to be "tangent to a surface."

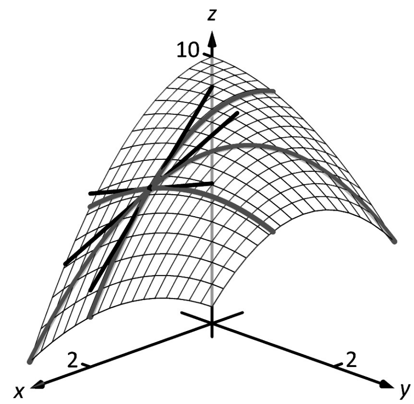


Figure 12.7.1: Showing various lines tangent to a surface.

Definition 12.7.1 Directional Tangent Line

Let $z = f(x, y)$ be differentiable on a set S containing (x_0, y_0) and let $\vec{u} = \langle u_1, u_2\rangle$ be a unit vector.

1. The line ℓ_x through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle 1, 0, f_x(x_0, y_0)\rangle$ is the **tangent line to f in the direction of x at** (x_0, y_0).
2. The line ℓ_y through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle 0, 1, f_y(x_0, y_0)\rangle$ is the **tangent line to f in the direction of y at** (x_0, y_0).
3. The line $\ell_{\vec{u}}$ through $\big(x_0, y_0, f(x_0, y_0)\big)$ parallel to $\langle u_1, u_2, D_{\vec{u}}f(x_0, y_0)\rangle$ is the **tangent line to f in the direction of $\vec{u}$ at** (x_0, y_0).

It is instructive to consider each of three directions given in the definition in terms of "slope." The direction of ℓ_x is $\langle 1, 0, f_x(x_0, y_0)\rangle$; that is, the "run" is one unit in the x-direction and the "rise" is $f_x(x_0, y_0)$ units in the z-direction. Note how the slope is just the partial derivative with respect to x. A similar statement can be made for ℓ_y. The direction of $\ell_{\vec{u}}$ is $\langle u_1, u_2, D_{\vec{u}}f(x_0, y_0)\rangle$; the "run" is one unit in the $\vec{u}$ direction (where $\vec{u}$ is a unit vector) and the "rise" is the directional derivative of z in that direction.

Definition 12.7.1 leads to the following parametric equations of directional tangent lines:

$$\ell_x(t) = \begin{cases} x = x_0 + t \\ y = y_0 \\ z = z_0 + f_x(x_0, y_0)t \end{cases}, \quad \ell_y(t) = \begin{cases} x = x_0 \\ y = y_0 + t \\ z = z_0 + f_y(x_0, y_0)t \end{cases} \quad \text{and} \quad \ell_{\vec{u}}(t) = \begin{cases} x = x_0 + u_1 t \\ y = y_0 + u_2 t \\ z = z_0 + D_{\vec{u}}f(x_0, y_0)t \end{cases}.$$

Notes:

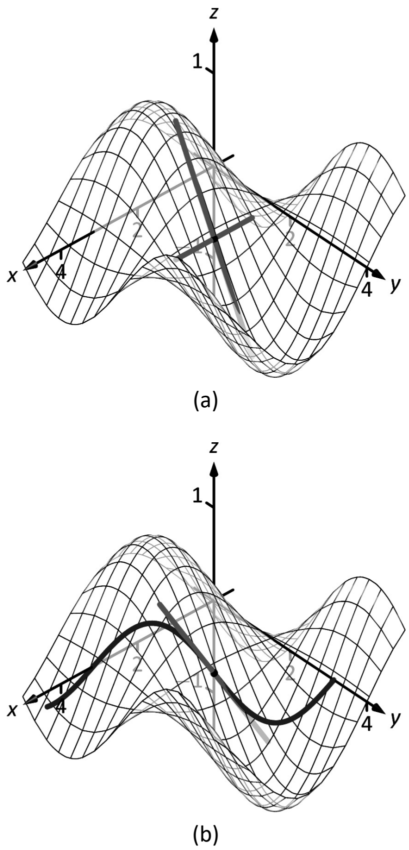


Figure 12.7.2: A surface and directional tangent lines in Example 12.7.1.

Example 12.7.1 Finding directional tangent lines

Find the lines tangent to the surface $z = \sin x \cos y$ at $(\pi/2, \pi/2)$ in the x and y directions and also in the direction of $\vec{v} = \langle -1, 1\rangle$.

Solution The partial derivatives with respect to x and y are:

$$\begin{aligned} f_x(x,y) &= \cos x \cos y \quad &\Rightarrow \quad f_x(\pi/2,\pi/2) = 0 \\ f_y(x,y) &= -\sin x \sin y \quad &\Rightarrow \quad f_y(\pi/2,\pi/2) = -1. \end{aligned}$$

At $(\pi/2, \pi/2)$, the z-value is 0.

Thus the parametric equations of the line tangent to f at $(\pi/2, \pi/2)$ in the directions of x and y are:

$$\ell_x(t) = \begin{cases} x = \pi/2 + t \\ y = \pi/2 \\ z = 0 \end{cases} \quad \text{and} \quad \ell_y(t) = \begin{cases} x = \pi/2 \\ y = \pi/2 + t \\ z = -t \end{cases}.$$

The two lines are shown with the surface in Figure 12.7.2(a). To find the equation of the tangent line in the direction of $\vec{v}$, we first find the unit vector in the direction of $\vec{v}$: $\vec{u} = \left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle$. The directional derivative at $(\pi/2, \pi, 2)$ in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(\pi/2, \pi, 2) = \langle 0, -1\rangle \cdot \left\langle -1/\sqrt{2}, 1/\sqrt{2}\right\rangle = -1/\sqrt{2}.$$

Thus the directional tangent line is

$$\ell_{\vec{u}}(t) = \begin{cases} x = \pi/2 - t/\sqrt{2} \\ y = \pi/2 + t/\sqrt{2} \\ z = -t/\sqrt{2} \end{cases}.$$

The curve through $(\pi/2, \pi/2, 0)$ in the direction of $\vec{v}$ is shown in Figure 12.7.2(b) along with $\ell_{\vec{u}}(t)$.

Example 12.7.2 Finding directional tangent lines

Let $f(x, y) = 4xy - x^4 - y^4$. Find the equations of *all* directional tangent lines to f at $(1, 1)$.

Solution First note that $f(1, 1) = 2$. We need to compute directional derivatives, so we need ∇f. We begin by computing partial derivatives.

$$f_x = 4y - 4x^3 \Rightarrow f_x(1,1) = 0; \quad f_y = 4x - 4y^3 \Rightarrow f_y(1,1) = 0.$$

Thus $\nabla f(1, 1) = \langle 0, 0\rangle$. Let $\vec{u} = \langle u_1, u_2\rangle$ be any unit vector. The directional derivative of f at $(1, 1)$ will be $D_{\vec{u}} f(1, 1) = \langle 0, 0\rangle \cdot \langle u_1, u_2\rangle = 0$. It does not matter

Notes:

what direction we choose; the directional derivative is always 0. Therefore

$$\ell_{\vec{u}}(t) = \begin{cases} x = 1 + u_1 t \\ y = 1 + u_2 t \\ z = 2 \end{cases}.$$

Figure 12.7.3 shows a graph of f and the point $(1, 1, 2)$. Note that this point comes at the top of a "hill," and therefore every tangent line through this point will have a "slope" of 0.

That is, consider any curve on the surface that goes through this point. Each curve will have a relative maximum at this point, hence its tangent line will have a slope of 0. The following section investigates the points on surfaces where all tangent lines have a slope of 0.

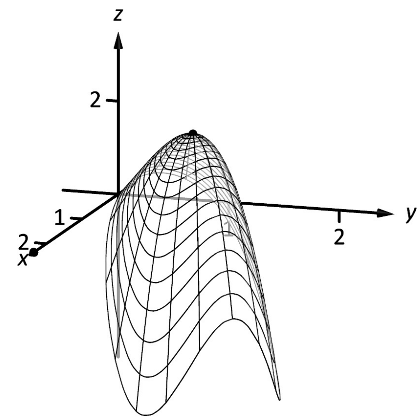


Figure 12.7.3: Graphing f in Example 12.7.2.

Normal Lines

When dealing with a function $y = f(x)$ of one variable, we stated that a line through $(c, f(c))$ was *tangent* to f if the line had a slope of $f'(c)$ and was *normal* (or, *perpendicular, orthogonal*) to f if it had a slope of $-1/f'(c)$. We extend the concept of normal, or orthogonal, to functions of two variables.

Let $z = f(x, y)$ be a differentiable function of two variables. By Definition 12.7.1, at (x_0, y_0), $\ell_x(t)$ is a line parallel to the vector $\vec{d}_x = \langle 1, 0, f_x(x_0, y_0)\rangle$ and $\ell_y(t)$ is a line parallel to $\vec{d}_y = \langle 0, 1, f_y(x_0, y_0)\rangle$. Since lines in these directions through $\big(x_0, y_0, f(x_0, y_0)\big)$ are *tangent* to the surface, a line through this point and orthogonal to these directions would be *orthogonal*, or *normal*, to the surface. We can use this direction to create a normal line.

The direction of the normal line is orthogonal to $\vec{d}_x$ and $\vec{d}_y$, hence the direction is parallel to $\vec{d}_n = \vec{d}_x \times \vec{d}_y$. It turns out this cross product has a very simple form:

$$\vec{d}_x \times \vec{d}_y = \langle 1, 0, f_x\rangle \times \langle 0, 1, f_y\rangle = \langle -f_x, -f_y, 1\rangle\,.$$

It is often more convenient to refer to the opposite of this direction, namely $\langle f_x, f_y, -1\rangle$. This leads to a definition.

Notes:

Definition 12.7.2 Normal Line

Let $z = f(x, y)$ be differentiable on a set S containing (x_0, y_0) where

$$a = f_x(x_0, y_0) \quad \text{and} \quad b = f_y(x_0, y_0)$$

are defined.

1. A nonzero vector parallel to $\vec{n} = \langle a, b, -1\rangle$ is **orthogonal to f at** $P = \big(x_0, y_0, f(x_0, y_0)\big)$.
2. The line ℓ_n through P with direction parallel to $\vec{n}$ is the **normal line to f at** P.

Thus the parametric equations of the normal line to a surface f at $\big(x_0, y_0, f(x_0, y_0)\big)$ is:

$$\ell_n(t) = \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = f(x_0, y_0) - t \end{cases}.$$

Example 12.7.3 Finding a normal line

Find the equation of the normal line to $z = -x^2 - y^2 + 2$ at $(0, 1)$.

Solution We find $z_x(x, y) = -2x$ and $z_y(x, y) = -2y$; at $(0, 1)$, we have $z_x = 0$ and $z_y = -2$. We take the direction of the normal line, following Definition 12.7.2, to be $\vec{n} = \langle 0, -2, -1\rangle$. The line with this direction going through the point $(0, 1, 1)$ is

$$\ell_n(t) = \begin{cases} x = 0 \\ y = -2t + 1 \\ z = -t + 1 \end{cases} \quad \text{or} \quad \ell_n(t) = \langle 0, -2, -1\rangle\, t + \langle 0, 1, 1\rangle .$$

The surface $z = -x^2 - y^2 + 2$, along with the found normal line, is graphed in Figure 12.7.4.

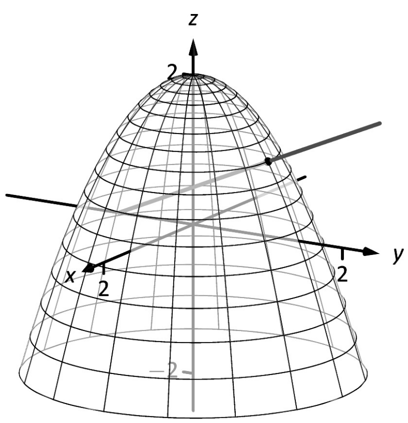


Figure 12.7.4: Graphing a surface with a normal line from Example 12.7.3.

The direction of the normal line has many uses, one of which is the definition of the **tangent plane** which we define shortly. Another use is in measuring distances from the surface to a point. Given a point Q in space, it is a general geometric concept to define the distance from Q to the surface as being the length of the shortest line segment $\overline{PQ}$ over all points P on the surface. This, in turn, implies that $\overrightarrow{PQ}$ will be orthogonal to the surface at P. Therefore we can measure the distance from Q to the surface f by finding a point P on the surface such that $\overrightarrow{PQ}$ is parallel to the normal line to f at P.

Notes:

Example 12.7.4 Finding the distance from a point to a surface

Let $f(x, y) = 2 - x^2 - y^2$ and let $Q = (2, 2, 2)$. Find the distance from Q to the surface defined by f.

Solution This surface is used in Example 12.7.2, so we know that at (x, y), the direction of the normal line will be $\vec{d}_n = \langle -2x, -2y, -1\rangle$. A point P on the surface will have coordinates $(x, y, 2-x^2-y^2)$, so $\overrightarrow{PQ} = \left\langle 2 - x, 2 - y, x^2 + y^2\right\rangle$. To find where $\overrightarrow{PQ}$ is parallel to $\vec{d}_n$, we need to find x, y and c such that $c\overrightarrow{PQ} = \vec{d}_n$.

$$\begin{aligned} c\overrightarrow{PQ} &= \vec{d}_n \\ c\left\langle 2 - x, 2 - y, x^2 + y^2\right\rangle &= \langle -2x, -2y, -1\rangle . \end{aligned}$$

This implies

$$\begin{aligned} c(2 - x) &= -2x \\ c(2 - y) &= -2y \\ c(x^2 + y^2) &= -1 \end{aligned}$$

In each equation, we can solve for c:

$$c = \frac{-2x}{2 - x} = \frac{-2y}{2 - y} = \frac{-1}{x^2 + y^2}.$$

The first two fractions imply $x = y$, and so the last fraction can be rewritten as $c = -1/(2x^2)$. Then

$$\begin{aligned} \frac{-2x}{2 - x} &= \frac{-1}{2x^2} \\ -2x(2x^2) &= -1(2 - x) \\ 4x^3 &= 2 - x \\ 4x^3 + x - 2 &= 0. \end{aligned}$$

This last equation is a cubic, which is not difficult to solve with a numeric solver. We find that $x = 0.689$, hence $P = (0.689, 0.689, 1.051)$. We find the distance from Q to the surface of f is

$$||\,\overrightarrow{PQ}\,|| = \sqrt{(2 - 0.689)^2 + (2 - 0.689)^2 + (2 - 1.051)^2} = 2.083.$$

We can take the concept of measuring the distance from a point to a surface to find a point Q a particular distance from a surface at a given point P on the surface.

Notes:

Example 12.7.5 Finding a point a set distance from a surface

Let $f(x,y) = x - y^2 + 3$. Let $P = \big(2, 1, f(2,1)\big) = (2, 1, 4)$. Find points Q in space that are 4 units from the surface of f at P. That is, find Q such that $||\,\vec{PQ}\,|| = 4$ and $\vec{PQ}$ is orthogonal to f at P.

SOLUTION We begin by finding partial derivatives:

$$\begin{aligned} f_x(x,y) &= 1 &\Rightarrow&& f_x(2,1) &= 1\\ f_y(x,y) &= -2y &\Rightarrow&& f_y(2,1) &= -2 \end{aligned}$$

The vector $\vec{n} = \langle 1, -2, -1\rangle$ is orthogonal to f at P. For reasons that will become more clear in a moment, we find the unit vector in the direction of $\vec{n}$:

$$\vec{u} = \frac{\vec{n}}{||\,\vec{n}\,||} = \left\langle 1/\sqrt{6}, -2/\sqrt{6}, -1/\sqrt{6}\right\rangle \approx \langle 0.408, -0.816, -0.408\rangle.$$

Thus a the normal line to f at P can be written as

$$\ell_n(t) = \langle 2, 1, 4\rangle + t\,\langle 0.408, -0.816, -0.408\rangle.$$

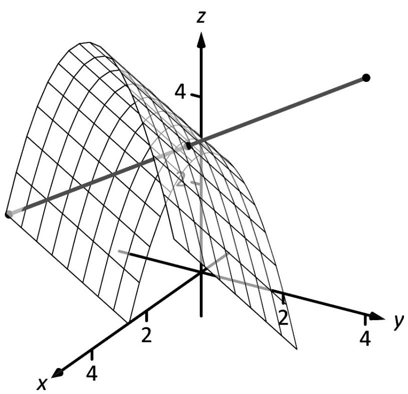


Figure 12.7.5: Graphing the surface in Example 12.7.5 along with points 4 units from the surface.

An advantage of this parametrization of the line is that letting $t = t_0$ gives a point on the line that is $|t_0|$ units from P. (This is because the direction of the line is given in terms of a unit vector.) There are thus two points in space 4 units from P:

$$\begin{aligned} Q_1 &= \ell_n(4) & Q_2 &= \ell_n(-4)\\ &\approx \langle 3.63, -2.27, 2.37\rangle & &\approx \langle 0.37, 4.27, 5.63\rangle \end{aligned}$$

width=150pt The surface is graphed along with points P, Q_1, Q_2 and a portion of the normal line to f at P.

Tangent Planes

We can use the direction of the normal line to define a plane. With $a = f_x(x_0, y_0)$, $b = f_y(x_0, y_0)$ and $P = \big(x_0, y_0, f(x_0, y_0)\big)$, the vector $\vec{n} = \langle a, b, -1\rangle$ is orthogonal to f at P. The plane through P with normal vector $\vec{n}$ is therefore **tangent** to f at P.

Notes:

Definition 12.7.3 Tangent Plane

Let $z = f(x,y)$ be differentiable on a set S containing (x_0, y_0), where $a = f_x(x_0,y_0)$, $b = f_y(x_0,y_0)$, $\vec{n} = \langle a, b, -1\rangle$ and $P = \big(x_0, y_0, f(x_0,y_0)\big)$.

The plane through P with normal vector $\vec{n}$ is the **tangent plane to f at P**. The standard form of this plane is

$$a(x-x_0) + b(y-y_0) - \big(z - f(x_0,y_0)\big) = 0.$$

Example 12.7.6 Finding tangent planes

Find the equation of the tangent plane to $z = -x^2 - y^2 + 2$ at $(0,1)$.

Solution Note that this is the same surface and point used in Example 12.7.3. There we found $\vec{n} = \langle 0, -2, -1\rangle$ and $P = (0,1,1)$. Therefore the equation of the tangent plane is

$$-2(y-1) - (z-1) = 0.$$

The surface $z = -x^2 - y^2 + 2$ and tangent plane are graphed in Figure 12.7.6.

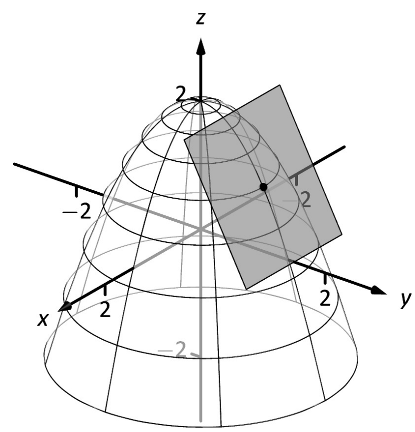


Figure 12.7.6: Graphing a surface with tangent plane from Example 12.7.6.

Example 12.7.7 Using the tangent plane to approximate function values

The point $(3,-1,4)$ lies on the surface of an unknown differentiable function f where $f_x(3,-1) = 2$ and $f_y(3,-1) = -1/2$. Find the equation of the tangent plane to f at P, and use this to approximate the value of $f(2.9,-0.8)$.

Solution Knowing the partial derivatives at $(3,-1)$ allows us to form the normal vector to the tangent plane, $\vec{n} = \langle 2, -1/2, -1\rangle$. Thus the equation of the tangent line to f at P is:

$$2(x-3) - 1/2(y+1) - (z-4) = 0 \quad \Rightarrow \quad z = 2(x-3) - 1/2(y+1) + 4. \tag{12.5}$$

Just as tangent lines provide excellent approximations of curves near their point of intersection, tangent planes provide excellent approximations of surfaces near their point of intersection. So $f(2.9,-0.8) \approx z(2.9,-0.8) = 3.7$.

This is not a new method of approximation. Compare the right hand expression for z in Equation (12.5) to the total differential:

$$dz = f_x dx + f_y dy \quad \text{and} \quad z = \underbrace{\underbrace{2}_{f_x}\underbrace{(x-3)}_{dx} + \underbrace{-1/2}_{f_y}\underbrace{(y+1)}_{dy}}_{dz} + 4.$$

Notes:

Thus the "new z-value" is the sum of the change in z (i.e., dz) and the old z-value (4). As mentioned when studying the total differential, it is not uncommon to know partial derivative information about a unknown function, and tangent planes are used to give accurate approximations of the function.

The Gradient and Normal Lines, Tangent Planes

The methods developed in this section so far give a straightforward method of finding equations of normal lines and tangent planes for surfaces with explicit equations of the form $z = f(x, y)$. However, they do not handle implicit equations well, such as $x^2 + y^2 + z^2 = 1$. There is a technique that allows us to find vectors orthogonal to these surfaces based on the **gradient**.

Definition 12.7.4 Gradient

Let $w = F(x, y, z)$ be differentiable on a set D that contains the point (x_0, y_0, z_0).

1. The **gradient of** F is $\nabla F(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$.
2. The **gradient of** F **at** (x_0, y_0, z_0) **is**

$$\nabla F(x_0, y_0, z_0) = \langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)\rangle .$$

Recall that when $z = f(x, y)$, the gradient $\nabla f = \langle f_x, f_y\rangle$ is orthogonal to level curves of f. An analogous statement can be made about the gradient ∇F, where $w = F(x, y, z)$. Given a point (x_0, y_0, z_0), let $c = F(x_0, y_0, z_0)$. Then $F(x, y, z) = c$ is a **level surface** that contains the point (x_0, y_0, z_0). The following theorem states that $\nabla F(x_0, y_0, z_0)$ is orthogonal to this level surface.

Theorem 12.7.1 The Gradient and Level Surfaces

Let $w = F(x, y, z)$ be differentiable on a set D containing (x_0, y_0, z_0) with gradient ∇F, where $F(x_0, y_0, z_0) = c$.

The vector $\nabla F(x_0, y_0, z_0)$ is orthogonal to the level surface $F(x, y, z) = c$ at (x_0, y_0, z_0).

The gradient at a point gives a vector orthogonal to the surface at that point. This direction can be used to find tangent planes and normal lines.

Notes:

Example 12.7.8 Using the gradient to find a tangent plane

Find the equation of the plane tangent to the ellipsoid $\dfrac{x^2}{12}+\dfrac{y^2}{6}+\dfrac{z^2}{4}=1$ at $P=(1,2,1)$.

Solution We consider the equation of the ellipsoid as a level surface of a function F of three variables, where $F(x,y,z)=\frac{x^2}{12}+\frac{y^2}{6}+\frac{z^2}{4}$. The gradient is:

$$\begin{aligned}\nabla F(x,y,z) &= \langle F_x, F_y, F_z\rangle\\ &= \left\langle \frac{x}{6}, \frac{y}{3}, \frac{z}{2}\right\rangle.\end{aligned}$$

At P, the gradient is $\nabla F(1,2,1)=\langle 1/6, 2/3, 1/2\rangle$. Thus the equation of the plane tangent to the ellipsoid at P is

$$\frac{1}{6}(x-1)+\frac{2}{3}(y-2)+\frac{1}{2}(z-1)=0.$$

The ellipsoid and tangent plane are graphed in Figure 12.7.7.

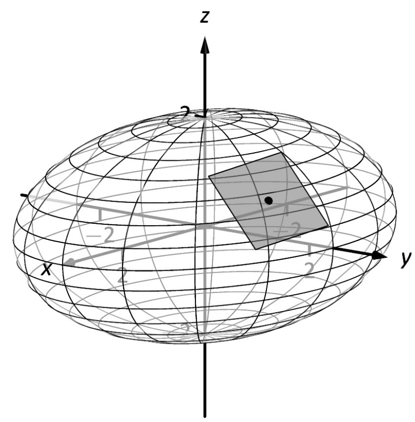


Figure 12.7.7: An ellipsoid and its tangent plane at a point.

Tangent lines and planes to surfaces have many uses, including the study of instantaneous rates of changes and making approximations. Normal lines also have many uses. In this section we focused on using them to measure distances from a surface. Another interesting application is in computer graphics, where the effects of light on a surface are determined using normal vectors.

The next section investigates another use of partial derivatives: determining relative extrema. When dealing with functions of the form $y=f(x)$, we found relative extrema by finding x where $f'(x)=0$. We can *start* finding relative extrema of $z=f(x,y)$ by setting f_x and f_y to 0, but it turns out that there is more to consider.

Notes:

Exercises 12.7

Terms and Concepts

1. Explain how the vector $\vec{v} = \langle 1, 0, 3\rangle$ can be thought of as having a "slope" of 3.

2. Explain how the vector $\vec{v} = \langle 0.6, 0.8, -2\rangle$ can be thought of as having a "slope" of -2.

3. T/F: Let $z = f(x, y)$ be differentiable at P. If $\vec{n}$ is a normal vector to the tangent plane of f at P, then $\vec{n}$ is orthogonal to ℓ_x and ℓ_y at P.

4. Explain in your own words why we do not refer to *the* tangent line to a surface at a point, but rather to *directional* tangent line*s* to a surface at a point.

Problems

In Exercises 5 – 8, a function $z = f(x, y)$, a vector $\vec{v}$ and a point P are given. Give the parametric equations of the following directional tangent lines to f at P:

(a) $\ell_x(t)$

(b) $\ell_y(t)$

(c) $\ell_{\vec{u}}(t)$**, where $\vec{u}$ is the unit vector in the direction of $\vec{v}$.**

5. $f(x, y) = 2x^2y - 4xy^2$, $\vec{v} = \langle 1, 3\rangle$, $P = (2, 3)$.

6. $f(x, y) = 3\cos x \sin y$, $\vec{v} = \langle 1, 2\rangle$, $P = (\pi/3, \pi/6)$.

7. $f(x, y) = 3x - 5y$, $\vec{v} = \langle 1, 1\rangle$, $P = (4, 2)$.

8. $f(x, y) = x^2 - 2x - y^2 + 4y$, $\vec{v} = \langle 1, 1\rangle$, $P = (1, 2)$.

In Exercises 9 – 12, a function $z = f(x, y)$ and a point P are given. Find the equation of the normal line to f at P. Note: these are the same functions as in Exercises 5 – 8.

9. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

10. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

11. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

12. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

In Exercises 13 – 16, a function $z = f(x, y)$ and a point P are given. Find the two points that are 2 units from the surface f at P. Note: these are the same functions as in Exercises 5 – 8.

13. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

14. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

15. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

16. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

In Exercises 17 – 20, a function $z = f(x, y)$ and a point P are given. Find the equation of the tangent plane to f at P. Note: these are the same functions as in Exercises 5 – 8.

17. $f(x, y) = 2x^2y - 4xy^2$, $P = (2, 3)$.

18. $f(x, y) = 3\cos x \sin y$, $P = (\pi/3, \pi/6)$.

19. $f(x, y) = 3x - 5y$, $P = (4, 2)$.

20. $f(x, y) = x^2 - 2x - y^2 + 4y$, $P = (1, 2)$.

In Exercises 21 – 24, an implicitly defined function of x, y and z is given along with a point P that lies on the surface. Use the gradient ∇F to:

(a) find the equation of the normal line to the surface at P, and

(b) find the equation of the plane tangent to the surface at P.

21. $\dfrac{x^2}{8} + \dfrac{y^2}{4} + \dfrac{z^2}{16} = 1$, at $P = (1, \sqrt{2}, \sqrt{6})$

22. $z^2 - \dfrac{x^2}{4} - \dfrac{y^2}{9} = 0$, at $P = (4, -3, \sqrt{5})$

23. $xy^2 - xz^2 = 0$, at $P = (2, 1, -1)$

24. $\sin(xy) + \cos(yz) = 0$, at $P = (2, \pi/12, 4)$

12.8 Extreme Values

Given a function $z = f(x, y)$, we are often interested in points where z takes on the largest or smallest values. For instance, if z represents a cost function, we would likely want to know what (x, y) values minimize the cost. If z represents the ratio of a volume to surface area, we would likely want to know where z is greatest. This leads to the following definition.

Definition 12.8.1 Relative and Absolute Extrema

Let $z = f(x, y)$ be defined on a set S containing the point $P = (x_0, y_0)$.

1. If $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in S, then f has an **absolute maximum** at P

   If $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in S, then f has an **absolute minimum** at P.

2. If there is an open disk D containing P such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that are in both D and S, then f has a **relative maximum** at P.

   If there is an open disk D containing P such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that are in both D and S, then f has a **relative minimum** at P.

3. If f has an absolute maximum or minimum at P, then f has an **absolute extrema** at P.

   If f has a relative maximum or minimum at P, then f has a **relative extrema** at P.

If f has a relative or absolute maximum at $P = (x_0, y_0)$, it means every curve on the surface of f through P will also have a relative or absolute maximum at P. Recalling what we learned in Section 3.1, the slopes of the tangent lines to these curves at P must be 0 or undefined. Since directional derivatives are computed using f_x and f_y, we are led to the following definition and theorem.

Definition 12.8.2 Critical Point

Let $z = f(x, y)$ be continuous on a set S. A **critical point** $P = (x_0, y_0)$ of f is a point in S such that, at P,

- $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or
- $f_x(x_0, y_0)$ and/or $f_y(x_0, y_0)$ is undefined.

Notes:

> **Theorem 12.8.1 Critical Points and Relative Extrema**
>
> Let $z = f(x, y)$ be defined on an open set S containing $P = (x_0, y_0)$. If f has a relative extrema at P, then P is a critical point of f.

Therefore, to find relative extrema, we find the critical points of f and determine which correspond to relative maxima, relative minima, or neither. The following examples demonstrate this process.

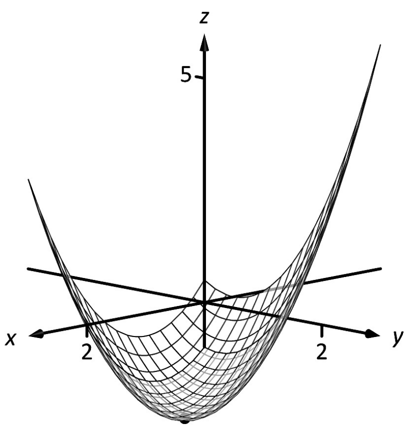


Figure 12.8.1: The surface in Example 12.8.1 with its absolute minimum indicated.

Example 12.8.1 Finding critical points and relative extrema

Let $f(x, y) = x^2 + y^2 - xy - x - 2$. Find the relative extrema of f.

Solution We start by computing the partial derivatives of f:

$$f_x(x, y) = 2x - y - 1 \qquad \text{and} \qquad f_y(x, y) = 2y - x.$$

Each is never undefined. A critical point occurs when f_x and f_y are simultaneously 0, leading us to solve the following system of linear equations:

$$2x - y - 1 = 0 \qquad \text{and} \qquad -x + 2y = 0.$$

This solution to this system is $x = 2/3$, $y = 1/3$. (Check that at $(2/3, 1/3)$, both f_x and f_y are 0.)

The graph in Figure 12.8.1 shows f along with this critical point. It is clear from the graph that this is a relative minimum; further consideration of the function shows that this is actually the absolute minimum.

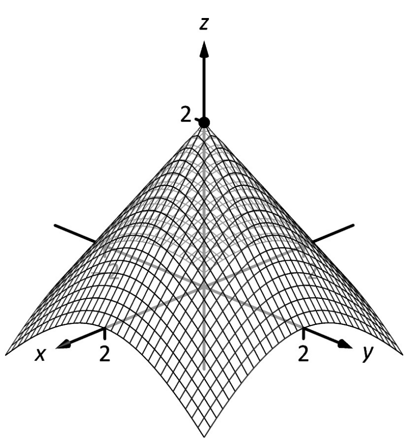


Figure 12.8.2: The surface in Example 12.8.2 with its absolute maximum indicated.

Example 12.8.2 Finding critical points and relative extrema

Let $f(x, y) = -\sqrt{x^2 + y^2} + 2$. Find the relative extrema of f.

Solution We start by computing the partial derivatives of f:

$$f_x(x, y) = \frac{-x}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad f_y(x, y) = \frac{-y}{\sqrt{x^2 + y^2}}.$$

It is clear that $f_x = 0$ when $x = 0$ & $y \neq 0$, and that $f_y = 0$ when $y = 0$ & $x \neq 0$. At $(0, 0)$, both f_x and f_y are *not* 0, but rather undefined. The point $(0, 0)$ is still a critical point, though, because the partial derivatives are undefined. This is the only critical point of f.

The surface of f is graphed in Figure 12.8.2 along with the point $(0, 0, 2)$. The graph shows that this point is the absolute maximum of f.

Notes:

In each of the previous two examples, we found a critical point of f and then determined whether or not it was a relative (or absolute) maximum or minimum by graphing. It would be nice to be able to determine whether a critical point corresponded to a max or a min without a graph. Before we develop such a test, we do one more example that sheds more light on the issues our test needs to consider.

Example 12.8.3 Finding critical points and relative extrema

Let $f(x,y) = x^3 - 3x - y^2 + 4y$. Find the relative extrema of f.

Solution Once again we start by finding the partial derivatives of f:

$$f_x(x,y) = 3x^2 - 3 \quad \text{and} \quad f_y(x,y) = -2y + 4.$$

Each is always defined. Setting each equal to 0 and solving for x and y, we find

$$\begin{aligned} f_x(x,y) = 0 \quad &\Rightarrow x = \pm 1 \\ f_y(x,y) = 0 \quad &\Rightarrow y = 2. \end{aligned}$$

We have two critical points: $(-1,2)$ and $(1,2)$. To determine if they correspond to a relative maximum or minimum, we consider the graph of f in Figure 12.8.3.

The critical point $(-1,2)$ clearly corresponds to a relative maximum. However, the critical point at $(1,2)$ is neither a maximum nor a minimum, displaying a different, interesting characteristic.

If one walks parallel to the y-axis towards this critical point, then this point becomes a relative maximum along this path. But if one walks towards this point parallel to the x-axis, this point becomes a relative minimum along this path. A point that seems to act as both a max and a min is a **saddle point**. A formal definition follows.

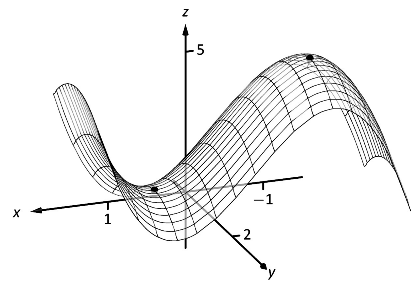


Figure 12.8.3: The surface in Example 12.8.3 with both critical points marked.

Definition 12.8.3 Saddle Point

Let $P = (x_0, y_0)$ be in the domain of f where $f_x = 0$ and $f_y = 0$ at P. We say P is a **saddle point** of f if, for every open disk D containing P, there are points (x_1, y_1) and (x_2, y_2) in D such that $f(x_0, y_0) > f(x_1, y_1)$ and $f(x_0, y_0) < f(x_2, y_2)$.

At a saddle point, the instantaneous rate of change in all directions is 0 and there are points nearby with z-values both less than and greater than the z-value of the saddle point.

Notes:

Before Example 12.8.3 we mentioned the need for a test to differentiate between relative maxima and minima. We now recognize that our test also needs to account for saddle points. To do so, we consider the second partial derivatives of f.

Recall that with single variable functions, such as $y = f(x)$, if $f''(c) > 0$, then if f is concave up at c, and if $f'(c) = 0$, then f has a relative minimum at $x = c$. (We called this the Second Derivative Test.) Note that at a saddle point, it seems the graph is "both" concave up and concave down, depending on which direction you are considering.

It would be nice if the following were true:

$$\begin{array}{rcc} f_{xx} \text{ and } f_{yy} > 0 & \Rightarrow & \text{relative minimum} \\ f_{xx} \text{ and } f_{yy} < 0 & \Rightarrow & \text{relative maximum} \\ f_{xx} \text{ and } f_{yy} \text{ have opposite signs} & \Rightarrow & \text{saddle point.} \end{array}$$

However, this is not the case. Functions f exist where f_{xx} and f_{yy} are both positive but a saddle point still exists. In such a case, while the concavity in the x-direction is up (i.e., $f_{xx} > 0$) and the concavity in the y-direction is also up (i.e., $f_{yy} > 0$), the concavity switches somewhere in between the x- and y-directions.

To account for this, consider $D = f_{xx}f_{yy} - f_{xy}f_{yx}$. Since f_{xy} and f_{yx} are equal when continuous (refer back to Theorem 12.3.1), we can rewrite this as $D = f_{xx}f_{yy} - f_{xy}^2$. D can be used to test whether the concavity at a point changes depending on direction. If $D > 0$, the concavity does not switch (i.e., at that point, the graph is concave up or down in all directions). If $D < 0$, the concavity does switch. If $D = 0$, our test fails to determine whether concavity switches or not. We state the use of D in the following theorem.

Theorem 12.8.2 Second Derivative Test

Let R be an open set on which a function $z = f(x, y)$ and all its first and second partial derivatives are defined, let $P = (x_0, y_0)$ be a critical point of f in R, and let

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0).$$

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a relative minimum at P.
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a relative maximum at P.
3. If $D < 0$, then f has a saddle point at P.
4. If $D = 0$, the test is inconclusive.

Notes:

We first practice using this test with the function in the previous example, where we visually determined we had a relative maximum and a saddle point.

Example 12.8.4 Using the Second Derivative Test

Let $f(x,y) = x^3 - 3x - y^2 + 4y$ as in Example 12.8.3. Determine whether the function has a relative minimum, maximum, or saddle point at each critical point.

SOLUTION We determined previously that the critical points of f are $(-1,2)$ and $(1,2)$. To use the Second Derivative Test, we must find the second partial derivatives of f:

$$f_{xx} = 6x; \qquad f_{yy} = -2; \qquad f_{xy} = 0.$$

Thus $D(x,y) = -12x$.

At $(-1,2)$: $D(-1,2) = 12 > 0$, and $f_{xx}(-1,2) = -6$. By the Second Derivative Test, f has a relative maximum at $(-1,2)$.

At $(1,2)$: $D(1,2) = -12 < 0$. The Second Derivative Test states that f has a saddle point at $(1,2)$.

The Second Derivative Test confirmed what we determined visually.

Example 12.8.5 Using the Second Derivative Test

Find the relative extrema of $f(x,y) = x^2y + y^2 + xy$.

SOLUTION We start by finding the first and second partial derivatives of f:

$$\begin{aligned} f_x &= 2xy + y & f_y &= x^2 + 2y + x \\ f_{xx} &= 2y & f_{yy} &= 2 \\ f_{xy} &= 2x + 1 & f_{yx} &= 2x + 1. \end{aligned}$$

We find the critical points by finding where f_x and f_y are simultaneously 0 (they are both never undefined). Setting $f_x = 0$, we have:

$$f_x = 0 \quad \Rightarrow \quad 2xy + y = 0 \quad \Rightarrow \quad y(2x+1) = 0.$$

This implies that for $f_x = 0$, either $y = 0$ or $2x + 1 = 0$.

Assume $y = 0$ then consider $f_y = 0$:

$$\begin{aligned} f_y &= 0 \\ x^2 + 2y + x &= 0, \qquad \text{and since } y = 0 \text{, we have} \\ x^2 + x &= 0 \\ x(x+1) &= 0. \end{aligned}$$

Thus if $y = 0$, we have either $x = 0$ or $x = -1$, giving two critical points: $(-1,0)$ and $(0,0)$.

Notes:

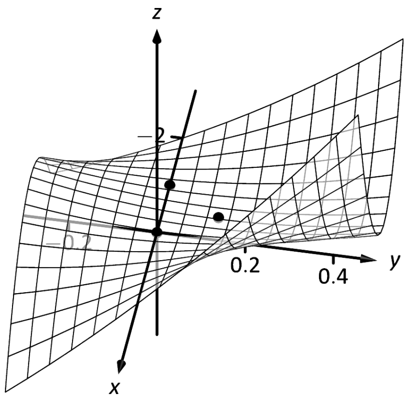


Figure 12.8.4: Graphing f from Example 12.8.5 and its relative extrema.

Going back to f_x, now assume $2x + 1 = 0$, i.e., that $x = -1/2$, then consider $f_y = 0$:

$$\begin{aligned} f_y &= 0 \\ x^2 + 2y + x &= 0, \qquad \text{and since } x = -1/2, \text{ we have} \\ 1/4 + 2y - 1/2 &= 0 \\ y &= 1/8. \end{aligned}$$

Thus if $x = -1/2$, $y = 1/8$ giving the critical point $(-1/2, 1/8)$.

With $D = 4y - (2x+1)^2$, we apply the Second Derivative Test to each critical point.

At $(-1, 0)$, $D < 0$, so $(-1, 0)$ is a saddle point.

At $(0, 0)$, $D < 0$, so $(0, 0)$ is also a saddle point.

At $(-1/2, 1/8)$, $D > 0$ and $f_{xx} > 0$, so $(-1/2, 1/8)$ is a relative minimum.

Figure 12.8.4 shows a graph of f and the three critical points. Note how this function does not vary much near the critical points – that is, visually it is difficult to determine whether a point is a saddle point or relative minimum (or even a critical point at all!). This is one reason why the Second Derivative Test is so important to have.

Constrained Optimization

When optimizing functions of one variable such as $y = f(x)$, we made use of Theorem 3.1.1, the Extreme Value Theorem, that said that over a closed interval I, a continuous function has both a maximum and minimum value. To find these maximum and minimum values, we evaluated f at all critical points in the interval, as well as at the endpoints (the "boundary") of the interval.

A similar theorem and procedure applies to functions of two variables. A continuous function over a closed set also attains a maximum and minimum value (see the following theorem). We can find these values by evaluating the function at the critical values in the set and over the boundary of the set. After formally stating this extreme value theorem, we give examples.

Theorem 12.8.3 Extreme Value Theorem

Let $z = f(x, y)$ be a continuous function on a closed, bounded set S. Then f has a maximum and minimum value on S.

Example 12.8.6 Finding extrema on a closed set

Let $f(x, y) = x^2 - y^2 + 5$ and let S be the triangle with vertices $(-1, -2)$, $(0, 1)$ and $(2, -2)$. Find the maximum and minimum values of f on S.

Notes:

Solution It can help to see a graph of f along with the set S. In Figure 12.8.5(a) the triangle defining S is shown in the x-y plane in a dashed line. Above it is the surface of f; we are only concerned with the portion of f enclosed by the "triangle" on its surface.

We begin by finding the critical points of f. With $f_x = 2x$ and $f_y = -2y$, we find only one critical point, at $(0, 0)$.

We now find the maximum and minimum values that f attains along the boundary of S, that is, along the edges of the triangle. In Figure 12.8.5(b) we see the triangle sketched in the plane with the equations of the lines forming its edges labeled.

Start with the bottom edge, along the line $y = -2$. If y is -2, then on the surface, we are considering points $f(x, -2)$; that is, our function reduces to $f(x, -2) = x^2 - (-2)^2 + 5 = x^2 + 1 = f_1(x)$. We want to maximize/minimize $f_1(x) = x^2 + 1$ on the interval $[-1, 2]$. To do so, we evaluate $f_1(x)$ at its critical points and at the endpoints.

The critical points of f_1 are found by setting its derivative equal to 0:

$$f_1'(x) = 0 \qquad \Rightarrow x = 0.$$

Evaluating f_1 at this critical point, and at the endpoints of $[-1, 2]$ gives:

$$\begin{aligned} f_1(-1) = 2 &\qquad \Rightarrow \qquad f(-1, -2) = 2 \\ f_1(0) = 1 &\qquad \Rightarrow \qquad f(0, -2) = 1 \\ f_1(2) = 5 &\qquad \Rightarrow \qquad f(2, -2) = 5. \end{aligned}$$

Notice how evaluating f_1 at a point is the same as evaluating f at its corresponding point.

We need to do this process twice more, for the other two edges of the triangle.

Along the left edge, along the line $y = 3x + 1$, we substitute $3x + 1$ in for y in $f(x, y)$:

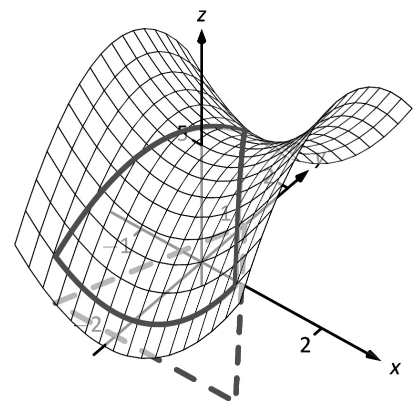
$$f(x, y) = f(x, 3x + 1) = x^2 - (3x + 1)^2 + 5 = -8x^2 - 6x + 4 = f_2(x).$$

We want the maximum and minimum values of f_2 on the interval $[-1, 0]$, so we evaluate f_2 at its critical points and the endpoints of the interval. We find the critical points:

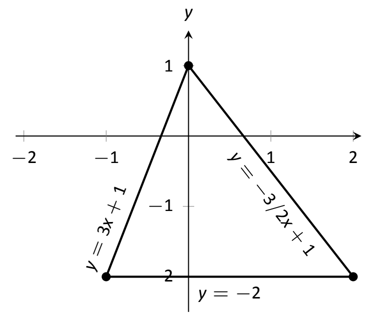
$$f_2'(x) = -16x - 6 = 0 \qquad \Rightarrow \qquad x = -3/8.$$

Evaluate f_2 at its critical point and the endpoints of $[-1, 0]$:

$$\begin{aligned} f_2(-1) = 2 &\qquad \Rightarrow \qquad f(-1, -2) = 2 \\ f_2(-3/8) = 41/8 = 5.125 &\qquad \Rightarrow \qquad f(-3/8, -0.125) = 5.125 \\ f_2(0) = 4 &\qquad \Rightarrow \qquad f(0, 1) = 4. \end{aligned}$$



(a)



(b)

Figure 12.8.5: Plotting the surface of f along with the restricted domain S in Example 12.8.6.

Notes:

Finally, we evaluate f along the right edge of the triangle, where $y = -3/2x + 1$.

$$f(x, y) = f(x, -3/2x + 1) = x^2 - (-3/2x + 1)^2 + 5 = -\frac{5}{4}x^2 + 3x + 4 = f_3(x).$$

The critical points of $f_3(x)$ are:

$$f_3'(x) = 0 \qquad \Rightarrow \qquad x = 6/5 = 1.2.$$

We evaluate f_3 at this critical point and at the endpoints of the interval $[0, 2]$:

$$\begin{aligned} f_3(0) = 4 &\qquad \Rightarrow \qquad f(0, 1) = 4 \\ f_3(1.2) = 5.8 &\qquad \Rightarrow \qquad f(1.2, -0.8) = 5.8 \\ f_3(2) = 5 &\qquad \Rightarrow \qquad f(2, -2) = 5. \end{aligned}$$

One last point to test: the critical point of f, $(0, 0)$. We find $f(0, 0) = 5$.

We have evaluated f at a total of 7 different places, all shown in Figure 12.8.6. We checked each vertex of the triangle twice, as each showed up as the endpoint of an interval twice. Of all the z-values found, the maximum is 5.8, found at $(1.2, -0.8)$; the minimum is 1, found at $(0, -2)$.

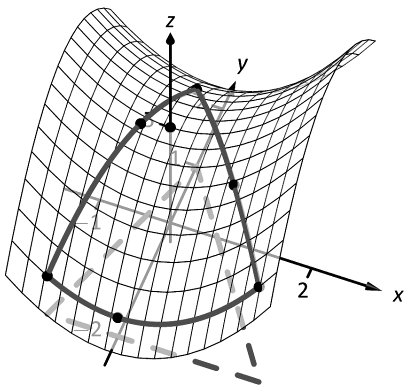


Figure 12.8.6: The surface of f along with important points along the boundary of S and the interior in Example 12.8.6.

This portion of the text is entitled "Constrained Optimization" because we want to optimize a function (i.e., find its maximum and/or minimum values) subject to a *constraint* – some limit to what values the function can attain. In the previous example, we constrained ourselves by considering a function only within the boundary of a triangle. This was largely arbitrary; the function and the boundary were chosen just as an example, with no real "meaning" behind the function or the chosen constraint.

However, solving constrained optimization problems is a very important topic in applied mathematics. The techniques developed here are the basis for solving larger problems, where more than two variables are involved.

We illustrate the technique once more with a classic problem.

Example 12.8.7 Constrained Optimization

The U.S. Postal Service states that the girth+length of Standard Post Package must not exceed 130". Given a rectangular box, the "length" is the longest side, and the "girth" is twice the width+height.

Given a rectangular box where the width and height are equal, what are the dimensions of the box that give the maximum volume subject to the constraint of the size of a Standard Post Package?

Solution Let w, h and ℓ denote the width, height and length of a rectangular box; we assume here that $w = h$. The girth is then $2(w + h) = 4w$. The

Notes:

volume of the box is $V(w,\ell) = wh\ell = w^2\ell$. We wish to maximize this volume subject to the constraint $4w + \ell \leq 130$, or $\ell \leq 130 - 4w$. (Common sense also indicates that $\ell > 0, w > 0$.)

We begin by finding the critical values of V. We find that $V_w = 2w\ell$ and $V_\ell = w^2$; these are simultaneously 0 only at $(0,0)$. This gives a volume of 0, so we can ignore this critical point.

We now consider the volume along the constraint $\ell = 130 - 4w$. Along this line, we have:

$$V(w,\ell) = V(w, 130 - 4w) = w^2(130 - 4w) = 130w^2 - 4w^3 = V_1(w).$$

The constraint is applicable on the w-interval $[0, 32.5]$ as indicated in the figure. Thus we want to maximize V_1 on $[0, 32.5]$.

Finding the critical values of V_1, we take the derivative and set it equal to 0:

$$V_1'(w) = 260w - 12w^2 = 0 \quad \Rightarrow \quad w(260 - 12w) = 0 \quad \Rightarrow \quad w = 0, \frac{260}{12} \approx 21.67.$$

We found two critical values: when $w = 0$ and when $w = 21.67$. We again ignore the $w = 0$ solution; the maximum volume, subject to the constraint, comes at $w = h = 21.67$, $\ell = 130 - 4(21.6) = 43.33$. This gives a volume of $V(21.67, 43.33) \approx 19,408\text{in}^3$.

The volume function $V(w,\ell)$ is shown in Figure 12.8.7 along with the constraint $\ell = 130 - 4w$. As done previously, the constraint is drawn dashed in the x-y plane and also along the surface of the function. The point where the volume is maximized is indicated.

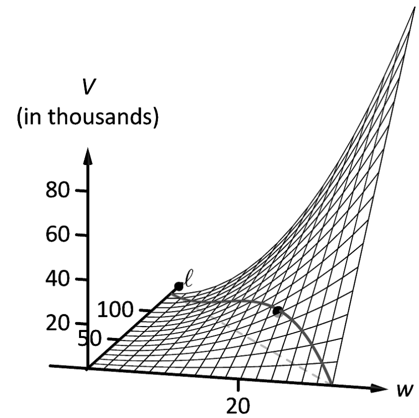


Figure 12.8.7: Graphing the volume of a box with girth $4w$ and length ℓ, subject to a size constraint.

It is hard to overemphasize the importance of optimization. In "the real world," we routinely seek to make *something* better. By expressing the *something* as a mathematical function, "making *something* better" means "optimize *some function*."

The techniques shown here are only the beginning of an incredibly important field. Many functions that we seek to optimize are incredibly complex, making the step of "find the gradient and set it equal to $\vec{0}$" highly nontrivial. Mastery of the principles here are key to being able to tackle these more complicated problems.

Notes:

Exercises 12.8

Terms and Concepts

1. T/F: Theorem 12.8.1 states that if f has a critical point at P, then f has a relative extrema at P.

2. T/F: A point P is a critical point of f if f_x and f_y are both 0 at P.

3. T/F: A point P is a critical point of f if f_x or f_y are undefined at P.

4. Explain what it means to "solve a constrained optimization" problem.

Problems

In Exercises 5 – 14, find the critical points of the given function. Use the Second Derivative Test to determine if each critical point corresponds to a relative maximum, minimum, or saddle point.

5. $f(x,y) = \frac{1}{2}x^2 + 2y^2 - 8y + 4x$

6. $f(x,y) = x^2 + 4x + y^2 - 9y + 3xy$

7. $f(x,y) = x^2 + 3y^2 - 6y + 4xy$

8. $f(x,y) = \dfrac{1}{x^2+y^2+1}$

9. $f(x,y) = x^2 + y^3 - 3y + 1$

10. $f(x,y) = \dfrac{1}{3}x^3 - x + \dfrac{1}{3}y^3 - 4y$

11. $f(x,y) = x^2y^2$

12. $f(x,y) = x^4 - 2x^2 + y^3 - 27y - 15$

13. $f(x,y) = \sqrt{16-(x-3)^2-y^2}$

14. $f(x,y) = \sqrt{x^2+y^2}$

In Exercises 15 – 18, find the absolute maximum and minimum of the function subject to the given constraint.

15. $f(x,y) = x^2 + y^2 + y + 1$, constrained to the triangle with vertices $(0,1)$, $(-1,-1)$ and $(1,-1)$.

16. $f(x,y) = 5x - 7y$, constrained to the region bounded by $y = x^2$ and $y = 1$.

17. $f(x,y) = x^2 + 2x + y^2 + 2y$, constrained to the region bounded by the circle $x^2 + y^2 = 4$.

18. $f(x,y) = 3y - 2x^2$, constrained to the region bounded by the parabola $y = x^2 + x - 1$ and the line $y = x$.