APEX CALCULUS

Version 4.0

Gregory Hartman, Ph.D.

Department of Applied Mathematics

Virginia Military Institute

Contributing Authors

Troy Siemers, Ph.D.

Department of Applied Mathematics

Virginia Military Institute

Brian Heinold, Ph.D.

Department of Mathematics and Computer Science

Mount Saint Mary's University

Dimplekumar Chalishajar, Ph.D.

Department of Applied Mathematics

Virginia Military Institute

Editor

Jennifer Bowen, Ph.D.

Department of Mathematics and Computer Science

The College of Wooster

Contents

7: Applications of Integration

We begin this chapter with a reminder of a few key concepts from Chapter 5. Let f be a continuous function on $[a,b]$ which is partitioned into n equally spaced subintervals as

$$a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b.$$

Let $\Delta x = (b-a)/n$ denote the length of the subintervals, and let c_i be any x-value in the i^{th} subinterval. Definition 5.3.2 states that the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x$$

is a *Riemann Sum.* Riemann Sums are often used to approximate some quantity (area, volume, work, pressure, etc.). The *approximation* becomes *exact* by taking the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x.$$

Theorem 5.3.2 connects limits of Riemann Sums to definite integrals:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx.$$

Finally, the Fundamental Theorem of Calculus states how definite integrals can be evaluated using antiderivatives.

This chapter employs the following technique to a variety of applications. Suppose the value Q of a quantity is to be calculated. We first approximate the value of Q using a Riemann Sum, then find the exact value via a definite integral. We spell out this technique in the following Key Idea.

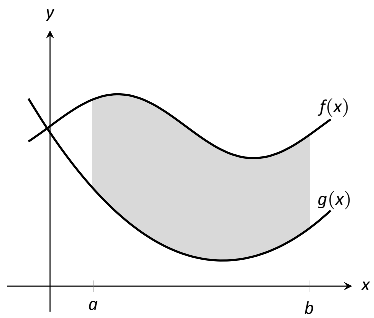
Key Idea 7.0.1 Application of Definite Integrals Strategy

Let a quantity be given whose value Q is to be computed.

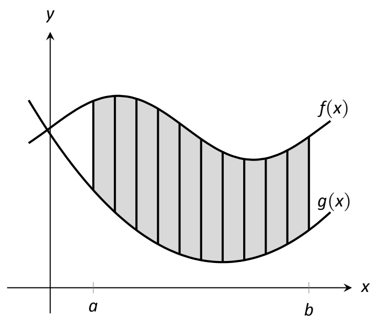
1. Divide the quantity into n smaller "subquantities" of value Q_i.
2. Identify a variable x and function $f(x)$ such that each subquantity can be approximated with the product $f(c_i)\Delta x$, where Δx represents a small change in x. Thus $Q_i \approx f(c_i)\Delta x$. A sample approximation $f(c_i)\Delta x$ of Q_i is called a *differential element*.
3. Recognize that $Q = \displaystyle\sum_{i=1}^{n} Q_i \approx \sum_{i=1}^{n} f(c_i)\Delta x$, which is a Riemann Sum.
4. Taking the appropriate limit gives $Q = \displaystyle\int_a^b f(x)\,dx$

This Key Idea will make more sense after we have had a chance to use it several times. We begin with Area Between Curves, which we addressed briefly in Section 5.4.

7.1 Area Between Curves



(a)



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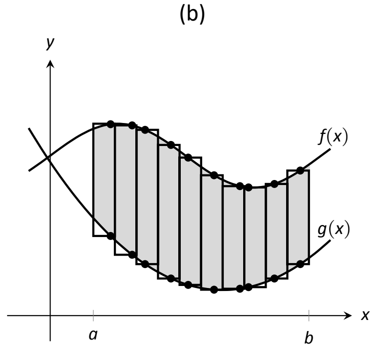


(c)

Figure 7.1.1: Subdividing a region into vertical slices and approximating the areas with rectangles.

We are often interested in knowing the area of a region. Forget momentarily that we addressed this already in Section 5.4 and approach it instead using the technique described in Key Idea 7.0.1.

Let Q be the area of a region bounded by continuous functions f and g. If we break the region into many subregions, we have an obvious equation:

Total Area = sum of the areas of the subregions.

The issue to address next is how to systematically break a region into subregions. A graph will help. Consider Figure 7.1.1 (a) where a region between two curves is shaded. While there are many ways to break this into subregions, one particularly efficient way is to "slice" it vertically, as shown in Figure 7.1.1 (b), into n equally spaced slices.

We now approximate the area of a slice. Again, we have many options, but using a rectangle seems simplest. Picking any x-value c_i in the i^{th} slice, we set the height of the rectangle to be $f(c_i) - g(c_i)$, the difference of the corresponding y-values. The width of the rectangle is a small difference in x-values, which we represent with Δx. Figure 7.1.1 (c) shows sample points c_i chosen in each subinterval and appropriate rectangles drawn. (Each of these rectangles represents a differential element.) Each slice has an area approximately equal to $\big(f(c_i) - g(c_i)\big)\Delta x$; hence, the total area is approximately the Riemann Sum

$$Q = \sum_{i=1}^{n} \big(f(c_i) - g(c_i)\big)\Delta x.$$

Taking the limit as $n \to \infty$ gives the exact area as $\int_a^b \big(f(x) - g(x)\big)\,dx$.

Theorem 7.1.1 Area Between Curves (restatement of Theorem 5.4.3)

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

Example 7.1.1 Finding area enclosed by curves

Find the area of the region bounded by $f(x) = \sin x + 2$, $g(x) = \frac{1}{2}\cos(2x) - 1$, $x = 0$ and $x = 4\pi$, as shown in Figure 7.1.2.

Notes:

Solution The graph verifies that the upper boundary of the region is given by f and the lower bound is given by g. Therefore the area of the region is the value of the integral

$$\begin{aligned}\int_0^{4\pi}\big(f(x)-g(x)\big)\,dx &= \int_0^{4\pi}\Big(\sin x+2-\big(\tfrac{1}{2}\cos(2x)-1\big)\Big)\,dx\\ &= -\cos x-\frac{1}{4}\sin(2x)+3x\Big|_0^{4\pi}\\ &= 12\pi\approx 37.7\text{ units}^2.\end{aligned}$$

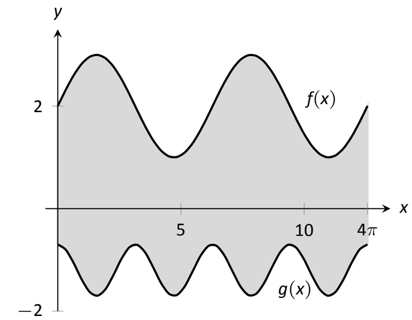


Figure 7.1.2: Graphing an enclosed region in Example 7.1.1.

Example 7.1.2 Finding total area enclosed by curves

Find the total area of the region enclosed by the functions $f(x)=-2x+5$ and $g(x)=x^3-7x^2+12x-3$ as shown in Figure 7.1.3.

Solution A quick calculation shows that $f=g$ at $x=1,2$ and 4. One can proceed thoughtlessly by computing $\displaystyle\int_1^4\big(f(x)-g(x)\big)\,dx$, but this ignores the fact that on $[1,2]$, $g(x)>f(x)$. (In fact, the thoughtless integration returns $-9/4$, hardly the expected value of an *area*.) Thus we compute the total area by breaking the interval $[1,4]$ into two subintervals, $[1,2]$ and $[2,4]$ and using the proper integrand in each.

$$\begin{aligned}\text{Total Area} &= \int_1^2\big(g(x)-f(x)\big)\,dx+\int_2^4\big(f(x)-g(x)\big)\,dx\\ &= \int_1^2\big(x^3-7x^2+14x-8\big)\,dx+\int_2^4\big(-x^3+7x^2-14x+8\big)\,dx\\ &= 5/12+8/3\\ &= 37/12=3.083\text{ units}^2.\end{aligned}$$

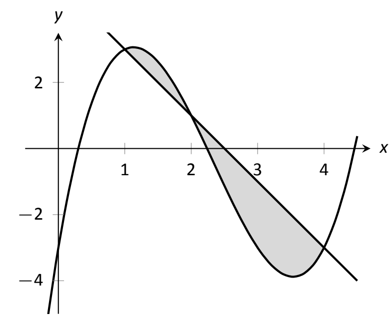


Figure 7.1.3: Graphing a region enclosed by two functions in Example 7.1.2.

The previous example makes note that we are expecting area to be *positive*. When first learning about the definite integral, we interpreted it as "signed area under the curve," allowing for "negative area." That doesn't apply here; area is to be positive.

The previous example also demonstrates that we often have to break a given region into subregions before applying Theorem 7.1.1. The following example shows another situation where this is applicable, along with an alternate view of applying the Theorem.

Example 7.1.3 Finding area: integrating with respect to *y*

Find the area of the region enclosed by the functions $y=\sqrt{x}+2$, $y=-(x-1)^2+3$ and $y=2$, as shown in Figure 7.1.4.

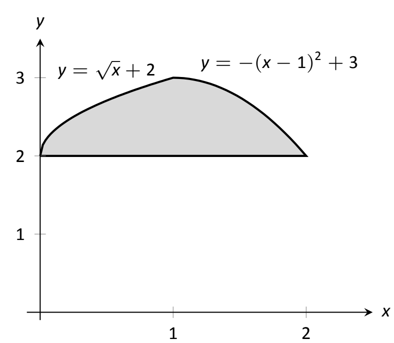


Figure 7.1.4: Graphing a region for Example 7.1.3.

Notes:

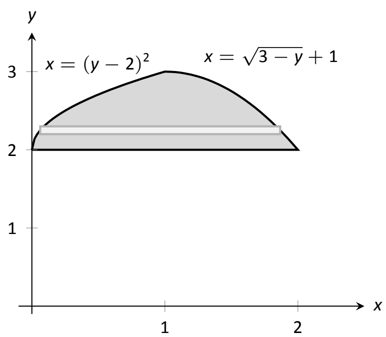


Figure 7.1.5: The region used in Example 7.1.3 with boundaries relabeled as functions of y.

Solution We give two approaches to this problem. In the first approach, we notice that the region's "top" is defined by two different curves. On $[0,1]$, the top function is $y=\sqrt{x}+2$; on $[1,2]$, the top function is $y=-(x-1)^2+3$. Thus we compute the area as the sum of two integrals:

$$\begin{aligned}\text{Total Area} &= \int_0^1 \Big(\big(\sqrt{x}+2\big)-2\Big)\,dx + \int_1^2 \Big(\big(-(x-1)^2+3\big)-2\Big)\,dx\\ &= 2/3+2/3\\ &= 4/3.\end{aligned}$$

The second approach is clever and very useful in certain situations. We are used to viewing curves as functions of x; we input an x-value and a y-value is returned. Some curves can also be described as functions of y: input a y-value and an x-value is returned. We can rewrite the equations describing the boundary by solving for x:

$$y=\sqrt{x}+2 \quad\Rightarrow\quad x=(y-2)^2$$

$$y=-(x-1)^2+3 \quad\Rightarrow\quad x=\sqrt{3-y}+1.$$

Figure 7.1.5 shows the region with the boundaries relabeled. A differential element, a horizontal rectangle, is also pictured. The width of the rectangle is a small change in y: Δy. The height of the rectangle is a difference in x-values. The "top" x-value is the largest value, i.e., the rightmost. The "bottom" x-value is the smaller, i.e., the leftmost. Therefore the height of the rectangle is

$$\big(\sqrt{3-y}+1\big)-(y-2)^2.$$

The area is found by integrating the above function with respect to y with the appropriate bounds. We determine these by considering the y-values the region occupies. It is bounded below by $y=2$, and bounded above by $y=3$. That is, both the "top" and "bottom" functions exist on the y interval $[2,3]$. Thus

$$\begin{aligned}\text{Total Area} &= \int_2^3 \big(\sqrt{3-y}+1-(y-2)^2\big)\,dy\\ &= \left(-\frac{2}{3}(3-y)^{3/2}+y-\frac{1}{3}(y-2)^3\right)\Bigg|_2^3\\ &= 4/3.\end{aligned}$$

This calculus–based technique of finding area can be useful even with shapes that we normally think of as "easy." Example 7.1.4 computes the area of a triangle. While the formula "$\frac{1}{2}\times$ base $\times$ height" is well known, in arbitrary triangles it can be nontrivial to compute the height. Calculus makes the problem simple.

Notes:

Example 7.1.4 Finding the area of a triangle

Compute the area of the regions bounded by the lines $y = x + 1$, $y = -2x + 7$ and $y = -\frac{1}{2}x + \frac{5}{2}$, as shown in Figure 7.1.6.

Solution Recognize that there are two "top" functions to this region, causing us to use two definite integrals.

$$\begin{aligned}\text{Total Area} &= \int_1^2 \big((x+1) - (-\frac{1}{2}x + \frac{5}{2})\big)\, dx + \int_2^3 \big((-2x+7) - (-\frac{1}{2}x + \frac{5}{2})\big)\, dx \\ &= 3/4 + 3/4 \\ &= 3/2.\end{aligned}$$

We can also approach this by converting each function into a function of y. This also requires 2 integrals, so there isn't really any advantage to doing so. We do it here for demonstration purposes.

The "top" function is always $x = \frac{7-y}{2}$ while there are two "bottom" functions. Being mindful of the proper integration bounds, we have

$$\begin{aligned}\text{Total Area} &= \int_1^2 \big(\frac{7-y}{2} - (5-2y)\big)\, dy + \int_2^3 \big(\frac{7-y}{2} - (y-1)\big)\, dy \\ &= 3/4 + 3/4 \\ &= 3/2.\end{aligned}$$

Of course, the final answer is the same. (It is interesting to note that the area of all 4 subregions used is 3/4. This is coincidental.)

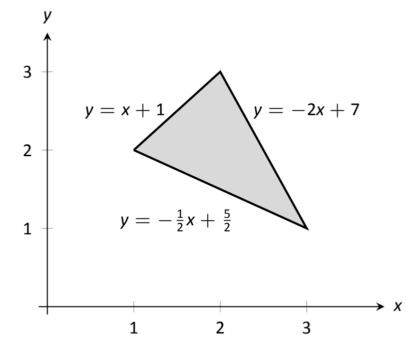


Figure 7.1.6: Graphing a triangular region in Example 7.1.4.

While we have focused on producing exact answers, we are also able to make approximations using the principle of Theorem 7.1.1. The integrand in the theorem is a distance ("top minus bottom"); integrating this distance function gives an area. By taking discrete measurements of distance, we can approximate an area using numerical integration techniques developed in Section 5.5. The following example demonstrates this.

Example 7.1.5 Numerically approximating area

To approximate the area of a lake, shown in Figure 7.1.7 (a), the "length" of the lake is measured at 200-foot increments as shown in Figure 7.1.7 (b), where the lengths are given in hundreds of feet. Approximate the area of the lake.

Solution The measurements of length can be viewed as measuring "top minus bottom" of two functions. The exact answer is found by integrating $\int_0^{12} \big(f(x) - g(x)\big)\, dx$, but of course we don't know the functions f and g. Our discrete measurements instead allow us to approximate.

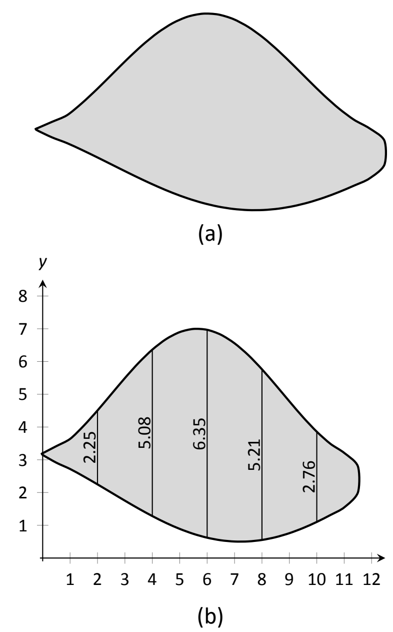


Figure 7.1.7: (a) A sketch of a lake, and (b) the lake with length measurements.

Notes:

We have the following data points:

$$(0,0),\ (2,2.25),\ (4,5.08),\ (6,6.35),\ (8,5.21),\ (10,2.76),\ (12,0).$$

We also have that $\Delta x = \frac{b-a}{n} = 2$, so Simpson's Rule gives

$$\begin{aligned}\text{Area} &\approx \frac{2}{3}\Big(1\cdot 0 + 4\cdot 2.25 + 2\cdot 5.08 + 4\cdot 6.35 + 2\cdot 5.21 + 4\cdot 2.76 + 1\cdot 0\Big)\\ &= 44.01\overline{3}\ \text{units}^2.\end{aligned}$$

Since the measurements are in hundreds of feet, $\text{units}^2 = (100\ \text{ft})^2 = 10,000\ \text{ft}^2$, giving a total area of $440,133\ \text{ft}^2$. (Since we are approximating, we'd likely say the area was about $440,000\ \text{ft}^2$, which is a little more than 10 acres.)

In the next section we apply our applications–of–integration techniques to finding the volumes of certain solids.

Notes:

Exercises 7.1

Terms and Concepts

1. T/F: The area between curves is always positive.

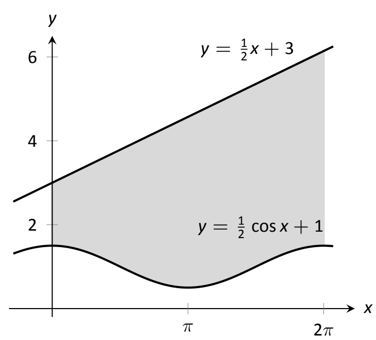
2. T/F: Calculus can be used to find the area of basic geometric shapes.

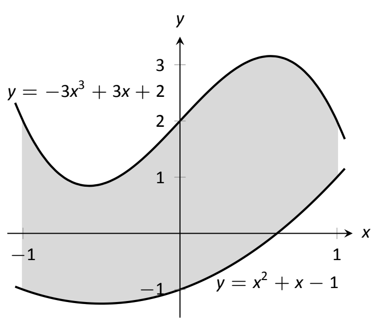
3. In your own words, describe how to find the total area enclosed by $y = f(x)$ and $y = g(x)$.

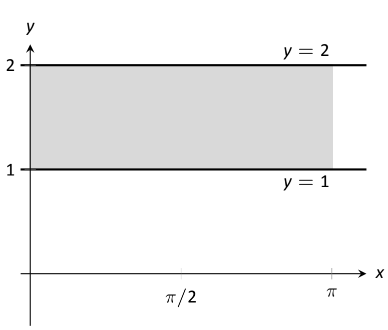
4. Describe a situation where it is advantageous to find an area enclosed by curves through integration with respect to y instead of x.

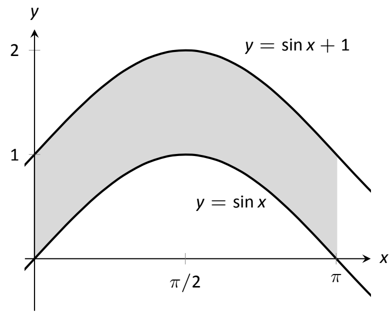
Problems

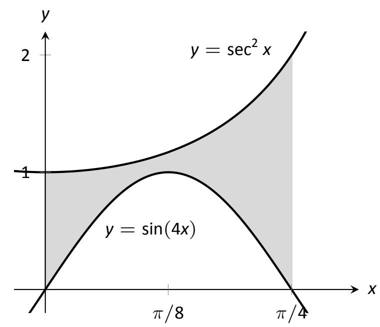
In Exercises 5 – 12, find the area of the shaded region in the given graph.

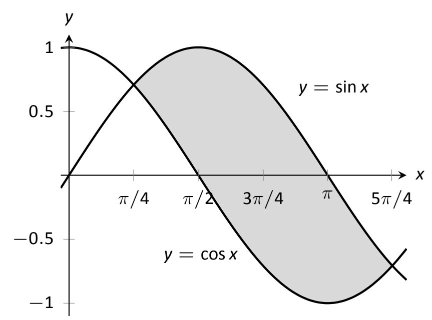
5. 

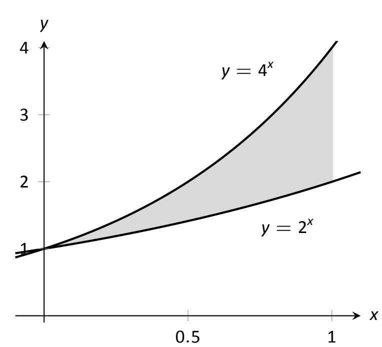
6. 

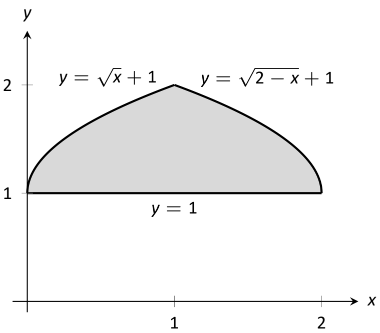
7. 

8. 

9. 

10. 

11. 

12. 

In Exercises 13 – 20, find the total area enclosed by the functions f and g.

13. $f(x) = 2x^2 + 5x - 3$, $g(x) = x^2 + 4x - 1$

14. $f(x) = x^2 - 3x + 2$, $g(x) = -3x + 3$

15. $f(x) = \sin x$, $g(x) = 2x/\pi$

16. $f(x) = x^3 - 4x^2 + x - 1$, $g(x) = -x^2 + 2x - 4$

17. $f(x) = x$, $g(x) = \sqrt{x}$

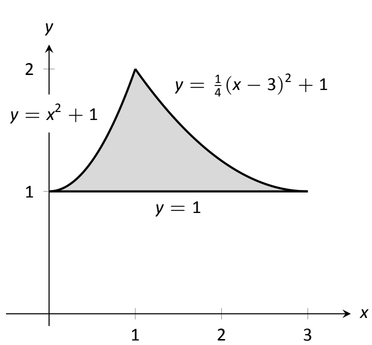
18. $f(x) = -x^3 + 5x^2 + 2x + 1$, $g(x) = 3x^2 + x + 3$

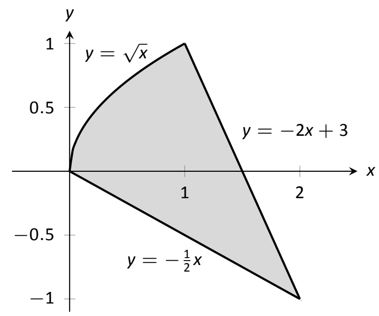
19. The functions $f(x) = \cos(x)$ and $g(x) = \sin x$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

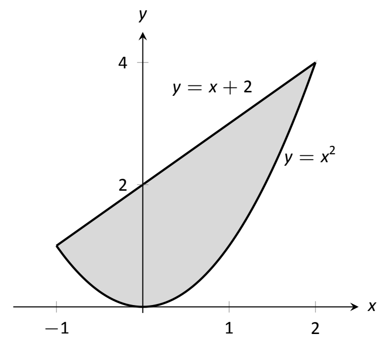
20. The functions $f(x) = \cos(2x)$ and $g(x) = \sin x$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

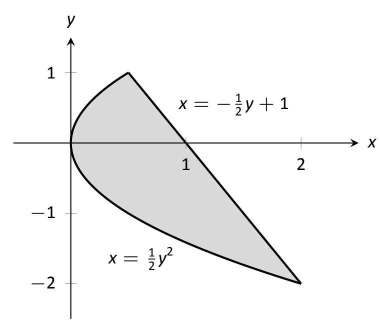
In Exercises 21 – 26, find the area of the enclosed region in two ways:

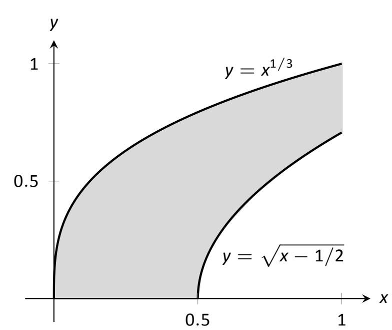
1. **by treating the boundaries as functions of x, and**
2. **by treating the boundaries as functions of y.**

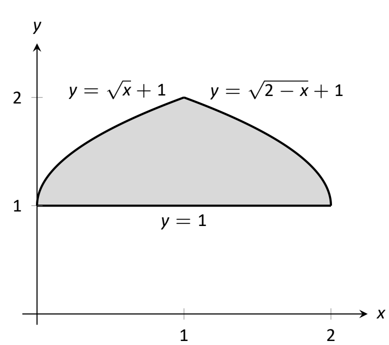
21. 

22. 

23. 

24. 

25. 

26. 

In Exercises 27 – 30, find the area triangle formed by the given three points.

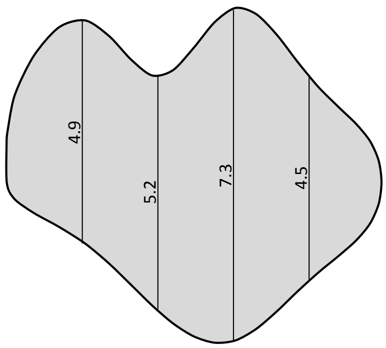
27. $(1,1)$, $(2,3)$, and $(3,3)$

28. $(-1,1)$, $(1,3)$, and $(2,-1)$

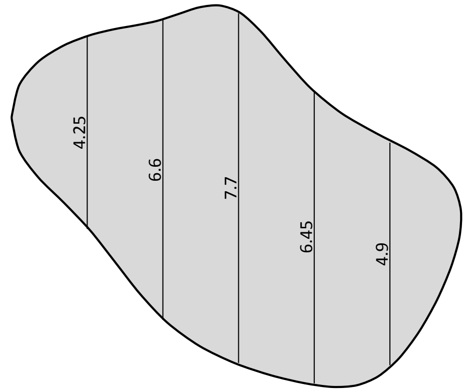
29. $(1,1)$, $(3,3)$, and $(3,3)$

30. $(0,0)$, $(2,5)$, and $(5,2)$

31. Use the Trapezoidal Rule to approximate the area of the pictured lake whose lengths, in hundreds of feet, are measured in 100-foot increments.



32. Use Simpson's Rule to approximate the area of the pictured lake whose lengths, in hundreds of feet, are measured in 200-foot increments.



7.2 Volume by Cross-Sectional Area; Disk and Washer Methods

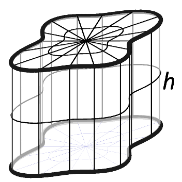


Figure 7.2.1: The volume of a general right cylinder

The volume of a general right cylinder, as shown in Figure 7.2.1, is

Area of the base $\times$ height.

We can use this fact as the building block in finding volumes of a variety of shapes.

Given an arbitrary solid, we can *approximate* its volume by cutting it into n thin slices. When the slices are thin, each slice can be approximated well by a general right cylinder. Thus the volume of each slice is approximately its cross-sectional area $\times$ thickness. (These slices are the differential elements.)

By orienting a solid along the x-axis, we can let $A(x_i)$ represent the cross-sectional area of the i^{th} slice, and let Δx_i represent the thickness of this slice (the thickness is a small change in x). The total volume of the solid is approximately:

$$\begin{aligned}\text{Volume} &\approx \sum_{i=1}^{n}\Big[\text{Area} \times \text{thickness}\Big]\\ &= \sum_{i=1}^{n} A(x_i)\Delta x_i.\end{aligned}$$

Recognize that this is a Riemann Sum. By taking a limit (as the thickness of the slices goes to 0) we can find the volume exactly.

Theorem 7.2.1 Volume By Cross-Sectional Area

The volume V of a solid, oriented along the x-axis with cross-sectional area $A(x)$ from $x = a$ to $x = b$, is

$$V = \int_a^b A(x)\,dx.$$

Example 7.2.1 Finding the volume of a solid

Find the volume of a pyramid with a square base of side length 10 in and a height of 5 in.

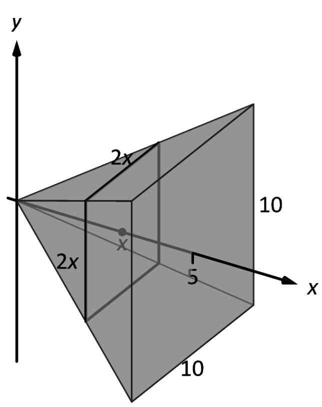


Figure 7.2.2: Orienting a pyramid along the x-axis in Example 7.2.1.

Solution There are many ways to "orient" the pyramid along the x-axis; Figure 7.2.2 gives one such way, with the pointed top of the pyramid at the origin and the x-axis going through the center of the base.

Each cross section of the pyramid is a square; this is a sample differential element. To determine its area $A(x)$, we need to determine the side lengths of

Notes:

the square.

When $x = 5$, the square has side length 10; when $x = 0$, the square has side length 0. Since the edges of the pyramid are lines, it is easy to figure that each cross-sectional square has side length $2x$, giving $A(x) = (2x)^2 = 4x^2$.

If one were to cut a slice out of the pyramid at $x = 3$, as shown in Figure 7.2.3, one would have a shape with square bottom and top with sloped sides. If the slice were thin, both the bottom and top squares would have sides lengths of about 6, and thus the cross–sectional area of the bottom and top would be about 36in^2. Letting Δx_i represent the thickness of the slice, the volume of this slice would then be about $36\Delta x_i$in^3.

Cutting the pyramid into n slices divides the total volume into n equally–spaced smaller pieces, each with volume $(2x_i)^2\Delta x$, where x_i is the approximate location of the slice along the x-axis and Δx represents the thickness of each slice. One can approximate total volume of the pyramid by summing up the volumes of these slices:

$$\text{Approximate volume } = \sum_{i=1}^{n}(2x_i)^2\Delta x.$$

Taking the limit as $n \to \infty$ gives the actual volume of the pyramid; recoginizing this sum as a Riemann Sum allows us to find the exact answer using a definite integral, matching the definite integral given by Theorem 7.2.1.

We have

$$\begin{aligned} V &= \lim_{n\to\infty}\sum_{i=1}^{n}(2x_i)^2\Delta x \\ &= \int_0^5 4x^2\,dx \\ &= \frac{4}{3}x^3\Big|_0^5 \\ &= \frac{500}{3}\text{ in}^3 \approx 166.67\text{ in}^3. \end{aligned}$$

We can check our work by consulting the general equation for the volume of a pyramid (see the back cover under "Volume of A General Cone"):

$$\tfrac{1}{3} \times \text{area of base} \times \text{height}.$$

Certainly, using this formula from geometry is faster than our new method, but the calculus–based method can be applied to much more than just cones.

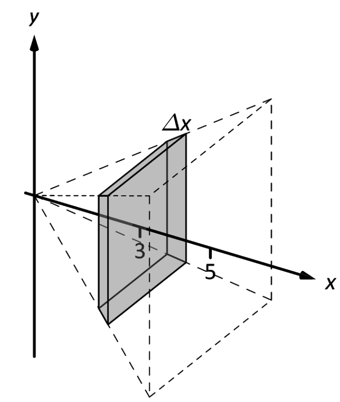


Figure 7.2.3: Cutting a slice in the pyramid in Example 7.2.1 at $x = 3$.

An important special case of Theorem 7.2.1 is when the solid is a **solid of revolution**, that is, when the solid is formed by rotating a shape around an axis.

Start with a function $y = f(x)$ from $x = a$ to $x = b$. Revolving this curve about a horizontal axis creates a three-dimensional solid whose cross sections

Notes:

are disks (thin circles). Let $R(x)$ represent the radius of the cross-sectional disk at x; the area of this disk is $\pi R(x)^2$. Applying Theorem 7.2.1 gives the Disk Method.

Key Idea 7.2.1 The Disk Method

Let a solid be formed by revolving the curve $y = f(x)$ from $x = a$ to $x = b$ around a horizontal axis, and let $R(x)$ be the radius of the cross-sectional disk at x. The volume of the solid is

$$V = \pi \int_a^b R(x)^2\,dx.$$

Example 7.2.2 Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, around the x-axis.

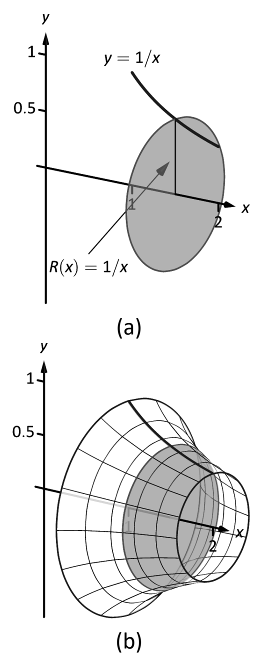


Figure 7.2.4: Sketching a solid in Example 7.2.2.

Solution A sketch can help us understand this problem. In Figure 7.2.4(a) the curve $y = 1/x$ is sketched along with the differential element – a disk – at x with radius $R(x) = 1/x$. In Figure 7.2.4 (b) the whole solid is pictured, along with the differential element.

The volume of the differential element shown in part (a) of the figure is approximately $\pi R(x_i)^2\Delta x$, where $R(x_i)$ is the radius of the disk shown and Δx is the thickness of that slice. The radius $R(x_i)$ is the distance from the x-axis to the curve, hence $R(x_i) = 1/x_i$.

Slicing the solid into n equally–spaced slices, we can approximate the total volume by adding up the approximate volume of each slice:

$$\text{Approximate volume } = \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x.$$

Taking the limit of the above sum as $n \to \infty$ gives the actual volume; recognizing this sum as a Riemann sum allows us to evaluate the limit with a definite integral, which matches the formula given in Key Idea 7.2.1:

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x \\ &= \pi \int_1^2 \left(\frac{1}{x}\right)^2 dx \\ &= \pi \int_1^2 \frac{1}{x^2}\,dx \end{aligned}$$

Notes:

$$= \pi \left[-\frac{1}{x}\right]\Big|_1^2$$
$$= \pi \left[-\frac{1}{2} - (-1)\right]$$
$$= \frac{\pi}{2} \text{ units}^3.$$

While Key Idea 7.2.1 is given in terms of functions of x, the principle involved can be applied to functions of y when the axis of rotation is vertical, not horizontal. We demonstrate this in the next example.

Example 7.2.3 Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, about the y-axis.

SOLUTION Since the axis of rotation is vertical, we need to convert the function into a function of y and convert the x-bounds to y-bounds. Since $y = 1/x$ defines the curve, we rewrite it as $x = 1/y$. The bound $x = 1$ corresponds to the y-bound $y = 1$, and the bound $x = 2$ corresponds to the y-bound $y = 1/2$.

Thus we are rotating the curve $x = 1/y$, from $y = 1/2$ to $y = 1$ about the y-axis to form a solid. The curve and sample differential element are sketched in Figure 7.2.5 (a), with a full sketch of the solid in Figure 7.2.5 (b). We integrate to find the volume:

$$V = \pi \int_{1/2}^{1} \frac{1}{y^2}\, dy$$
$$= -\frac{\pi}{y}\Big|_{1/2}^{1}$$
$$= \pi \text{ units}^3.$$

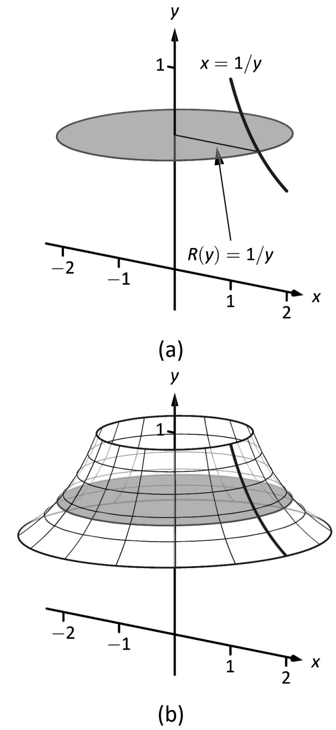


Figure 7.2.5: Sketching a solid in Example 7.2.3.

We can also compute the volume of solids of revolution that have a hole in the center. The general principle is simple: compute the volume of the solid irrespective of the hole, then subtract the volume of the hole. If the outside radius of the solid is $R(x)$ and the inside radius (defining the hole) is $r(x)$, then the volume is

$$V = \pi \int_a^b R(x)^2\, dx - \pi \int_a^b r(x)^2\, dx = \pi \int_a^b \left(R(x)^2 - r(x)^2\right)\, dx.$$

One can generate a solid of revolution with a hole in the middle by revolving a region about an axis. Consider Figure 7.2.6(a), where a region is sketched along

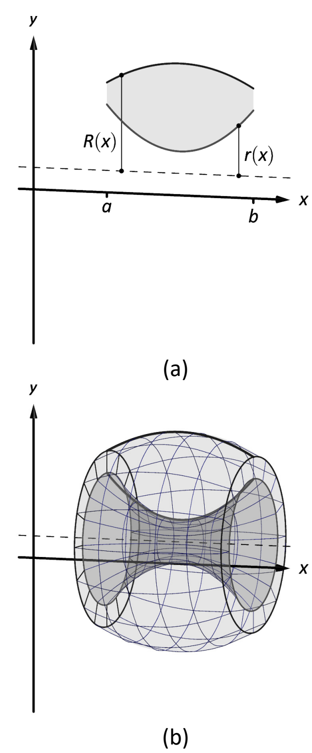


Figure 7.2.6: Establishing the Washer Method; see also Figure 7.2.7.

Notes:

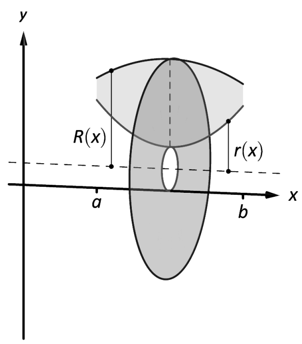


Figure 7.2.7: Establishing the Washer Method; see also Figure 7.2.6.

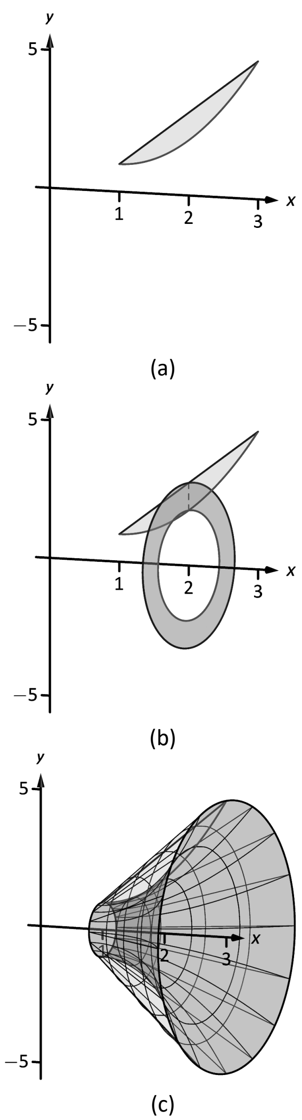


Figure 7.2.8: Sketching the differential element and solid in Example 7.2.4.

with a dashed, horizontal axis of rotation. By rotating the region about the axis, a solid is formed as sketched in Figure 7.2.6(b). The outside of the solid has radius $R(x)$, whereas the inside has radius $r(x)$. Each cross section of this solid will be a washer (a disk with a hole in the center) as sketched in Figure 7.2.7. This leads us to the Washer Method.

Key Idea 7.2.2 The Washer Method

Let a region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ be rotated about a horizontal axis that does not intersect the region, forming a solid. Each cross section at x will be a washer with outside radius $R(x)$ and inside radius $r(x)$. The volume of the solid is

$$V = \pi \int_a^b \Big(R(x)^2 - r(x)^2\Big)\,dx.$$

Even though we introduced it first, the Disk Method is just a special case of the Washer Method with an inside radius of $r(x) = 0$.

Example 7.2.4 Finding volume with the Washer Method

Find the volume of the solid formed by rotating the region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$ about the x-axis.

Solution A sketch of the region will help, as given in Figure 7.2.8(a). Rotating about the x-axis will produce cross sections in the shape of washers, as shown in Figure 7.2.8(b); the complete solid is shown in part (c). The outside radius of this washer is $R(x) = 2x + 1$; the inside radius is $r(x) = x^2 - 2x + 2$. As the region is bounded from $x = 1$ to $x = 3$, we integrate as follows to compute the volume.

$$\begin{aligned} V &= \pi \int_1^3 \Big((2x-1)^2 - (x^2-2x+2)^2\Big)\,dx \\ &= \pi \int_1^3 \Big(-x^4 + 4x^3 - 4x^2 + 4x - 3\Big)\,dx \\ &= \pi \Big[-\frac{1}{5}x^5 + x^4 - \frac{4}{3}x^3 + 2x^2 - 3x\Big]\Big|_1^3 \\ &= \frac{104}{15}\pi \approx 21.78 \text{ units}^3. \end{aligned}$$

When rotating about a vertical axis, the outside and inside radius functions must be functions of y.

Notes:

Example 7.2.5 Finding volume with the Washer Method

Find the volume of the solid formed by rotating the triangular region with vertices at $(1,1)$, $(2,1)$ and $(2,3)$ about the y-axis.

Solution The triangular region is sketched in Figure 7.2.9(a); the differential element is sketched in (b) and the full solid is drawn in (c). They help us establish the outside and inside radii. Since the axis of rotation is vertical, each radius is a function of y.

The outside radius $R(y)$ is formed by the line connecting $(2,1)$ and $(2,3)$; it is a constant function, as regardless of the y-value the distance from the line to the axis of rotation is 2. Thus $R(y) = 2$.

The inside radius is formed by the line connecting $(1,1)$ and $(2,3)$. The equation of this line is $y = 2x - 1$, but we need to refer to it as a function of y. Solving for x gives $r(y) = \frac{1}{2}(y+1)$.

We integrate over the y-bounds of $y = 1$ to $y = 3$. Thus the volume is

$$\begin{aligned}
V &= \pi \int_1^3 \left(2^2 - \left(\frac{1}{2}(y+1)\right)^2\right)\,dy \\
&= \pi \int_1^3 \left(-\frac{1}{4}y^2 - \frac{1}{2}y + \frac{15}{4}\right)\,dy \\
&= \pi\left[-\frac{1}{12}y^3 - \frac{1}{4}y^2 + \frac{15}{4}y\right]\Bigg|_1^3 \\
&= \frac{10}{3}\pi \approx 10.47 \text{ units}^3.
\end{aligned}$$

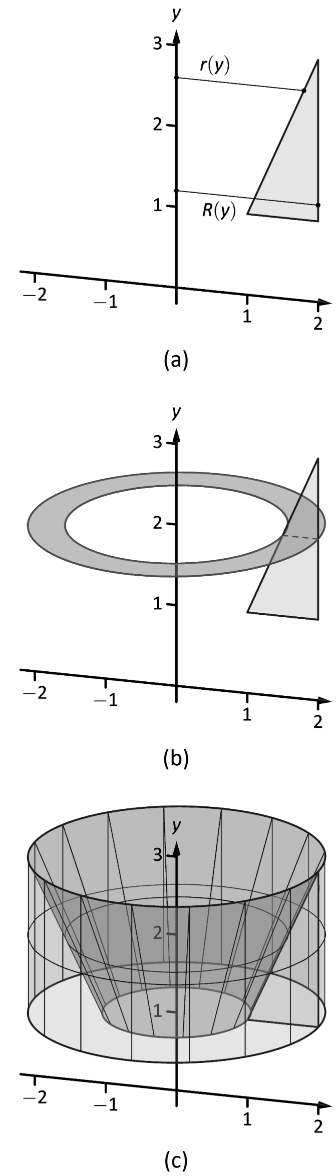


Figure 7.2.9: Sketching the solid in Example 7.2.5.

This section introduced a new application of the definite integral. Our default view of the definite integral is that it gives "the area under the curve." However, we can establish definite integrals that represent other quantities; in this section, we computed volume.

The ultimate goal of this section is not to compute volumes of solids. That can be useful, but what is more useful is the understanding of this basic principle of integral calculus, outlined in Key Idea 7.0.1: to find the exact value of some quantity,

- we start with an approximation (in this section, slice the solid and approximate the volume of each slice),
- then make the approximation better by refining our original approximation (i.e., use more slices),
- then use limits to establish a definite integral which gives the exact value.

We practice this principle in the next section where we find volumes by slicing solids in a different way.

Notes:

Exercises 7.2

Terms and Concepts

1. T/F: A solid of revolution is formed by revolving a shape around an axis.

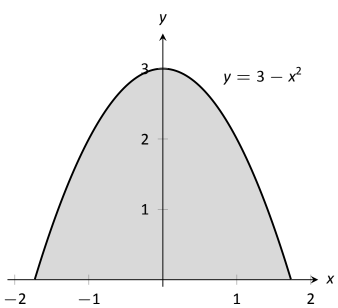
2. In your own words, explain how the Disk and Washer Methods are related.

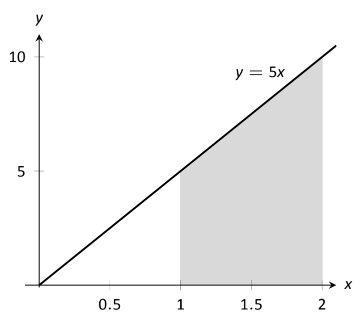
3. Explain the how the units of volume are found in the integral of Theorem 7.2.1: if $A(x)$ has units of in^2, how does $\int A(x)\, dx$ have units of in^3?

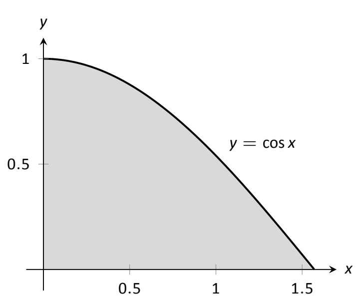
4. A fundamental principle of this section is "________ can be found by integrating an area function."

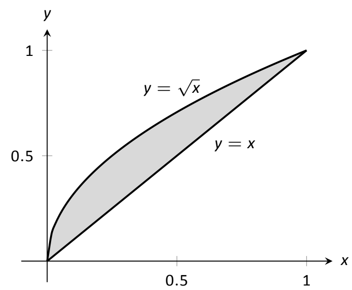
Problems

In Exercises 5 – 8, a region of the Cartesian plane is shaded. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the region about the *x*-axis.

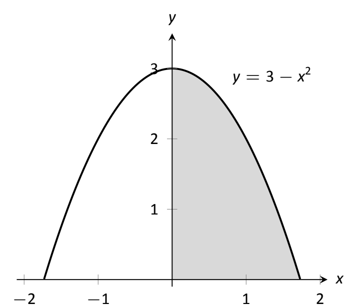
5. 

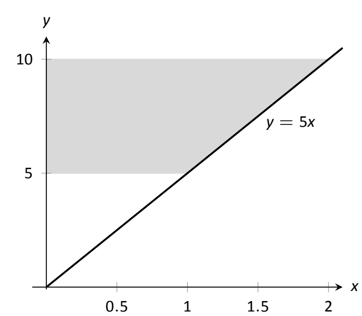
6. 

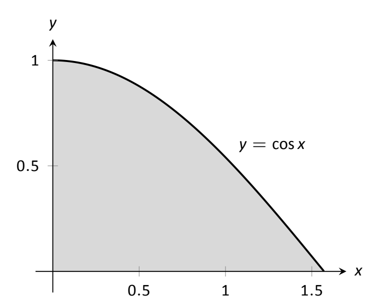
7. 

8. 

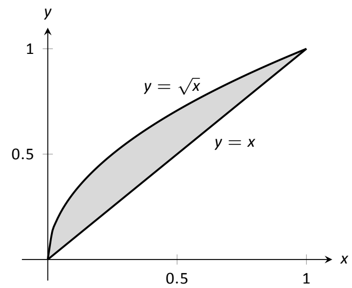
In Exercises 9 – 12, a region of the Cartesian plane is shaded. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the region about the *y*-axis.

9. 

10. 

11. 

(Hint: Integration By Parts will be necessary, twice. First let $u = \arccos^2 x$, then let $u = \arccos x$.)

12. 

In Exercises 13 – 18, a region of the Cartesian plane is described. Use the Disk/Washer Method to find the volume of the solid of revolution formed by rotating the region about each of the given axes.

13. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.
Rotate about:

    (a) the x-axis (c) the y-axis
    (b) $y = 1$ (d) $x = 1$

14. Region bounded by: $y = 4 - x^2$ and $y = 0$.
Rotate about:

    (a) the x-axis (c) $y = -1$
    (b) $y = 4$ (d) $x = 2$

15. The triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$.
Rotate about:

    (a) the x-axis (c) the y-axis
    (b) $y = 2$ (d) $x = 1$

16. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.
Rotate about:

    (a) the x-axis (c) $y = 5$
    (b) $y = 1$

17. Region bounded by $y = 1/\sqrt{x^2 + 1}$, $x = -1$, $x = 1$ and the x-axis.
Rotate about:

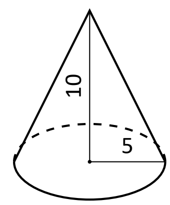
    (a) the x-axis (c) $y = -1$
    (b) $y = 1$

18. Region bounded by $y = 2x$, $y = x$ and $x = 2$.
Rotate about:

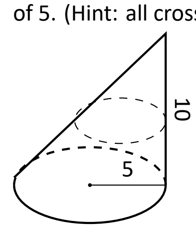
    (a) the x-axis (c) the y-axis
    (b) $y = 4$ (d) $x = 2$

In Exercises 19 – 22, a solid is described. Orient the solid along the x-axis such that a cross-sectional area function $A(x)$ can be obtained, then apply Theorem 7.2.1 to find the volume of the solid.

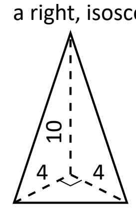
19. A right circular cone with height of 10 and base radius of 5.



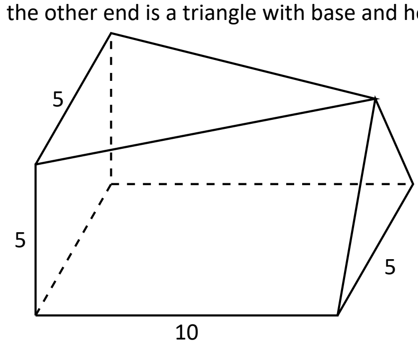
20. A skew right circular cone with height of 10 and base radius of 5. (Hint: all cross-sections are circles.)



21. A right triangular cone with height of 10 and whose base is a right, isosceles triangle with side length 4.



22. A solid with length 10 with a rectangular base and triangular top, wherein one end is a square with side length 5 and the other end is a triangle with base and height of 5.



7.3 The Shell Method

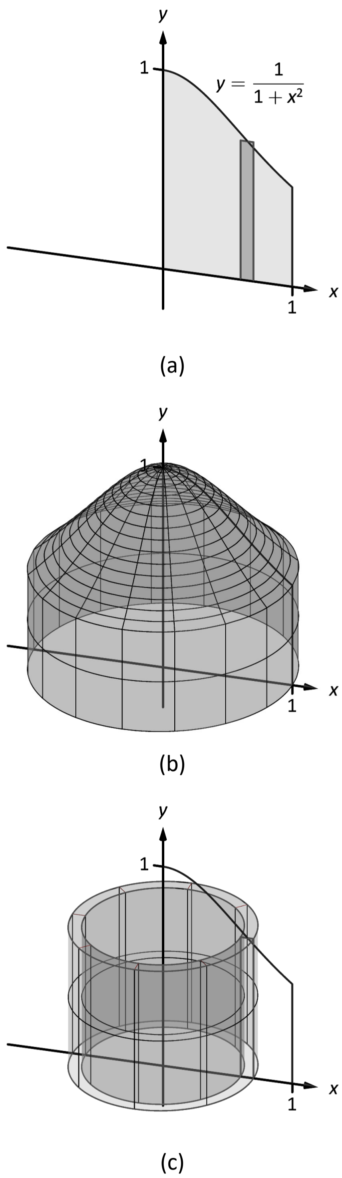


Figure 7.3.1: Introducing the Shell Method.

Often a given problem can be solved in more than one way. A particular method may be chosen out of convenience, personal preference, or perhaps necessity. Ultimately, it is good to have options.

The previous section introduced the Disk and Washer Methods, which computed the volume of solids of revolution by integrating the cross–sectional area of the solid. This section develops another method of computing volume, the **Shell Method.** Instead of slicing the solid perpendicular to the axis of rotation creating cross-sections, we now slice it parallel to the axis of rotation, creating "shells."

Consider Figure 7.3.1, where the region shown in (a) is rotated around the y-axis forming the solid shown in (b). A small slice of the region is drawn in (a), parallel to the axis of rotation. When the region is rotated, this thin slice forms a **cylindrical shell**, as pictured in part (c) of the figure. The previous section approximated a solid with lots of thin disks (or washers); we now approximate a solid with many thin cylindrical shells.

To compute the volume of one shell, first consider the paper label on a soup can with radius r and height h. What is the area of this label? A simple way of determining this is to cut the label and lay it out flat, forming a rectangle with height h and length $2\pi r$. Thus the area is $A = 2\pi rh$; see Figure 7.3.2(a).

Do a similar process with a cylindrical shell, with height h, thickness Δx, and approximate radius r. Cutting the shell and laying it flat forms a rectangular solid with length $2\pi r$, height h and depth Δx. Thus the volume is $V \approx 2\pi rh\Delta x$; see Figure 7.3.2(b). (We say "approximately" since our radius was an approximation.)

By breaking the solid into n cylindrical shells, we can approximate the volume of the solid as

$$V \approx \sum_{i=1}^{n} 2\pi r_i h_i \Delta x_i,$$

where r_i, h_i and Δx_i are the radius, height and thickness of the i^{th} shell, respectively.

This is a Riemann Sum. Taking a limit as the thickness of the shells approaches 0 leads to a definite integral.

Notes:

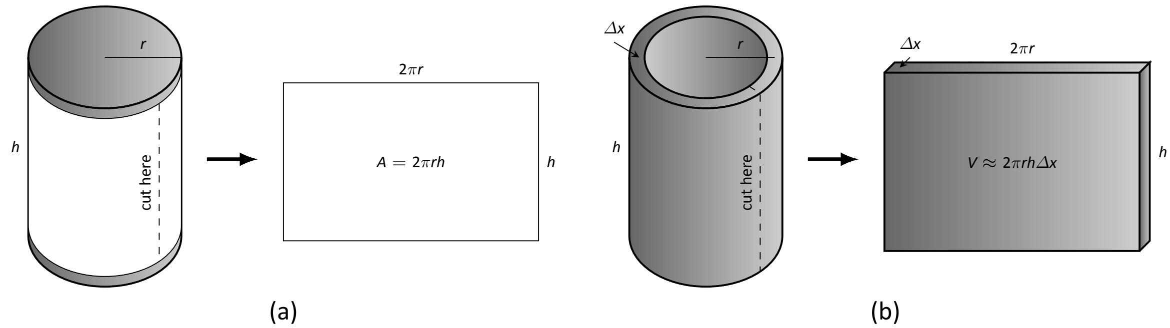


Figure 7.3.2: Determining the volume of a thin cylindrical shell.

Key Idea 7.3.1 The Shell Method

Let a solid be formed by revolving a region R, bounded by $x = a$ and $x = b$, around a vertical axis. Let $r(x)$ represent the distance from the axis of rotation to x (i.e., the radius of a sample shell) and let $h(x)$ represent the height of the solid at x (i.e., the height of the shell). The volume of the solid is

$$V = 2\pi \int_a^b r(x)h(x)\,dx.$$

Special Cases:

1. When the region R is bounded above by $y = f(x)$ and below by $y = g(x)$, then $h(x) = f(x) - g(x)$.
2. When the axis of rotation is the y-axis (i.e., $x = 0$) then $r(x) = x$.

Let's practice using the Shell Method.

Example 7.3.1 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the region bounded by $y = 0$, $y = 1/(1 + x^2)$, $x = 0$ and $x = 1$ about the y-axis.

SOLUTION This is the region used to introduce the Shell Method in Figure 7.3.1, but is sketched again in Figure 7.3.3 for closer reference. A line is drawn in the region parallel to the axis of rotation representing a shell that will

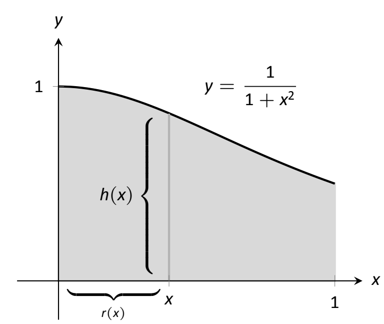


Figure 7.3.3: Graphing a region in Example 7.3.1.

Notes:

be carved out as the region is rotated about the y-axis. (This is the differential element.)

The distance this line is from the axis of rotation determines $r(x)$; as the distance from x to the y-axis is x, we have $r(x) = x$. The height of this line determines $h(x)$; the top of the line is at $y = 1/(1+x^2)$, whereas the bottom of the line is at $y = 0$. Thus $h(x) = 1/(1+x^2) - 0 = 1/(1+x^2)$. The region is bounded from $x = 0$ to $x = 1$, so the volume is

$$V = 2\pi \int_0^1 \frac{x}{1+x^2}\,dx.$$

This requires substitution. Let $u = 1 + x^2$, so $du = 2x\,dx$. We also change the bounds: $u(0) = 1$ and $u(1) = 2$. Thus we have:

$$\begin{aligned} &= \pi \int_1^2 \frac{1}{u}\,du \\ &= \pi \ln u \Big|_1^2 \\ &= \pi \ln 2 \approx 2.178 \text{ units}^3. \end{aligned}$$

Note: in order to find this volume using the Disk Method, two integrals would be needed to account for the regions above and below $y = 1/2$.

With the Shell Method, nothing special needs to be accounted for to compute the volume of a solid that has a hole in the middle, as demonstrated next.

Example 7.3.2 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the triangular region determined by the points $(0, 1)$, $(1, 1)$ and $(1, 3)$ about the line $x = 3$.

Solution The region is sketched in Figure 7.3.4(a) along with the differential element, a line within the region parallel to the axis of rotation. In part (b) of the figure, we see the shell traced out by the differential element, and in part (c) the whole solid is shown.

The height of the differential element is the distance from $y = 1$ to $y = 2x + 1$, the line that connects the points $(0, 1)$ and $(1, 3)$. Thus $h(x) = 2x+1-1 = 2x$. The radius of the shell formed by the differential element is the distance from x to $x = 3$; that is, it is $r(x) = 3 - x$. The x-bounds of the region are $x = 0$ to

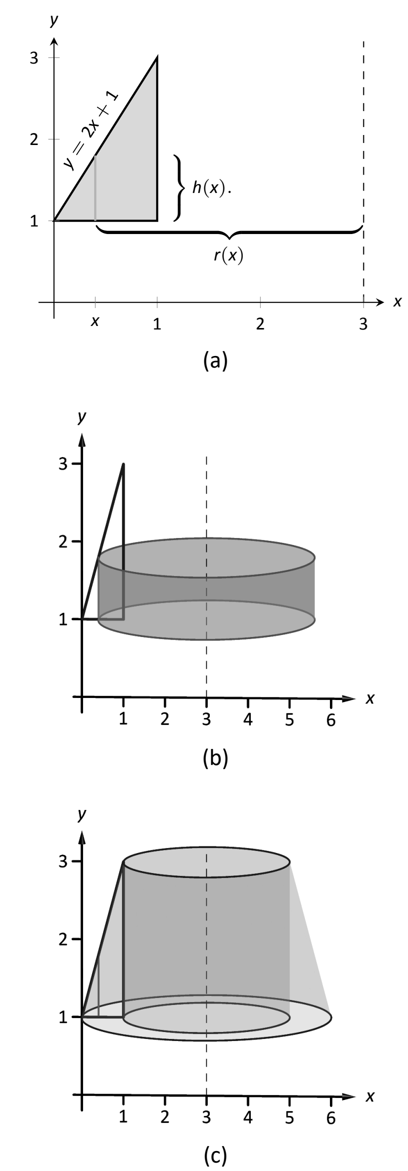


Figure 7.3.4: Graphing a region in Example 7.3.2.

Notes:

$x = 1$, giving

$$\begin{aligned} V &= 2\pi \int_0^1 (3-x)(2x)\,dx \\ &= 2\pi \int_0^1 \left(6x - 2x^2\right)\,dx \\ &= 2\pi \left(3x^2 - \frac{2}{3}x^3\right)\Big|_0^1 \\ &= \frac{14}{3}\pi \approx 14.66 \text{ units}^3. \end{aligned}$$

When revolving a region around a horizontal axis, we must consider the radius and height functions in terms of y, not x.

Example 7.3.3 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the region given in Example 7.3.2 about the x-axis.

Solution The region is sketched in Figure 7.3.5(a) with a sample differential element. In part (b) of the figure the shell formed by the differential element is drawn, and the solid is sketched in (c). (Note that the triangular region looks "short and wide" here, whereas in the previous example the same region looked "tall and narrow." This is because the bounds on the graphs are different.)

The height of the differential element is an x-distance, between $x = \frac{1}{2}y - \frac{1}{2}$ and $x = 1$. Thus $h(y) = 1 - (\frac{1}{2}y - \frac{1}{2}) = -\frac{1}{2}y + \frac{3}{2}$. The radius is the distance from y to the x-axis, so $r(y) = y$. The y bounds of the region are $y = 1$ and $y = 3$, leading to the integral

$$\begin{aligned} V &= 2\pi \int_1^3 \left[y\left(-\frac{1}{2}y + \frac{3}{2}\right)\right]\,dy \\ &= 2\pi \int_1^3 \left[-\frac{1}{2}y^2 + \frac{3}{2}y\right]\,dy \\ &= 2\pi \left[-\frac{1}{6}y^3 + \frac{3}{4}y^2\right]\Big|_1^3 \\ &= 2\pi \left[\frac{9}{4} - \frac{7}{12}\right] \\ &= \frac{10}{3}\pi \approx 10.472 \text{ units}^3. \end{aligned}$$

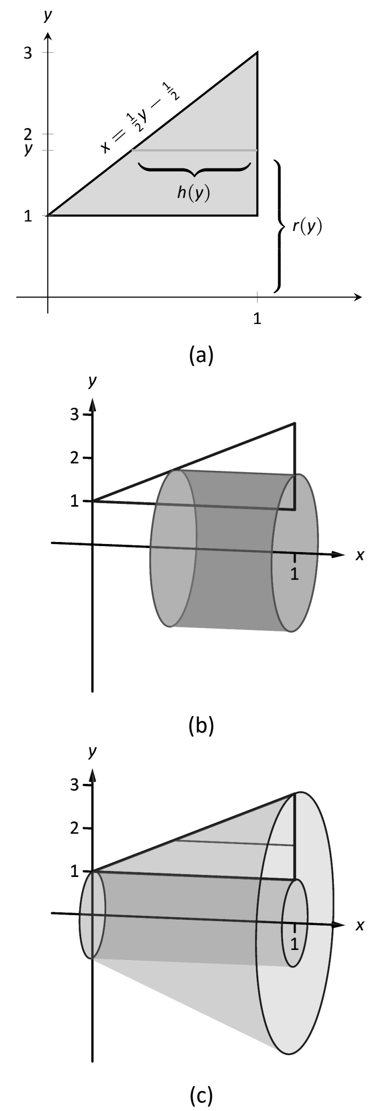


Figure 7.3.5: Graphing a region in Example 7.3.3.

Notes:

At the beginning of this section it was stated that "it is good to have options." The next example finds the volume of a solid rather easily with the Shell Method, but using the Washer Method would be quite a chore.

Example 7.3.4 Finding volume using the Shell Method

Find the volume of the solid formed by revolving the region bounded by $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ about the y-axis.

Solution The region and a differential element, the shell formed by this differential element, and the resulting solid are given in Figure 7.3.6. The radius of a sample shell is $r(x) = x$; the height of a sample shell is $h(x) = \sin x$, each from $x = 0$ to $x = \pi$. Thus the volume of the solid is

$$V = 2\pi \int_0^{\pi} x \sin x \, dx.$$

This requires Integration By Parts. Set $u = x$ and $dv = \sin x \, dx$; we leave it to the reader to fill in the rest. We have:

$$\begin{aligned}
&= 2\pi\Big[-x\cos x\Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx\Big] \\
&= 2\pi\Big[\pi + \sin x\Big|_0^{\pi}\Big] \\
&= 2\pi\Big[\pi + 0\Big] \\
&= 2\pi^2 \approx 19.74 \text{ units}^3.
\end{aligned}$$

Note that in order to use the Washer Method, we would need to solve $y = \sin x$ for x, requiring the use of the arcsine function. We leave it to the reader to verify that the outside radius function is $R(y) = \pi - \arcsin y$ and the inside radius function is $r(y) = \arcsin y$. Thus the volume can be computed as

$$\pi \int_0^1 \Big[(\pi - \arcsin y)^2 - (\arcsin y)^2\Big] \, dy.$$

This integral isn't terrible given that the $\arcsin^2 y$ terms cancel, but it is more onerous than the integral created by the Shell Method.

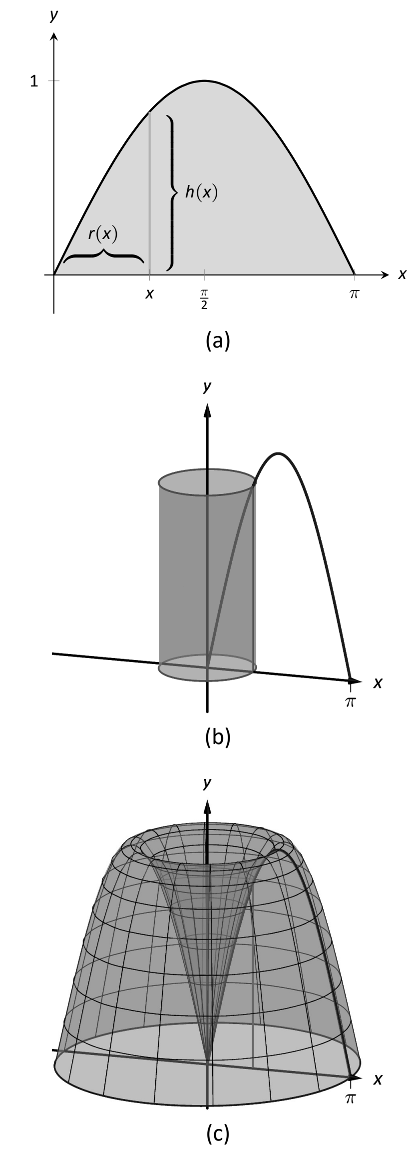


Figure 7.3.6: Graphing a region in Example 7.3.4.

We end this section with a table summarizing the usage of the Washer and Shell Methods.

Notes:

Key Idea 7.3.2 Summary of the Washer and Shell Methods

Let a region R be given with x-bounds $x = a$ and $x = b$ and y-bounds $y = c$ and $y = d$.

	Washer Method	Shell Method
Horizontal Axis	$\pi \int_a^b \left(R(x)^2 - r(x)^2\right)\, dx$	$2\pi \int_c^d r(y)h(y)\, dy$
Vertical Axis	$\pi \int_c^d \left(R(y)^2 - r(y)^2\right)\, dy$	$2\pi \int_a^b r(x)h(x)\, dx$

As in the previous section, the real goal of this section is not to be able to compute volumes of certain solids. Rather, it is to be able to solve a problem by first approximating, then using limits to refine the approximation to give the exact value. In this section, we approximate the volume of a solid by cutting it into thin cylindrical shells. By summing up the volumes of each shell, we get an approximation of the volume. By taking a limit as the number of equally spaced shells goes to infinity, our summation can be evaluated as a definite integral, giving the exact value.

We use this same principle again in the next section, where we find the length of curves in the plane.

Notes:

Exercises 7.3

Terms and Concepts

1. T/F: A solid of revolution is formed by revolving a shape around an axis.

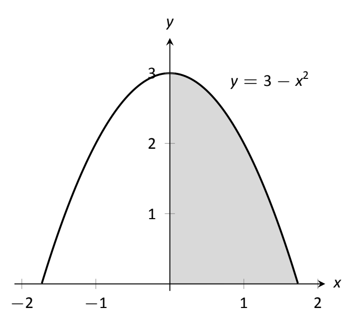
2. T/F: The Shell Method can only be used when the Washer Method fails.

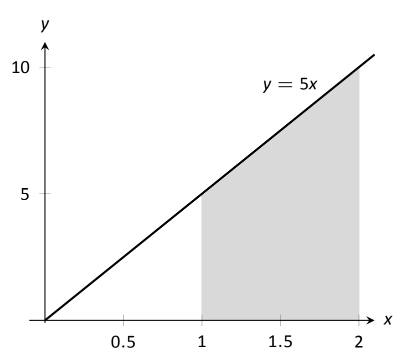
3. T/F: The Shell Method works by integrating cross–sectional areas of a solid.

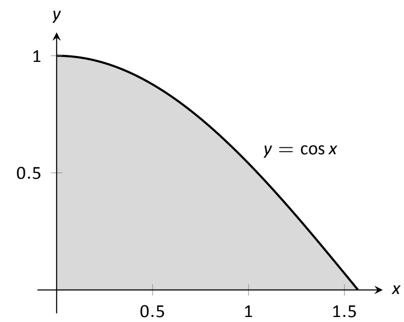
4. T/F: When finding the volume of a solid of revolution that was revolved around a vertical axis, the Shell Method integrates with respect to x.

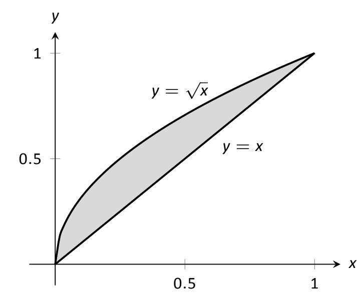
Problems

In Exercises 5 – 8, a region of the Cartesian plane is shaded. Use the Shell Method to find the volume of the solid of revolution formed by revolving the region about the y-axis.

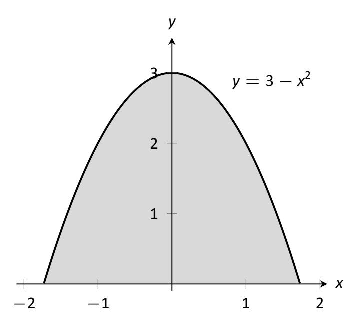
5. 

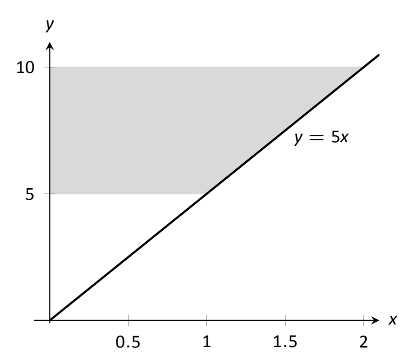
6. 

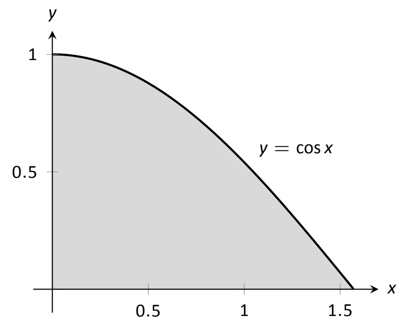
7. 

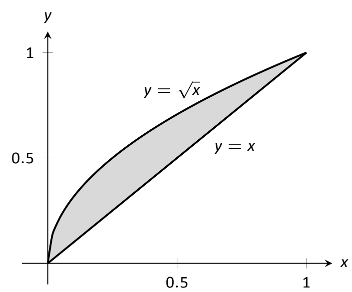
8. 

In Exercises 9 – 12, a region of the Cartesian plane is shaded. Use the Shell Method to find the volume of the solid of revolution formed by revolving the region about the x-axis.

9. 

10. 

11. 

12. 

In Exercises 13 – 18, a region of the Cartesian plane is described. Use the Shell Method to find the volume of the solid of revolution formed by rotating the region about each of the given axes.

13. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.
Rotate about:
(a) the y-axis
(b) $x = 1$
(c) the x-axis
(d) $y = 1$

14. Region bounded by: $y = 4 - x^2$ and $y = 0$.
Rotate about:
(a) $x = 2$
(b) $x = -2$
(c) the x-axis
(d) $y = 4$

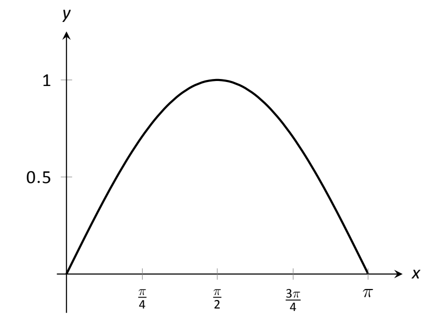
15. The triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$.
Rotate about:
(a) the y-axis
(b) $x = 1$
(c) the x-axis
(d) $y = 2$

16. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.
Rotate about:
(a) the y-axis
(b) $x = 1$
(c) $x = -1$

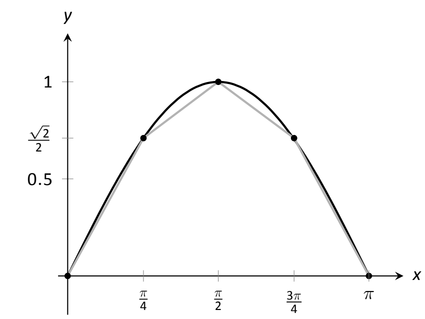
17. Region bounded by $y = 1/\sqrt{x^2 + 1}$, $x = 1$ and the x and y-axes.
Rotate about:
(a) the y-axis
(b) $x = 1$

18. Region bounded by $y = 2x$, $y = x$ and $x = 2$.
Rotate about:
(a) the y-axis
(b) $x = 2$
(c) the x-axis
(d) $y = 4$

7.4 Arc Length and Surface Area



(a)



(b)

Figure 7.4.1: Graphing $y = \sin x$ on $[0, \pi]$ and approximating the curve with line segments.

In previous sections we have used integration to answer the following questions:

1. Given a region, what is its area?
2. Given a solid, what is its volume?

In this section, we address a related question: Given a curve, what is its length? This is often referred to as **arc length**.

Consider the graph of $y = \sin x$ on $[0, \pi]$ given in Figure 7.4.1(a). How long is this curve? That is, if we were to use a piece of string to exactly match the shape of this curve, how long would the string be?

As we have done in the past, we start by approximating; later, we will refine our answer using limits to get an exact solution.

The length of straight–line segments is easy to compute using the Distance Formula. We can approximate the length of the given curve by approximating the curve with straight lines and measuring their lengths.

In Figure 7.4.1(b), the curve $y = \sin x$ has been approximated with 4 line segments (the interval $[0, \pi]$ has been divided into 4 equally–lengthed subintervals). It is clear that these four line segments approximate $y = \sin x$ very well on the first and last subinterval, though not so well in the middle. Regardless, the sum of the lengths of the line segments is 3.79, so we approximate the arc length of $y = \sin x$ on $[0, \pi]$ to be 3.79.

In general, we can approximate the arc length of $y = f(x)$ on $[a, b]$ in the following manner. Let $a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b$ be a partition of $[a, b]$ into n subintervals. Let Δx_i represent the length of the i^{th} subinterval $[x_i, x_{i+1}]$.

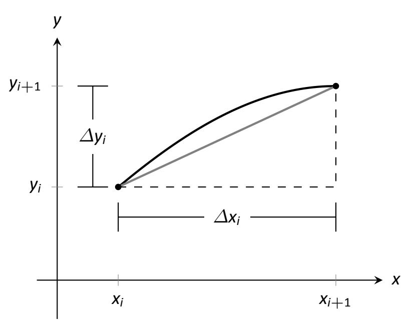


Figure 7.4.2: Zooming in on the i^{th} subinterval $[x_i, x_{i+1}]$ of a partition of $[a, b]$.

Figure 7.4.2 zooms in on the i^{th} subinterval where $y = f(x)$ is approximated by a straight line segment. The dashed lines show that we can view this line segment as the hypotenuse of a right triangle whose sides have length Δx_i and Δy_i. Using the Pythagorean Theorem, the length of this line segment is $\sqrt{\Delta x_i^2 + \Delta y_i^2}$. Summing over all subintervals gives an arc length approximation

$$L \approx \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}.$$

As shown here, this is *not* a Riemann Sum. While we could conclude that taking a limit as the subinterval length goes to zero gives the exact arc length, we would not be able to compute the answer with a definite integral. We need first to do a little algebra.

Notes:

In the above expression factor out a Δx_i^2 term:

$$\sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_{i=1}^{n} \sqrt{\Delta x_i^2 \left(1 + \frac{\Delta y_i^2}{\Delta x_i^2}\right)}.$$

Now pull the Δx_i^2 term out of the square root:

$$= \sum_{i=1}^{n} \sqrt{1 + \frac{\Delta y_i^2}{\Delta x_i^2}}\ \Delta x_i.$$

This is nearly a Riemann Sum. Consider the $\Delta y_i^2/\Delta x_i^2$ term. The expression $\Delta y_i/\Delta x_i$ measures the "change in y/change in x," that is, the "rise over run" of f on the i^{th} subinterval. The Mean Value Theorem of Differentiation (Theorem 3.2.1) states that there is a c_i in the i^{th} subinterval where $f'(c_i) = \Delta y_i/\Delta x_i$. Thus we can rewrite our above expression as:

$$= \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2}\ \Delta x_i.$$

This *is* a Riemann Sum. As long as f' is continuous, we can invoke Theorem 5.3.2 and conclude

$$= \int_a^b \sqrt{1 + f'(x)^2}\ dx.$$

Theorem 7.4.1 Arc Length

Let f be differentiable on $[a, b]$, where f' is also continuous on $[a, b]$. Then the arc length of f from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\ dx.$$

Note: This is our first use of differentiability on a closed interval since Section 2.1.

The theorem also requires that f' be continuous on $[a, b]$; while examples are arcane, it is possible for f to be differentiable yet f' is not continuous.

As the integrand contains a square root, it is often difficult to use the formula in Theorem 7.4.1 to find the length exactly. When exact answers are difficult to come by, we resort to using numerical methods of approximating definite integrals. The following examples will demonstrate this.

Notes:

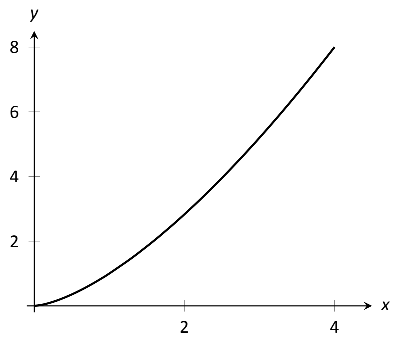


Figure 7.4.3: A graph of $f(x) = x^{3/2}$ from Example 7.4.1.

Example 7.4.1 Finding arc length

Find the arc length of $f(x) = x^{3/2}$ from $x = 0$ to $x = 4$.

Solution We find $f'(x) = \frac{3}{2}x^{1/2}$; note that on $[0, 4]$, f is differentiable and f' is also continuous. Using the formula, we find the arc length L as

$$\begin{aligned} L &= \int_0^4 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx \\ &= \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx \\ &= \int_0^4 \left(1+\frac{9}{4}x\right)^{1/2}\,dx \\ &= \frac{2}{3}\cdot\frac{4}{9}\cdot\left(1+\frac{9}{4}x\right)^{3/2}\Big|_0^4 \\ &= \frac{8}{27}\left(10^{3/2}-1\right) \approx 9.07\text{units}. \end{aligned}$$

A graph of f is given in Figure 7.4.3.

Example 7.4.2 Finding arc length

Find the arc length of $f(x) = \frac{1}{8}x^2 - \ln x$ from $x = 1$ to $x = 2$.

Solution This function was chosen specifically because the resulting integral can be evaluated exactly. We begin by finding $f'(x) = x/4 - 1/x$. The arc length is

$$\begin{aligned} L &= \int_1^2 \sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}\,dx \\ &= \int_1^2 \sqrt{1+\frac{x^2}{16}-\frac{1}{2}+\frac{1}{x^2}}\,dx \\ &= \int_1^2 \sqrt{\frac{x^2}{16}+\frac{1}{2}+\frac{1}{x^2}}\,dx \\ &= \int_1^2 \sqrt{\left(\frac{x}{4}+\frac{1}{x}\right)^2}\,dx \end{aligned}$$

Notes:

$$= \int_1^2 \left(\frac{x}{4} + \frac{1}{x}\right)\, dx$$
$$= \left(\frac{x^2}{8} + \ln x\right)\Bigg|_1^2$$
$$= \frac{3}{8} + \ln 2 \approx 1.07 \text{ units.}$$

A graph of f is given in Figure 7.4.4; the portion of the curve measured in this problem is in bold.

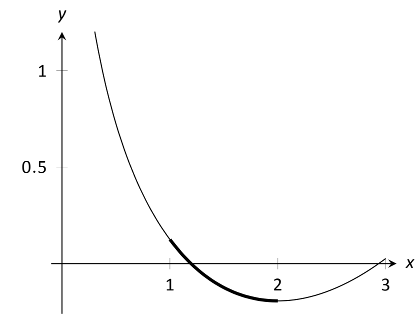


Figure 7.4.4: A graph of $f(x) = \frac{1}{8}x^2 - \ln x$ from Example 7.4.2.

The previous examples found the arc length exactly through careful choice of the functions. In general, exact answers are much more difficult to come by and numerical approximations are necessary.

Example 7.4.3 Approximating arc length numerically

Find the length of the sine curve from $x = 0$ to $x = \pi$.

SOLUTION This is somewhat of a mathematical curiosity; in Example 5.4.3 we found the area under one "hump" of the sine curve is 2 square units; now we are measuring its arc length.

The setup is straightforward: $f(x) = \sin x$ and $f'(x) = \cos x$. Thus

$$L = \int_0^\pi \sqrt{1 + \cos^2 x}\, dx.$$

This integral *cannot* be evaluated in terms of elementary functions so we will approximate it with Simpson's Method with $n = 4$. Figure 7.4.5 gives $\sqrt{1 + \cos^2 x}$ evaluated at 5 evenly spaced points in $[0, \pi]$. Simpson's Rule then states that

$$\int_0^\pi \sqrt{1 + \cos^2 x}\, dx \approx \frac{\pi - 0}{4 \cdot 3}\left(\sqrt{2} + 4\sqrt{3/2} + 2(1) + 4\sqrt{3/2} + \sqrt{2}\right)$$
$$= 3.82918.$$

Using a computer with $n = 100$ the approximation is $L \approx 3.8202$; our approximation with $n = 4$ is quite good.

x	$\sqrt{1 + \cos^2 x}$
0	$\sqrt{2}$
$\pi/4$	$\sqrt{3/2}$
$\pi/2$	1
$3\pi/4$	$\sqrt{3/2}$
π	$\sqrt{2}$

Figure 7.4.5: A table of values of $y = \sqrt{1 + \cos^2 x}$ to evaluate a definite integral in Example 7.4.3.

Notes:

Surface Area of Solids of Revolution

We have already seen how a curve $y = f(x)$ on $[a, b]$ can be revolved around an axis to form a solid. Instead of computing its volume, we now consider its surface area.

We begin as we have in the previous sections: we partition the interval $[a, b]$ with n subintervals, where the i^{th} subinterval is $[x_i, x_{i+1}]$. On each subinterval, we can approximate the curve $y = f(x)$ with a straight line that connects $f(x_i)$ and $f(x_{i+1})$ as shown in Figure 7.4.6(a). Revolving this line segment about the x-axis creates part of a cone (called a *frustum* of a cone) as shown in Figure 7.4.6(b). The surface area of a frustum of a cone is

$$2\pi \cdot \text{ length } \cdot \text{average of the two radii } R \text{ and } r.$$

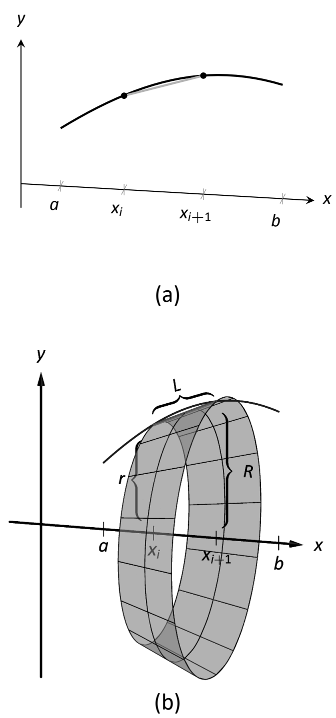


Figure 7.4.6: Establishing the formula for surface area.

The length is given by L; we use the material just covered by arc length to state that

$$L \approx \sqrt{1 + f'(c_i)^2}\Delta x_i$$

for some c_i in the i^{th} subinterval. The radii are just the function evaluated at the endpoints of the interval. That is,

$$R = f(x_{i+1}) \quad \text{and} \quad r = f(x_i).$$

Thus the surface area of this sample frustum of the cone is approximately

$$2\pi \frac{f(x_i) + f(x_{i+1})}{2}\sqrt{1 + f'(c_i)^2}\Delta x_i.$$

Since f is a continuous function, the Intermediate Value Theorem states there is some d_i in $[x_i, x_{i+1}]$ such that $f(d_i) = \dfrac{f(x_i) + f(x_{i+1})}{2}$; we can use this to rewrite the above equation as

$$2\pi f(d_i)\sqrt{1 + f'(c_i)^2}\Delta x_i.$$

Summing over all the subintervals we get the total surface area to be approximately

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(d_i)\sqrt{1 + f'(c_i)^2}\Delta x_i,$$

which is a Riemann Sum. Taking the limit as the subinterval lengths go to zero gives us the exact surface area, given in the following theorem.

Notes:

Theorem 7.4.2 Surface Area of a Solid of Revolution

Let f be differentiable on $[a, b]$, where f' is also continuous on $[a, b]$.

1. The surface area of the solid formed by revolving the graph of $y = f(x)$, where $f(x) \geq 0$, about the x-axis is

$$\text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx.$$

2. The surface area of the solid formed by revolving the graph of $y = f(x)$ about the y-axis, where $a, b \geq 0$, is

$$\text{Surface Area} = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\, dx.$$

(When revolving $y = f(x)$ about the y-axis, the radii of the resulting frustum are x_i and x_{i+1}; their average value is simply the midpoint of the interval. In the limit, this midpoint is just x. This gives the second part of Theorem 7.4.2.)

Example 7.4.4 Finding surface area of a solid of revolution

Find the surface area of the solid formed by revolving $y = \sin x$ on $[0, \pi]$ around the x-axis, as shown in Figure 7.4.7.

Solution The setup is relatively straightforward. Using Theorem 7.4.2, we have the surface area SA is:

$$\begin{aligned} SA &= 2\pi \int_0^\pi \sin x\sqrt{1 + \cos^2 x}\, dx \\ &= -2\pi\frac{1}{2}\left(\sinh^{-1}(\cos x) + \cos x\sqrt{1 + \cos^2 x}\right)\Bigg|_0^\pi \\ &= 2\pi\left(\sqrt{2} + \sinh^{-1} 1\right) \approx 14.42 \text{ units}^2. \end{aligned}$$

The integration step above is nontrivial, utilizing an integration method called Trigonometric Substitution.

It is interesting to see that the surface area of a solid, whose shape is defined by a trigonometric function, involves both a square root and an inverse hyperbolic trigonometric function.

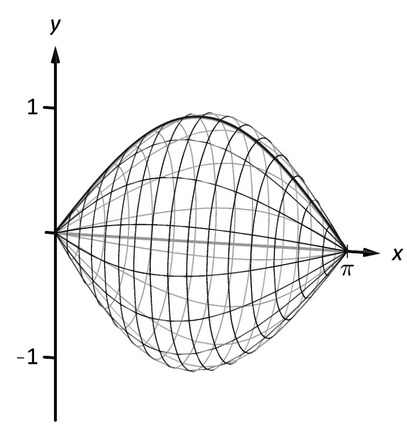


Figure 7.4.7: Revolving $y = \sin x$ on $[0, \pi]$ about the x-axis.

Example 7.4.5 Finding surface area of a solid of revolution

Find the surface area of the solid formed by revolving the curve $y = x^2$ on $[0, 1]$ about the x-axis and the y-axis.

Notes:

Solution About the x-axis: the integral is straightforward to setup:

$$SA = 2\pi \int_0^1 x^2\sqrt{1+(2x)^2}\,dx.$$

Like the integral in Example 7.4.4, this requires Trigonometric Substitution.

$$\begin{aligned} &= \frac{\pi}{32}\left(2(8x^3+x)\sqrt{1+4x^2} - \sinh^{-1}(2x)\right)\Big|_0^1 \\ &= \frac{\pi}{32}\left(18\sqrt{5} - \sinh^{-1} 2\right) \\ &\approx 3.81 \text{ units}^2. \end{aligned}$$

The solid formed by revolving $y = x^2$ around the x-axis is graphed in Figure 7.4.8 (a).

About the y-axis: since we are revolving around the y-axis, the "radius" of the solid is not $f(x)$ but rather x. Thus the integral to compute the surface area is:

$$SA = 2\pi \int_0^1 x\sqrt{1+(2x)^2}\,dx.$$

This integral can be solved using substitution. Set $u = 1 + 4x^2$; the new bounds are $u = 1$ to $u = 5$. We then have

$$\begin{aligned} &= \frac{\pi}{4}\int_1^5 \sqrt{u}\,du \\ &= \frac{\pi}{4}\frac{2}{3}u^{3/2}\Big|_1^5 \\ &= \frac{\pi}{6}\left(5\sqrt{5} - 1\right) \\ &\approx 5.33 \text{ units}^2. \end{aligned}$$

The solid formed by revolving $y = x^2$ about the y-axis is graphed in Figure 7.4.8 (b).

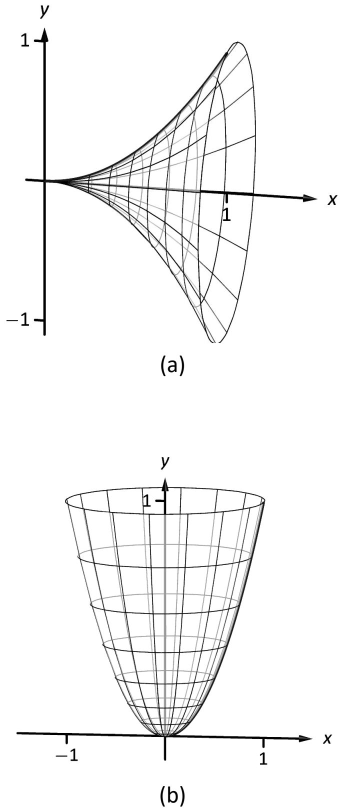


Figure 7.4.8: The solids used in Example 7.4.5.

Our final example is a famous mathematical "paradox."

Example 7.4.6 The surface area and volume of Gabriel's Horn

Consider the solid formed by revolving $y = 1/x$ about the x-axis on $[1, \infty)$. Find the volume and surface area of this solid. (This shape, as graphed in Figure 7.4.9, is known as "Gabriel's Horn" since it looks like a very long horn that only a supernatural person, such as an angel, could play.)

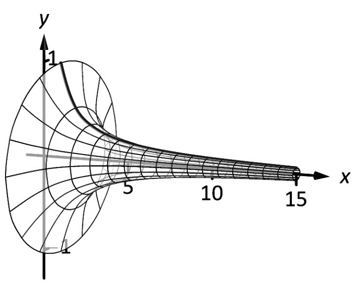


Figure 7.4.9: A graph of Gabriel's Horn.

Notes:

Solution To compute the volume it is natural to use the Disk Method. We have:

$$\begin{aligned} V &= \pi \int_1^\infty \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi \int_1^b \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi \left(\frac{-1}{x}\right)\Bigg|_1^b \\ &= \lim_{b\to\infty} \pi \left(1-\frac{1}{b}\right) \\ &= \pi \text{ units}^3. \end{aligned}$$

Gabriel's Horn has a finite volume of π cubic units. Since we have already seen that regions with infinite length can have a finite area, this is not too difficult to accept.

We now consider its surface area. The integral is straightforward to setup:

$$SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}\,dx.$$

Integrating this expression is not trivial. We can, however, compare it to other improper integrals. Since $1 < \sqrt{1+1/x^4}$ on $[1,\infty)$, we can state that

$$2\pi \int_1^\infty \frac{1}{x}\,dx < 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}\,dx.$$

By Key Idea 6.8.1, the improper integral on the left diverges. Since the integral on the right is larger, we conclude it also diverges, meaning Gabriel's Horn has infinite surface area.

Hence the "paradox": we can fill Gabriel's Horn with a finite amount of paint, but since it has infinite surface area, we can never paint it.

Somehow this paradox is striking when we think about it in terms of volume and area. However, we have seen a similar paradox before, as referenced above. We know that the area under the curve $y = 1/x^2$ on $[1,\infty)$ is finite, yet the shape has an infinite perimeter. Strange things can occur when we deal with the infinite.

A standard equation from physics is "Work = force $\times$ distance", when the force applied is constant. In the next section we learn how to compute work when the force applied is variable.

Notes:

Exercises 7.4

Terms and Concepts

1. T/F: The integral formula for computing Arc Length was found by first approximating arc length with straight line segments.

2. T/F: The integral formula for computing Arc Length includes a square–root, meaning the integration is probably easy.

Problems

In Exercises 3 – 12, find the arc length of the function on the given interval.

3. $f(x) = x$ on $[0, 1]$.

4. $f(x) = \sqrt{8}x$ on $[-1, 1]$.

5. $f(x) = \frac{1}{3}x^{3/2} - x^{1/2}$ on $[0, 1]$.

6. $f(x) = \frac{1}{12}x^3 + \frac{1}{x}$ on $[1, 4]$.

7. $f(x) = 2x^{3/2} - \frac{1}{6}\sqrt{x}$ on $[0, 9]$.

8. $f(x) = \cosh x$ on $[-\ln 2, \ln 2]$.

9. $f(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$ on $[0, \ln 5]$.

10. $f(x) = \frac{1}{12}x^5 + \frac{1}{5x^3}$ on $[.1, 1]$.

11. $f(x) = \ln\big(\sin x\big)$ on $[\pi/6, \pi/2]$.

12. $f(x) = \ln\big(\cos x\big)$ on $[0, \pi/4]$.

In Exercises 13 – 20, set up the integral to compute the arc length of the function on the given interval. Do not evaluate the integral.

13. $f(x) = x^2$ on $[0, 1]$.

14. $f(x) = x^{10}$ on $[0, 1]$.

15. $f(x) = \sqrt{x}$ on $[0, 1]$.

16. $f(x) = \ln x$ on $[1, e]$.

17. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$. (Note: this describes the top half of a circle with radius 1.)

18. $f(x) = \sqrt{1 - x^2/9}$ on $[-3, 3]$. (Note: this describes the top half of an ellipse with a major axis of length 6 and a minor axis of length 2.)

19. $f(x) = \frac{1}{x}$ on $[1, 2]$.

20. $f(x) = \sec x$ on $[-\pi/4, \pi/4]$.

In Exercises 21 – 28, use Simpson's Rule, with $n = 4$, to approximate the arc length of the function on the given interval. Note: these are the same problems as in Exercises 13–20.

21. $f(x) = x^2$ on $[0, 1]$.

22. $f(x) = x^{10}$ on $[0, 1]$.

23. $f(x) = \sqrt{x}$ on $[0, 1]$. (Note: $f'(x)$ is not defined at $x = 0$.)

24. $f(x) = \ln x$ on $[1, e]$.

25. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$. (Note: $f'(x)$ is not defined at the endpoints.)

26. $f(x) = \sqrt{1 - x^2/9}$ on $[-3, 3]$. (Note: $f'(x)$ is not defined at the endpoints.)

27. $f(x) = \frac{1}{x}$ on $[1, 2]$.

28. $f(x) = \sec x$ on $[-\pi/4, \pi/4]$.

In Exercises 29 – 33, find the surface area of the described solid of revolution.

29. The solid formed by revolving $y = 2x$ on $[0, 1]$ about the x-axis.

30. The solid formed by revolving $y = x^2$ on $[0, 1]$ about the y-axis.

31. The solid formed by revolving $y = x^3$ on $[0, 1]$ about the x-axis.

32. The solid formed by revolving $y = \sqrt{x}$ on $[0, 1]$ about the x-axis.

33. The sphere formed by revolving $y = \sqrt{1 - x^2}$ on $[-1, 1]$ about the x-axis.

7.5 Work

Work is the scientific term used to describe the action of a force which moves an object. When a constant force F is applied to move an object a distance d, the amount of work performed is $W = F \cdot d$.

The SI unit of force is the Newton, (kg·m/s^2), and the SI unit of distance is a meter (m). The fundamental unit of work is one Newton–meter, or a joule (J). That is, applying a force of one Newton for one meter performs one joule of work. In Imperial units (as used in the United States), force is measured in pounds (lb) and distance is measured in feet (ft), hence work is measured in ft–lb.

When force is constant, the measurement of work is straightforward. For instance, lifting a 200 lb object 5 ft performs $200 \cdot 5 = 1000$ ft–lb of work.

What if the force applied is variable? For instance, imagine a climber pulling a 200 ft rope up a vertical face. The rope becomes lighter as more is pulled in, requiring less force and hence the climber performs less work.

In general, let $F(x)$ be a force function on an interval $[a, b]$. We want to measure the amount of work done applying the force F from $x = a$ to $x = b$. We can approximate the amount of work being done by partitioning $[a, b]$ into subintervals $a = x_1 < x_2 < \cdots < x_{n+1} = b$ and assuming that F is constant on each subinterval. Let c_i be a value in the i^{th} subinterval $[x_i, x_{i+1}]$. Then the work done on this interval is approximately $W_i \approx F(c_i) \cdot (x_{i+1} - x_i) = F(c_i)\Delta x_i$, a constant force $\times$ the distance over which it is applied. The total work is

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(c_i)\Delta x_i.$$

This, of course, is a Riemann sum. Taking a limit as the subinterval lengths go to zero gives an exact value of work which can be evaluated through a definite integral.

Key Idea 7.5.1 Work

Let $F(x)$ be a continuous function on $[a, b]$ describing the amount of force being applied to an object in the direction of travel from distance $x = a$ to distance $x = b$. The total work W done on $[a, b]$ is

$$W = \int_a^b F(x)\, dx.$$

Note: *Mass* and *weight* are closely related, yet different, concepts. The mass m of an object is a quantitative measure of that object's resistance to acceleration. The weight w of an object is a measurement of the force applied to the object by the acceleration of gravity g.

Since the two measurements are proportional, $w = m \cdot g$, they are often used interchangeably in everyday conversation. When computing work, one must be careful to note which is being referred to. When mass is given, it must be multiplied by the acceleration of gravity to reference the related force.

Notes:

Example 7.5.1 Computing work performed: applying variable force

A 60m climbing rope is hanging over the side of a tall cliff. How much work is performed in pulling the rope up to the top, where the rope has a mass of 66g/m?

SOLUTION We need to create a force function $F(x)$ on the interval $[0, 60]$. To do so, we must first decide what x is measuring: it is the length of the rope still hanging or is it the amount of rope pulled in? As long as we are consistent, either approach is fine. We adopt for this example the convention that x is the amount of rope pulled in. This seems to match intuition better; pulling up the first 10 meters of rope involves $x = 0$ to $x = 10$ instead of $x = 60$ to $x = 50$.

As x is the amount of rope pulled in, the amount of rope still hanging is $60-x$. This length of rope has a mass of 66 g/m, or 0.066 kg/m. The mass of the rope still hanging is $0.066(60 - x)$ kg; multiplying this mass by the acceleration of gravity, 9.8 m/s^2, gives our variable force function

$$F(x) = (9.8)(0.066)(60 - x) = 0.6468(60 - x).$$

Thus the total work performed in pulling up the rope is

$$W = \int_0^{60} 0.6468(60 - x)\,dx = 1,164.24 \text{ J}.$$

By comparison, consider the work done in lifting the entire rope 60 meters. The rope weighs $60 \times 0.066 \times 9.8 = 38.808$ N, so the work applying this force for 60 meters is $60 \times 38.808 = 2,328.48$ J. This is exactly twice the work calculated before (and we leave it to the reader to understand why.)

Example 7.5.2 Computing work performed: applying variable force

Consider again pulling a 60 m rope up a cliff face, where the rope has a mass of 66 g/m. At what point is exactly half the work performed?

SOLUTION From Example 7.5.1 we know the total work performed is $1,164.24$ J. We want to find a height h such that the work in pulling the rope from a height of $x = 0$ to a height of $x = h$ is 582.12, half the total work. Thus we want to solve the equation

$$\int_0^{h} 0.6468(60 - x)\,dx = 582.12$$

for h.

Notes:

$$\int_0^h 0.6468(60-x)\,dx = 582.12$$
$$\left(38.808x - 0.3234x^2\right)\Big|_0^h = 582.12$$
$$38.808h - 0.3234h^2 = 582.12$$
$$-0.3234h^2 + 38.808h - 582.12 = 0.$$

Apply the Quadratic Formula:

$$h = 17.57 \text{ and } 102.43$$

As the rope is only 60m long, the only sensible answer is $h = 17.57$. Thus about half the work is done pulling up the first 17.5m the other half of the work is done pulling up the remaining 42.43m.

Note: In Example 7.5.2, we find that half of the work performed in pulling up a 60 m rope is done in the last 42.43 m. Why is it not coincidental that $60/\sqrt{2} = 42.43$?

Example 7.5.3 Computing work performed: applying variable force

A box of 100 lb of sand is being pulled up at a uniform rate a distance of 50 ft over 1 minute. The sand is leaking from the box at a rate of 1 lb/s. The box itself weighs 5 lb and is pulled by a rope weighing .2 lb/ft.

1. How much work is done lifting just the rope?
2. How much work is done lifting just the box and sand?
3. What is the total amount of work performed?

Solution

1. We start by forming the force function $F_r(x)$ for the rope (where the subscript denotes we are considering the rope). As in the previous example, let x denote the amount of rope, in feet, pulled in. (This is the same as saying x denotes the height of the box.) The weight of the rope with x feet pulled in is $F_r(x) = 0.2(50-x) = 10 - 0.2x$. (Note that we do not have to include the acceleration of gravity here, for the *weight* of the rope per foot is given, not its *mass* per meter as before.) The work performed lifting the rope is

$$W_r = \int_0^{50} (10-0.2x)\,dx = 250 \text{ ft–lb}.$$

Notes:

2. The sand is leaving the box at a rate of 1 lb/s. As the vertical trip is to take one minute, we know that 60 lb will have left when the box reaches its final height of 50 ft. Again letting x represent the height of the box, we have two points on the line that describes the weight of the sand: when $x = 0$, the sand weight is 100 lb, producing the point $(0, 100)$; when $x = 50$, the sand in the box weighs 40 lb, producing the point $(50, 40)$. The slope of this line is $\frac{100-40}{0-50} = -1.2$, giving the equation of the weight of the sand at height x as $w(x) = -1.2x + 100$. The box itself weighs a constant 5 lb, so the total force function is $F_b(x) = -1.2x + 105$. Integrating from $x = 0$ to $x = 50$ gives the work performed in lifting box and sand:

$$W_b = \int_0^{50} (-1.2x + 105)\, dx = 3750 \text{ ft–lb.}$$

3. The total work is the sum of W_r and W_b: $250 + 3750 = 4000$ ft–lb. We can also arrive at this via integration:

$$\begin{aligned} W &= \int_0^{50} (F_r(x) + F_b(x))\, dx \\ &= \int_0^{50} (10 - 0.2x - 1.2x + 105)\, dx \\ &= \int_0^{50} (-1.4x + 115)\, dx \\ &= 4000 \text{ ft–lb.} \end{aligned}$$

Hooke's Law and Springs

Hooke's Law states that the force required to compress or stretch a spring x units from its natural length is proportional to x; that is, this force is $F(x) = kx$ for some constant k. For example, if a force of 1 N stretches a given spring 2 cm, then a force of 5 N will stretch the spring 10 cm. Converting the distances to meters, we have that stretching this spring 0.02 m requires a force of $F(0.02) = k(0.02) = 1$ N, hence $k = 1/0.02 = 50$ N/m.

Example 7.5.4 Computing work performed: stretching a spring

A force of 20 lb stretches a spring from a natural length of 7 inches to a length of 12 inches. How much work was performed in stretching the spring to this length?

Solution In many ways, we are not at all concerned with the actual length of the spring, only with the amount of its change. Hence, we do not care

Notes:

that 20 lb of force stretches the spring to a length of 12 inches, but rather that a force of 20 lb stretches the spring by 5 in. This is illustrated in Figure 7.5.1; we only measure the change in the spring's length, not the overall length of the spring.

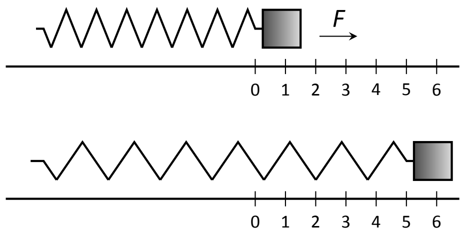


Figure 7.5.1: Illustrating the important aspects of stretching a spring in computing work in Example 7.5.4.

Converting the units of length to feet, we have

$$F(5/12) = 5/12k = 20 \text{ lb}.$$

Thus $k = 48$ lb/ft and $F(x) = 48x$.

We compute the total work performed by integrating $F(x)$ from $x = 0$ to $x = 5/12$:

$$\begin{aligned} W &= \int_0^{5/12} 48x\, dx \\ &= 24x^2\Big|_0^{5/12} \\ &= 25/6 \approx 4.1667 \text{ ft–lb}. \end{aligned}$$

Pumping Fluids

Another useful example of the application of integration to compute work comes in the pumping of fluids, often illustrated in the context of emptying a storage tank by pumping the fluid out the top. This situation is different than our previous examples for the forces involved are constant. After all, the force required to move one cubic foot of water (about 62.4 lb) is the same regardless of its location in the tank. What is variable is the distance that cubic foot of water has to travel; water closer to the top travels less distance than water at the bottom, producing less work.

We demonstrate how to compute the total work done in pumping a fluid out of the top of a tank in the next two examples.

Fluid	lb/ft^3	kg/m^3
Concrete	150	2400
Fuel Oil	55.46	890.13
Gasoline	45.93	737.22
Iodine	307	4927
Methanol	49.3	791.3
Mercury	844	13546
Milk	63.6–65.4	1020 – 1050
Water	62.4	1000

Figure 7.5.2: Weight and Mass densities

Notes:

Example 7.5.5 Computing work performed: pumping fluids

A cylindrical storage tank with a radius of 10 ft and a height of 30 ft is filled with water, which weighs approximately 62.4 lb/ft^3. Compute the amount of work performed by pumping the water up to a point 5 feet above the top of the tank.

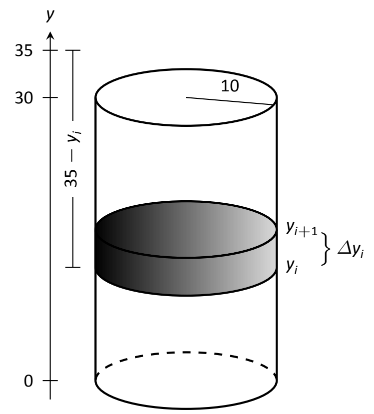


Figure 7.5.3: Illustrating a water tank in order to compute the work required to empty it in Example 7.5.5.

Solution We will refer often to Figure 7.5.3 which illustrates the salient aspects of this problem.

We start as we often do: we partition an interval into subintervals. We orient our tank vertically since this makes intuitive sense with the base of the tank at $y = 0$. Hence the top of the water is at $y = 30$, meaning we are interested in subdividing the y-interval $[0, 30]$ into n subintervals as

$$0 = y_1 < y_2 < \cdots < y_{n+1} = 30.$$

Consider the work W_i of pumping only the water residing in the i^{th} subinterval, illustrated in Figure 7.5.3. The force required to move this water is equal to its weight which we calculate as volume $\times$ density. The volume of water in this subinterval is $V_i = 10^2\pi\Delta y_i$; its density is 62.4 lb/ft^3. Thus the required force is $6240\pi\Delta y_i$ lb.

We approximate the distance the force is applied by using any y-value contained in the i^{th} subinterval; for simplicity, we arbitrarily use y_i for now (it will not matter later on). The water will be pumped to a point 5 feet above the top of the tank, that is, to the height of $y = 35$ ft. Thus the distance the water at height y_i travels is $35 - y_i$ ft.

In all, the approximate work W_i peformed in moving the water in the i^{th} subinterval to a point 5 feet above the tank is

$$W_i \approx 6240\pi\Delta y_i(35 - y_i).$$

To approximate the total work performed in pumping out all the water from the tank, we sum all the work W_i performed in pumping the water from each of the n subintervals of $[0, 30]$:

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} 6240\pi\Delta y_i(35 - y_i).$$

This is a Riemann sum. Taking the limit as the subinterval length goes to 0 gives

$$\begin{aligned} W &= \int_0^{30} 6240\pi(35 - y)\,dy \\ &= 6240\pi\left(35y - 1/2y^2\right)\Big|_0^{30} \\ &= 11,762,123 \text{ ft–lb} \\ &\approx 1.176 \times 10^7 \text{ ft–lb}. \end{aligned}$$

Notes:

We can "streamline" the above process a bit as we may now recognize what the important features of the problem are. Figure 7.5.4 shows the tank from Example 7.5.5 without the i^{th} subinterval identified. Instead, we just draw one differential element. This helps establish the height a small amount of water must travel along with the force required to move it (where the force is volume $\times$ density).

We demonstrate the concepts again in the next examples.

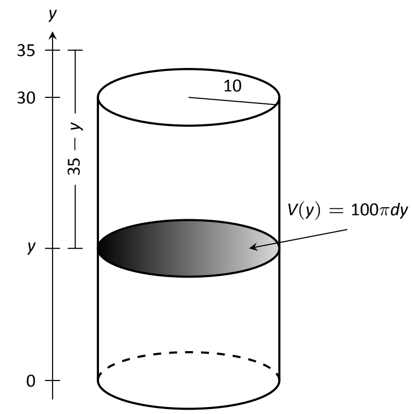


Figure 7.5.4: A simplified illustration for computing work.

Example 7.5.6 Computing work performed: pumping fluids

A conical water tank has its top at ground level and its base 10 feet below ground. The radius of the cone at ground level is 2 ft. It is filled with water weighing 62.4 lb/ft^3 and is to be emptied by pumping the water to a spigot 3 feet above ground level. Find the total amount of work performed in emptying the tank.

Solution The conical tank is sketched in Figure 7.5.5. We can orient the tank in a variety of ways; we could let $y = 0$ represent the base of the tank and $y = 10$ represent the top of the tank, but we choose to keep the convention of the wording given in the problem and let $y = 0$ represent ground level and hence $y = -10$ represents the bottom of the tank. The actual "height" of the water does not matter; rather, we are concerned with the distance the water travels.

The figure also sketches a differential element, a cross–sectional circle. The radius of this circle is variable, depending on y. When $y = -10$, the circle has radius 0; when $y = 0$, the circle has radius 2. These two points, $(-10, 0)$ and $(0, 2)$, allow us to find the equation of the line that gives the radius of the cross–sectional circle, which is $r(y) = 1/5y + 2$. Hence the volume of water at this height is $V(y) = \pi(1/5y + 2)^2dy$, where dy represents a very small height of the differential element. The force required to move the water at height y is $F(y) = 62.4 \times V(y)$.

The distance the water at height y travels is given by $h(y) = 3 - y$. Thus the total work done in pumping the water from the tank is

$$
\begin{aligned}
W &= \int_{-10}^{0} 62.4\pi(1/5y + 2)^2(3 - y)\, dy \\
&= 62.4\pi \int_{-10}^{0} \left(-\frac{1}{25}y^3 - \frac{17}{25}y^2 - \frac{8}{5}y + 12\right)\, dy \\
&= 62.2\pi \cdot \frac{220}{3} \approx 14,376 \text{ ft–lb.}
\end{aligned}
$$

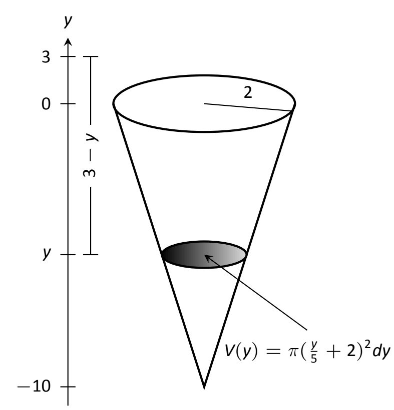


Figure 7.5.5: A graph of the conical water tank in Example 7.5.6.

Notes:

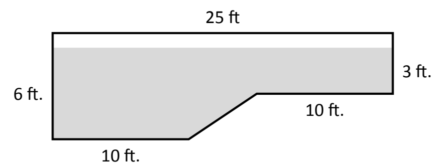


Figure 7.5.6: The cross–section of a swimming pool filled with water in Example 7.5.7.

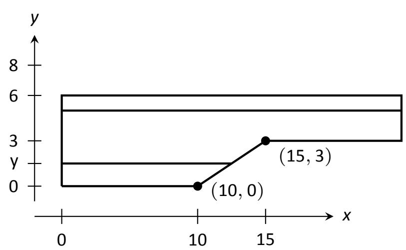


Figure 7.5.7: Orienting the pool and showing differential elements for Example 7.5.7.

Example 7.5.7 Computing work performed: pumping fluids

A rectangular swimming pool is 20 ft wide and has a 3 ft "shallow end" and a 6 ft "deep end." It is to have its water pumped out to a point 2 ft above the current top of the water. The cross–sectional dimensions of the water in the pool are given in Figure 7.5.6; note that the dimensions are for the water, not the pool itself. Compute the amount of work performed in draining the pool.

Solution For the purposes of this problem we choose to set $y = 0$ to represent the bottom of the pool, meaning the top of the water is at $y = 6$. Figure 7.5.7 shows the pool oriented with this y-axis, along with 2 differential elements as the pool must be split into two different regions.

The top region lies in the y-interval of $[3, 6]$, where the length of the differential element is 25 ft as shown. As the pool is 20 ft wide, this differential element represents a thin slice of water with volume $V(y) = 20 \cdot 25 \cdot dy$. The water is to be pumped to a height of $y = 8$, so the height function is $h(y) = 8 - y$. The work done in pumping this top region of water is

$$W_t = 62.4 \int_3^6 500(8 - y)\, dy = 327,600 \text{ ft–lb}.$$

The bottom region lies in the y-interval of $[0, 3]$; we need to compute the length of the differential element in this interval.

One end of the differential element is at $x = 0$ and the other is along the line segment joining the points $(10, 0)$ and $(15, 3)$. The equation of this line is $y = 3/5(x-10)$; as we will be integrating with respect to y, we rewrite this equation as $x = 5/3y + 10$. So the length of the differential element is a difference of x-values: $x = 0$ and $x = 5/3y + 10$, giving a length of $x = 5/3y + 10$.

Again, as the pool is 20 ft wide, this differential element represents a thin slice of water with volume $V(y) = 20 \cdot (5/3y + 10) \cdot dy$; the height function is the same as before at $h(y) = 8 - y$. The work performed in emptying this part of the pool is

$$W_b = 62.4 \int_0^3 20(5/3y + 10)(8 - y)\, dy = 299,520 \text{ ft–lb}.$$

The total work in empyting the pool is

$$W = W_b + W_t = 327,600 + 299,520 = 627,120 \text{ ft–lb}.$$

Notice how the emptying of the bottom of the pool performs almost as much work as emptying the top. The top portion travels a shorter distance but has more water. In the end, this extra water produces more work.

The next section introduces one final application of the definite integral, the calculation of fluid force on a plate.

Notes:

Exercises 7.5

Terms and Concepts

1. What are the typical units of work?

2. If a man has a mass of 80 kg on Earth, will his mass on the moon be bigger, smaller, or the same?

3. If a woman weighs 130 lb on Earth, will her weight on the moon be bigger, smaller, or the same?

4. Fill in the blanks:

   Some integrals in this section are set up by multiplying a variable ________ by a constant distance; others are set up by multiplying a constant force by a variable ________.

Problems

5. A 100 ft rope, weighing 0.1 lb/ft, hangs over the edge of a tall building.

   (a) How much work is done pulling the entire rope to the top of the building?

   (b) How much rope is pulled in when half of the total work is done?

6. A 50 m rope, with a mass density of 0.2 kg/m, hangs over the edge of a tall building.

   (a) How much work is done pulling the entire rope to the top of the building?

   (b) How much work is done pulling in the first 20 m?

7. A rope of length ℓ ft hangs over the edge of tall cliff. (Assume the cliff is taller than the length of the rope.) The rope has a weight density of d lb/ft.

   (a) How much work is done pulling the entire rope to the top of the cliff?

   (b) What percentage of the total work is done pulling in the first half of the rope?

   (c) How much rope is pulled in when half of the total work is done?

8. A 20 m rope with mass density of 0.5 kg/m hangs over the edge of a 10 m building. How much work is done pulling the rope to the top?

9. A crane lifts a 2,000 lb load vertically 30 ft with a 1” cable weighing 1.68 lb/ft.

   (a) How much work is done lifting the cable alone?

   (b) How much work is done lifting the load alone?

   (c) Could one conclude that the work done lifting the cable is negligible compared to the work done lifting the load?

10. A 100 lb bag of sand is lifted uniformly 120 ft in one minute. Sand leaks from the bag at a rate of 1/4 lb/s. What is the total work done in lifting the bag?

11. A box weighing 2 lb lifts 10 lb of sand vertically 50 ft. A crack in the box allows the sand to leak out such that 9 lb of sand is in the box at the end of the trip. Assume the sand leaked out at a uniform rate. What is the total work done in lifting the box and sand?

12. A force of 1000 lb compresses a spring 3 in. How much work is performed in compressing the spring?

13. A force of 2 N stretches a spring 5 cm. How much work is performed in stretching the spring?

14. A force of 50 lb compresses a spring from a natural length of 18 in to 12 in. How much work is performed in compressing the spring?

15. A force of 20 lb stretches a spring from a natural length of 6 in to 8 in. How much work is performed in stretching the spring?

16. A force of 7 N stretches a spring from a natural length of 11 cm to 21 cm. How much work is performed in stretching the spring from a length of 16 cm to 21 cm?

17. A force of f N stretches a spring d m from its natural length. How much work is performed in stretching the spring?

18. A 20 lb weight is attached to a spring. The weight rests on the spring, compressing the spring from a natural length of 1 ft to 6 in.

    How much work is done in lifting the box 1.5 ft (i.e, the spring will be stretched 1 ft beyond its natural length)?

19. A 20 lb weight is attached to a spring. The weight rests on the spring, compressing the spring from a natural length of 1 ft to 6 in.

    How much work is done in lifting the box 6 in (i.e, bringing the spring back to its natural length)?

20. A 5 m tall cylindrical tank with radius of 2 m is filled with 3 m of gasoline, with a mass density of 737.22 kg/m^3. Compute the total work performed in pumping all the gasoline to the top of the tank.

21. A 6 ft cylindrical tank with a radius of 3 ft is filled with water, which has a weight density of 62.4 lb/ft^3. The water is to be pumped to a point 2 ft above the top of the tank.

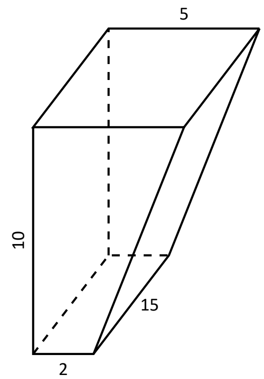
    (a) How much work is performed in pumping all the water from the tank?

    (b) How much work is performed in pumping 3 ft of water from the tank?

    (c) At what point is 1/2 of the total work done?

22. A gasoline tanker is filled with gasoline with a weight density of 45.93 lb/ft^3. The dispensing valve at the base is jammed shut, forcing the operator to empty the tank via pumping the gas to a point 1 ft above the top of the tank. Assume the tank is a perfect cylinder, 20 ft long with a diameter of 7.5 ft. How much work is performed in pumping all the gasoline from the tank?

23. A fuel oil storage tank is 10 ft deep with trapezoidal sides, 5 ft at the top and 2 ft at the bottom, and is 15 ft wide (see diagram below). Given that fuel oil weighs 55.46 lb/ft^3, find the work performed in pumping all the oil from the tank to a point 3 ft above the top of the tank.



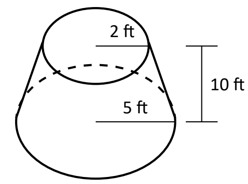
24. A conical water tank is 5 m deep with a top radius of 3 m. (This is similar to Example 7.5.6.) The tank is filled with pure water, with a mass density of 1000 kg/m^3.

    (a) Find the work performed in pumping all the water to the top of the tank.

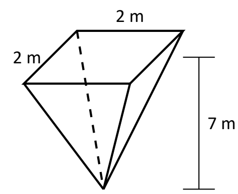
    (b) Find the work performed in pumping the top 2.5 m of water to the top of the tank.

    (c) Find the work performed in pumping the top half of the water, by volume, to the top of the tank.

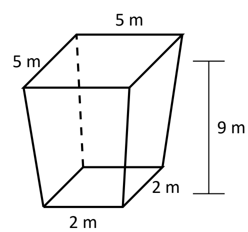
25. A water tank has the shape of a truncated cone, with dimensions given below, and is filled with water with a weight density of 62.4 lb/ft^3. Find the work performed in pumping all water to a point 1 ft above the top of the tank.



26. A water tank has the shape of an inverted pyramid, with dimensions given below, and is filled with water with a mass density of 1000 kg/m^3. Find the work performed in pumping all water to a point 5 m above the top of the tank.



27. A water tank has the shape of an truncated, inverted pyramid, with dimensions given below, and is filled with water with a mass density of 1000 kg/m^3. Find the work performed in pumping all water to a point 1 m above the top of the tank.



7.6 Fluid Forces

In the unfortunate situation of a car driving into a body of water, the conventional wisdom is that the water pressure on the doors will quickly be so great that they will be effectively unopenable. (Survival techniques suggest immediately opening the door, rolling down or breaking the window, or waiting until the water fills up the interior at which point the pressure is equalized and the door will open. See Mythbusters episode #72 to watch Adam Savage test these options.)

How can this be true? How much force does it take to open the door of a submerged car? In this section we will find the answer to this question by examining the forces exerted by fluids.

We start with **pressure**, which is related to **force** by the following equations:

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} \quad \Leftrightarrow \quad \text{Force} = \text{Pressure} \times \text{Area}.$$

In the context of fluids, we have the following definition.

> **Definition 7.6.1 Fluid Pressure**
>
> Let w be the weight–density of a fluid. The **pressure** p exerted on an object at depth d in the fluid is $p = w \cdot d$.

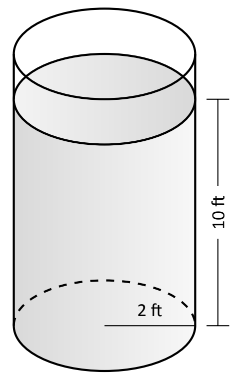
We use this definition to find the **force** exerted on a horizontal sheet by considering the sheet's area.

Example 7.6.1 Computing fluid force

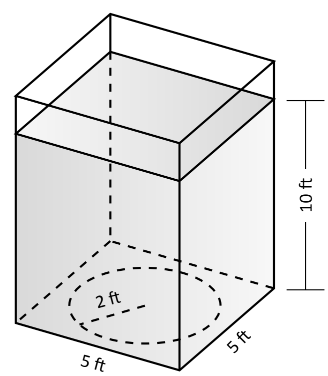
1. A cylindrical storage tank has a radius of 2 ft and holds 10 ft of a fluid with a weight–density of 50 lb/ft^3. (See Figure 7.6.1(a).) What is the force exerted on the base of the cylinder by the fluid?
2. A rectangular tank whose base is a 5 ft square has a circular hatch at the bottom with a radius of 2 ft. The tank holds 10 ft of a fluid with a weight–density of 50 lb/ft^3. (See Figure 7.6.1(b).) What is the force exerted on the hatch by the fluid?

SOLUTION

1. Using Definition 7.6.1, we calculate that the pressure exerted on the cylinder's base is $w \cdot d = 50\text{ lb/ft}^3 \times 10\text{ ft} = 500\text{ lb/ft}^2$. The area of the base is



(a)



(b)

Figure 7.6.1: The cylindrical and rectangular tank in Example 7.6.1.

Notes:

$\pi \cdot 2^2 = 4\pi$ ft^2. So the force exerted by the fluid is

$$F = 500 \times 4\pi = 6283 \text{ lb}.$$

Note that we effectively just computed the *weight* of the fluid in the tank.

2. The dimensions of the tank in this problem are irrelevant. All we are concerned with are the dimensions of the hatch and the depth of the fluid. Since the dimensions of the hatch are the same as the base of the tank in the previous part of this example, as is the depth, we see that the fluid force is the same. That is, $F = 6283$ lb.

   A key concept to understand here is that we are effectively measuring the weight of a 10 ft column of water above the hatch. The size of the tank holding the fluid does not matter.

The previous example demonstrates that computing the force exerted on a horizontally oriented plate is relatively easy to compute. What about a vertically oriented plate? For instance, suppose we have a circular porthole located on the side of a submarine. How do we compute the fluid force exerted on it?

Pascal's Principle states that the pressure exerted by a fluid at a depth is equal in all directions. Thus the pressure on any portion of a plate that is 1 ft below the surface of water is the same no matter how the plate is oriented. (Thus a hollow cube submerged at a great depth will not simply be "crushed" from above, but the sides will also crumple in. The fluid will exert force on *all* sides of the cube.)

So consider a vertically oriented plate as shown in Figure 7.6.2 submerged in a fluid with weight–density w. What is the total fluid force exerted on this plate? We find this force by first approximating the force on small horizontal strips.

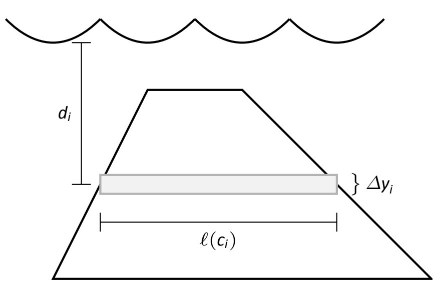


Figure 7.6.2: A thin, vertically oriented plate submerged in a fluid with weight–density w.

Let the top of the plate be at depth b and let the bottom be at depth a. (For now we assume that surface of the fluid is at depth 0, so if the bottom of the plate is 3 ft under the surface, we have $a = -3$. We will come back to this later.) We partition the interval $[a, b]$ into n subintervals

$$a = y_1 < y_2 < \cdots < y_{n+1} = b,$$

with the i^{th} subinterval having length Δy_i. The force F_i exerted on the plate in the i^{th} subinterval is $F_i = \text{Pressure} \times \text{Area}$.

The pressure is depth $\times w$. We approximate the depth of this thin strip by choosing any value d_i in $[y_i, y_{i+1}]$; the depth is approximately $-d_i$. (Our convention has d_i being a negative number, so $-d_i$ is positive.) For convenience, we let d_i be an endpoint of the subinterval; we let $d_i = y_i$.

The area of the thin strip is approximately length $\times$ width. The width is Δy_i. The length is a function of some y-value c_i in the i^{th} subinterval. We state the

Notes:

length is $\ell(c_i)$. Thus

$$\begin{aligned} F_i &= \text{Pressure} \times \text{Area} \\ &= -y_i \cdot w \times \ell(c_i) \cdot \Delta y_i. \end{aligned}$$

To approximate the total force, we add up the approximate forces on each of the n thin strips:

$$F = \sum_{i=1}^{n} F_i \approx \sum_{i=1}^{n} -w \cdot y_i \cdot \ell(c_i) \cdot \Delta y_i.$$

This is, of course, another Riemann Sum. We can find the exact force by taking a limit as the subinterval lengths go to 0; we evaluate this limit with a definite integral.

Key Idea 7.6.1 Fluid Force on a Vertically Oriented Plate

Let a vertically oriented plate be submerged in a fluid with weight–density w where the top of the plate is at $y = b$ and the bottom is at $y = a$. Let $\ell(y)$ be the length of the plate at y.

1. If $y = 0$ corresponds to the surface of the fluid, then the force exerted on the plate by the fluid is

$$F = \int_a^b w \cdot (-y) \cdot \ell(y)\, dy.$$

2. In general, let $d(y)$ represent the distance between the surface of the fluid and the plate at y. Then the force exerted on the plate by the fluid is

$$F = \int_a^b w \cdot d(y) \cdot \ell(y)\, dy.$$

Example 7.6.2 Finding fluid force

Consider a thin plate in the shape of an isosceles triangle as shown in Figure 7.6.3 submerged in water with a weight–density of 62.4 lb/ft^3. If the bottom of the plate is 10 ft below the surface of the water, what is the total fluid force exerted on this plate?

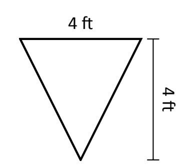


Figure 7.6.3: A thin plate in the shape of an isosceles triangle in Example 7.6.2.

Solution We approach this problem in two different ways to illustrate the different ways Key Idea 7.6.1 can be implemented. First we will let $y = 0$ represent the surface of the water, then we will consider an alternate convention.

Notes:

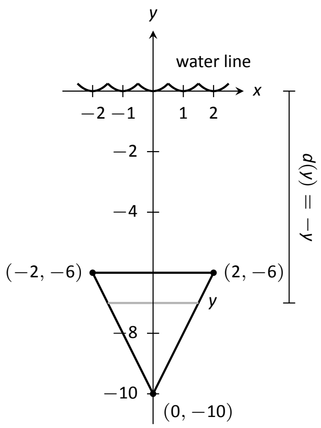


Figure 7.6.4: Sketching the triangular plate in Example 7.6.2 with the convention that the water level is at $y = 0$.

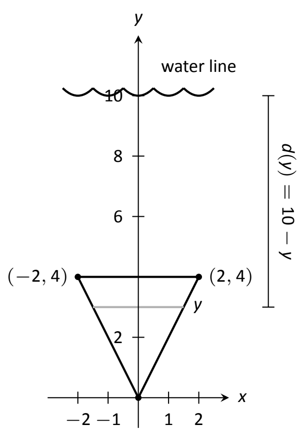


Figure 7.6.5: Sketching the triangular plate in Example 7.6.2 with the convention that the base of the triangle is at $(0, 0)$.

1. We let $y = 0$ represent the surface of the water; therefore the bottom of the plate is at $y = -10$. We center the triangle on the y-axis as shown in Figure 7.6.4. The depth of the plate at y is $-y$ as indicated by the Key Idea. We now consider the length of the plate at y.

   We need to find equations of the left and right edges of the plate. The right hand side is a line that connects the points $(0, -10)$ and $(2, -6)$: that line has equation $x = 1/2(y + 10)$. (Find the equation in the familiar $y = mx+b$ format and solve for x.) Likewise, the left hand side is described by the line $x = -1/2(y + 10)$. The total length is the distance between these two lines: $\ell(y) = 1/2(y + 10) - (-1/2(y + 10)) = y + 10$.

   The total fluid force is then:

$$\begin{aligned} F &= \int_{-10}^{-6} 62.4(-y)(y + 10)\, dy \\ &= 62.4 \cdot \frac{176}{3} \approx 3660.8 \text{ lb.} \end{aligned}$$

2. Sometimes it seems easier to orient the thin plate nearer the origin. For instance, consider the convention that the bottom of the triangular plate is at $(0, 0)$, as shown in Figure 7.6.5. The equations of the left and right hand sides are easy to find. They are $y = 2x$ and $y = -2x$, respectively, which we rewrite as $x = 1/2y$ and $x = -1/2y$. Thus the length function is $\ell(y) = 1/2y - (-1/2y) = y$.

   As the surface of the water is 10 ft above the base of the plate, we have that the surface of the water is at $y = 10$. Thus the depth function is the distance between $y = 10$ and y; $d(y) = 10 - y$. We compute the total fluid force as:

$$\begin{aligned} F &= \int_{0}^{4} 62.4(10 - y)(y)\, dy \\ &\approx 3660.8 \text{ lb.} \end{aligned}$$

The correct answer is, of course, independent of the placement of the plate in the coordinate plane as long as we are consistent.

Example 7.6.3 Finding fluid force

Find the total fluid force on a car door submerged up to the bottom of its window in water, where the car door is a rectangle 40” long and 27” high (based on the dimensions of a 2005 Fiat Grande Punto.)

Solution The car door, as a rectangle, is drawn in Figure 7.6.6. Its length is $10/3$ ft and its height is 2.25 ft. We adopt the convention that the top

Notes:

of the door is at the surface of the water, both of which are at $y = 0$. Using the weight–density of water of 62.4 lb/ft^3, we have the total force as

$$\begin{aligned} F &= \int_{-2.25}^{0} 62.4(-y)10/3\,dy \\ &= \int_{-2.25}^{0} -208y\,dy \\ &= -104y^2\Big|_{-2.25}^{0} \\ &= 526.5 \text{ lb.} \end{aligned}$$

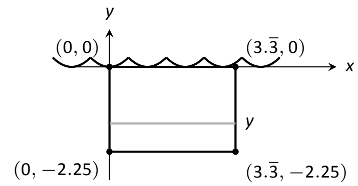


Figure 7.6.6: Sketching a submerged car door in Example 7.6.3.

Most adults would find it very difficult to apply over 500 lb of force to a car door while seated inside, making the door effectively impossible to open. This is counter–intuitive as most assume that the door would be relatively easy to open. The truth is that it is not, hence the survival tips mentioned at the beginning of this section.

Example 7.6.4 Finding fluid force

An underwater observation tower is being built with circular viewing portholes enabling visitors to see underwater life. Each vertically oriented porthole is to have a 3 ft diameter whose center is to be located 50 ft underwater. Find the total fluid force exerted on each porthole. Also, compute the fluid force on a horizontally oriented porthole that is under 50 ft of water.

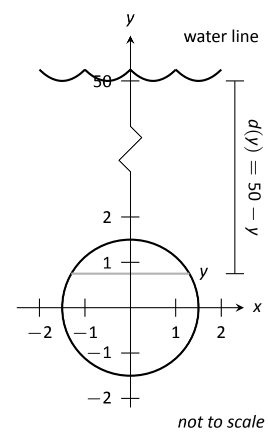


Figure 7.6.7: Measuring the fluid force on an underwater porthole in Example 7.6.4.

Solution We place the center of the porthole at the origin, meaning the surface of the water is at $y = 50$ and the depth function will be $d(y) = 50 - y$; see Figure 7.6.7

The equation of a circle with a radius of 1.5 is $x^2 + y^2 = 2.25$; solving for x we have $x = \pm\sqrt{2.25 - y^2}$, where the positive square root corresponds to the right side of the circle and the negative square root corresponds to the left side of the circle. Thus the length function at depth y is $\ell(y) = 2\sqrt{2.25 - y^2}$. Integrating on $[-1.5, 1.5]$ we have:

$$\begin{aligned} F &= 62.4\int_{-1.5}^{1.5} 2(50-y)\sqrt{2.25-y^2}\,dy \\ &= 62.4\int_{-1.5}^{1.5} \left(100\sqrt{2.25-y^2} - 2y\sqrt{2.25-y^2}\right)dy \\ &= 6240\int_{-1.5}^{1.5} \left(\sqrt{2.25-y^2}\right)dy - 62.4\int_{-1.5}^{1.5} \left(2y\sqrt{2.25-y^2}\right)dy. \end{aligned}$$

Notes:

The second integral above can be evaluated using substitution. Let $u = 2.25 - y^2$ with $du = -2y\,dy$. The new bounds are: $u(-1.5) = 0$ and $u(1.5) = 0$; the new integral will integrate from $u = 0$ to $u = 0$, hence the integral is 0.

The first integral above finds the area of half a circle of radius 1.5, thus the first integral evaluates to $6240 \cdot \pi \cdot 1.5^2/2 = 22,054$. Thus the total fluid force on a vertically oriented porthole is $22,054$ lb.

Finding the force on a horizontally oriented porthole is more straightforward:

$$F = \text{Pressure} \times \text{Area} = 62.4 \cdot 50 \times \pi \cdot 1.5^2 = 22,054 \text{ lb.}$$

That these two forces are equal is not coincidental; it turns out that the fluid force applied to a vertically oriented circle whose center is at depth d is the same as force applied to a horizontally oriented circle at depth d.

We end this chapter with a reminder of the true skills meant to be developed here. We are not truly concerned with an ability to find fluid forces or the volumes of solids of revolution. Work done by a variable force is important, though measuring the work done in pulling a rope up a cliff is probably not.

What we are actually concerned with is the ability to solve certain problems by first approximating the solution, then refining the approximation, then recognizing if/when this refining process results in a definite integral through a limit. Knowing the formulas found inside the special boxes within this chapter is beneficial as it helps solve problems found in the exercises, and other mathematical skills are strengthened by properly applying these formulas. However, more importantly, understand how each of these formulas was constructed. Each is the result of a summation of approximations; each summation was a Riemann sum, allowing us to take a limit and find the exact answer through a definite integral.

The next chapter addresses an entirely different topic: sequences and series. In short, a sequence is a list of numbers, where a series is the summation of a list of numbers. These seemingly–simple ideas lead to very powerful mathematics.

Notes:

Exercises 7.6

Terms and Concepts

1. State in your own words Pascal's Principle.

2. State in your own words how pressure is different from force.

Problems

In Exercises 3 – 12, find the fluid force exerted on the given plate, submerged in water with a weight density of 62.4 lb/ft^3.

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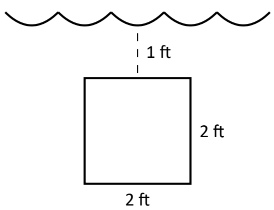

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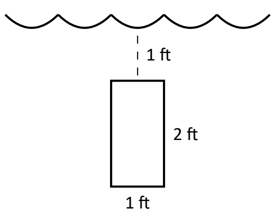

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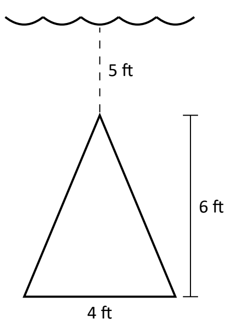

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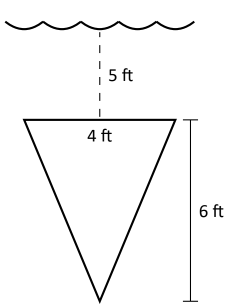

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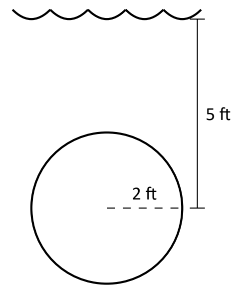

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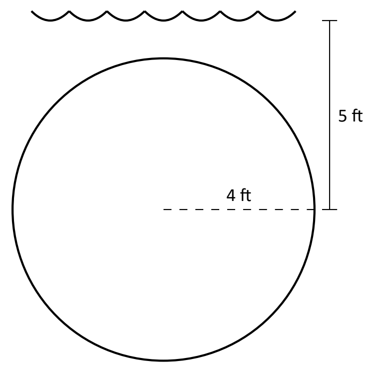

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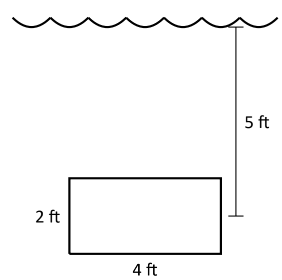

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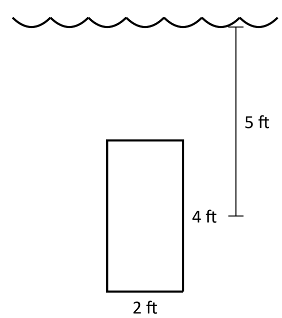

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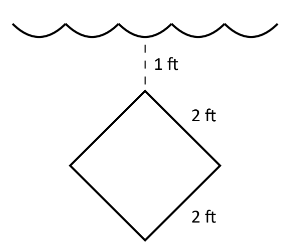

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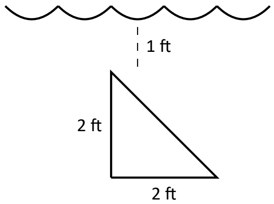
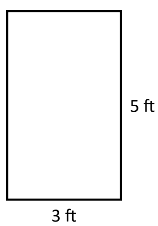


In Exercises 13 – 18, the side of a container is pictured. Find the fluid force exerted on this plate when the container is full of:

1. **water, with a weight density of 62.4 lb/ft^3, and**

2. **concrete, with a weight density of 150 lb/ft^3.**

13.



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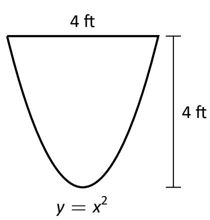

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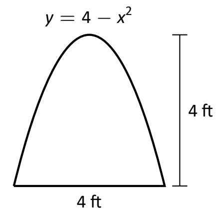

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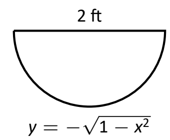

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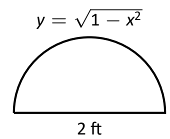
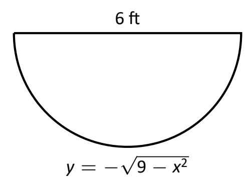


18.



19. How deep must the center of a vertically oriented circular plate with a radius of 1 ft be submerged in water, with a weight density of 62.4 lb/ft^3, for the fluid force on the plate to reach 1,000 lb?

20. How deep must the center of a vertically oriented square plate with a side length of 2 ft be submerged in water, with a weight density of 62.4 lb/ft^3, for the fluid force on the plate to reach 1,000 lb?