Index of the slides:

Natural Deduction and Sequent Calculus

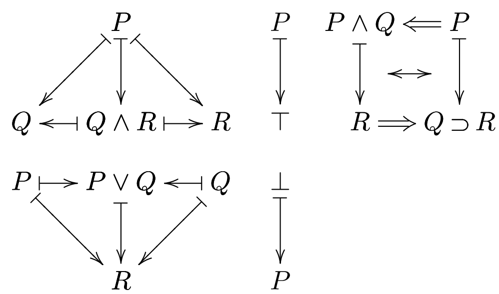
$$\dfrac{\dfrac{\dfrac{[P\&Q]^1}{P} \qquad \dfrac{\dfrac{[P\&Q]^1}{Q} \qquad Q \supset R}{R}}{P\&R}}{P\&Q \supset P\&R}\;1$$

$$\dfrac{\dfrac{\dfrac{P\&Q \vdash P\&Q}{P\&Q \vdash P} \qquad \dfrac{\dfrac{P\&Q \vdash P\&Q}{P\&Q \vdash Q} \qquad Q \supset R \vdash Q \supset R}{Q \supset R, P\&Q \vdash R}}{Q \supset R, P\&Q \vdash P\&R}}{Q \supset R \vdash P\&Q \supset P\&R}$$

Heyting Algebras
A *Heyting Algebra* is a 7-uple
$(\Omega, \top, \bot, \land, \lor, \supset, \vdash)$, where:

$$\begin{array}{rcl} \top, \bot & \in & \Omega, \\ \land, \lor, \supset & : & \Omega \times \Omega \to \Omega, \\ \vdash & \subseteq & \Omega \end{array}$$

and the relation $\vdash$ respects the following "derivation rules":

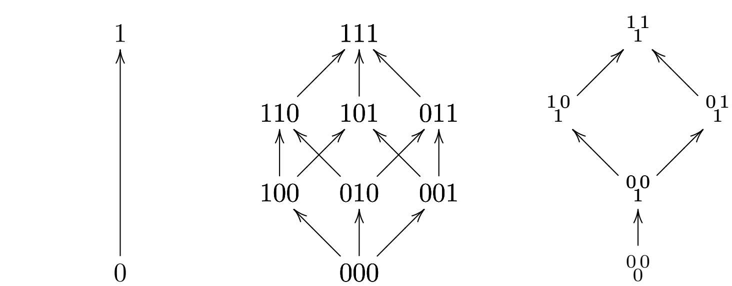


$$\frac{P \vdash Q \quad Q \vdash R}{P \vdash R} \qquad \frac{}{P \vdash P}$$

$$\frac{P \vdash Q \quad P \vdash R}{P \vdash Q \land R} \qquad \frac{}{Q \land R \vdash Q} \qquad \frac{}{Q \land R \vdash R} \qquad \frac{}{P \vdash \top} \qquad \frac{P \land Q \vdash R}{P \vdash Q \supset R} \ (\updownarrow)$$

$$\frac{P \vdash R \quad Q \vdash R}{P \lor Q \vdash R} \qquad \frac{}{P \vdash P \lor Q} \qquad \frac{}{Q \vdash P \lor Q} \qquad \frac{}{\bot \vdash P}$$

Here are three Heyting Algebras:



Note: consider the partial order induced by the DAGs above - i.e., the reflexive/transitive closure of the DAGs.

Some DAGs are Heyting Algebras

Theorem: if
$(\Omega, \top, \bot, \wedge, \vee, \supset, \vdash)$ and
$(\Omega, \top', \bot', \wedge', \vee', \supset', \vdash)$ are Heyting Algebras, then
we have
$\top \leftrightarrow \top'$,
$\bot \leftrightarrow \bot$,
and for any $P, Q \in \Omega$,
$P \wedge Q \leftrightarrow P \wedge' Q$,
$P \vee Q \leftrightarrow P \vee' Q$,
$P \supset Q \leftrightarrow P \supset' Q$.

Proof (half of it):

$$\overline{\top \vdash \top'} \qquad \overline{\bot \vdash \bot'}$$

$$\frac{\overline{P \vdash P \vee' Q} \quad \overline{Q \vdash P \vee' Q}}{P \vee Q \vdash P \vee' Q} \qquad \frac{\overline{P \vdash P \vee' Q} \quad \overline{Q \vdash P \vee' Q}}{P \vee Q \vdash P \vee' Q}$$

$$\cfrac{\cfrac{\overline{\overline{(P \supset Q) \&' P \vdash (P \supset Q) \& P}} \quad \cfrac{\overline{P \supset Q \vdash P \supset Q}}{(P \supset Q) \& P \vdash Q}}{(P \supset Q) \&' P \vdash Q}}{P \supset Q \vdash P \supset' Q}$$

Theorem: if
$(\Omega, \top, \bot, \wedge, \vee, \supset, \vdash)$ and
$(\Omega, \top', \bot', \wedge', \vee', \supset', \vdash)$ are Heyting Algebras and
$(\Omega, \vdash)$ is a DAG, then
$\top = \top'$, $\bot = \bot'$, $\wedge = \wedge'$, $\vee = \vee'$, $\supset = \supset'$.

So, if a DAG $(\Omega, \vdash)$ is Heyting Algebra,
then it is a Heyting Algebra in a unique way:
$\top, \bot, \wedge, \vee, \supset$ are well-defined.

Amazing fact:
For any topological space $(X, \mathcal{O}(X))$,
the DAG $(\mathcal{O}(X), \subseteq)$ is a Heyting Algebra.

Calculating $V \supset W$

What is $V \supset W$?

Idea: look at all Us such that $U \& V \vdash W$.

$$\begin{array}{ccc} ? \wedge V & \longleftarrow\!\dashv & ? \\ \downarrow & \leftrightarrow & \downarrow \\ W & \longmapsto & V \supset W \end{array}$$

$$\begin{array}{ccc} \emptyset \wedge V & \longleftarrow\!\dashv & \emptyset \\ \downarrow & \leftrightarrow & \downarrow \\ W & \longmapsto & V \supset W \end{array} \qquad \begin{array}{ccc} W \wedge V & \longleftarrow\!\dashv & W \\ \downarrow & \leftrightarrow & \downarrow \\ W & \longmapsto & V \supset W \end{array} \qquad \begin{array}{ccc} U \wedge V & \longleftarrow\!\dashv & U \\ \downarrow & \leftrightarrow & \downarrow \\ W & \longmapsto & V \supset W \end{array}$$

In $\mathcal{O}(\mathbb{V})$, this works for these open sets: $\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \\ & 1 & \end{smallmatrix}$.

Define $V \supset W$ as the greatest of them.

More formally:

$V \supset W := \sup\{\, U \mid U \& V \vdash W \,\}$

$V \supset W := \bigcup \{\, U \mid U \& V \vdash W \,\}$

$V \supset W := \bigcup \{\, U \mid U \cap V \subseteq W \,\}$

$V \supset W := \bigcup \{\, A^\circ \mid A^\circ \cap V \subseteq W \,\}$

$V \supset W := \bigcup \{\, A^\circ \mid A^\circ \subseteq W \cup (X \backslash V) \,\}$

$V \supset W := (W \cup (X \backslash V))^\circ$

"Mundo funcional" e "Mundo lógico" (Curry-Howard)

Compare a prova abaixo à esquerda, em Dedução Natural, de que $Q \supset R$ implica $P \land Q \supset P \land R$, com a construção do termo $\lambda d{:}A{\times}B.\langle \pi d, f(\pi' d)\rangle$: $(A{\times}B \to A{\times}C)$ em λ-cálculo simplesmente tipado:

$$\dfrac{\dfrac{\dfrac{[P\land Q]^1}{P} \qquad \dfrac{\dfrac{[P\land Q]^1}{Q} \qquad Q\supset R}{R}}{P\land R}}{(P\land Q\supset P\land R)}\;1 \qquad\qquad \dfrac{\dfrac{\dfrac{[d{:}A{\times}B]^1}{\pi d{:}A} \qquad \dfrac{\dfrac{[d{:}A{\times}B]^1}{\pi' d{:}B} \qquad f{:}B\to C}{f(\pi' d){:}C}}{\langle \pi d, f(\pi' d)\rangle{:}A{\times}C}}{\lambda d{:}A{\times}B.\langle \pi d, f(\pi' d)\rangle : A{\times}B\to A{\times}C}\;1$$

As duas têm exatamente a mesma estrutura. Isto é um exemplo do Isomorfismo de Curry-Howard em funcionamento; ele diz que há uma bijeção natural entre derivações em Dedução Natural e termos de λ-cálculo simplesmente tipado. Repare que na árvore um λ-cálculo os termos sempre crescem à medida que descemos; se usamos uma nova notação — "downcased types" — podemos não só manter os termos pequenos, como suprimir os tipos — os tipos podem ser reconstruídos "convertendo para maiúsculas" os termos. Note que os "conectivos" também têm que ser convertidos: ',' convertido para maiúscula vira '$\times$', e '$\mapsto$' convertido para maiúscula vira '$\to$'.

$$\dfrac{\dfrac{\dfrac{[a,b]^1}{a} \qquad \dfrac{\dfrac{[a,b]^1}{b} \qquad b{\mapsto}c}{c}}{a,c}}{a,b{\mapsto}a,c} \qquad\qquad \begin{array}{rcl} a,b &:=& d \\ b{\mapsto}c &:=& f \\ && \\ b &:=& \pi'(a,b) \\ c &:=& (b{\mapsto}c)(b) \\ a,c &:=& \langle a,c\rangle \\ a,b{\mapsto}a,c &:=& \lambda(a,b).(a,c) \end{array}$$

Agrora cada barra da árvore define um novo termo a partir de termos anteriores; isto gera o dicionário à direita... e a semântica de cada barra passa a ser: "se eu sei o significado dos termos acima da barra, eu sei o significado do termo abaixo da barra", ou: "se eu sei 'a' e sei 'c' eu sei 'a,c'", "se eu sei 'b' e '$b{\mapsto}c$' eu sei 'c'", etc.

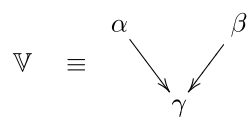
Os "termos" em DNC funcionam de um modo bem diferente dos termos de λ-cálculo. Em DNC nós permitimos nomes longos para variáveis (por exemplo, 'a,b'), a distinção sintática entre variáveis e termos não-primitivos não existe, e, aliás, sem o dicionário não é nem possível determinar só pelos nomes de dois termos qual é "mais primitivo" que o outro: por exemplo, $b{\mapsto}c$ é mais primitivo que c mas $a,b{\mapsto}a,c$ é menos primitivo que a,c.

Preamble: DGs and topologies
A *directed graph* is a set of worlds, W,
and a relation $R \subseteq W \times W$.

Important fact:
DGs induce topologies,
topologies induce DGs,
and in the finite case (which is what matters to us)
the correspondence **DG** $\leftrightarrows$ **Top** is especially well-behaved.

This will be our archetypical DAG:

$$\mathbb{V} := (W, R) := (\{\,\alpha, \beta, \gamma\,\}, \{\,\alpha \to \gamma, \beta \to \gamma\,\})$$



This will be the topological space induced by $\mathbb{V}$:

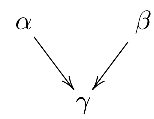
$$(\mathbb{V}, \mathcal{O}(\mathbb{V})) := (\{\,\alpha, \beta, \gamma\,\}, \{\,\emptyset, \{\,\gamma\,\}, \{\,\alpha, \gamma\,\}, \{\,\beta, \gamma\,\}, \{\,\alpha, \beta, \gamma\,\}\,\})$$

We will use the correspondence mainly to represent
finite topological spaces by their associated DGs (or DAGs).

Preamble: each DG induces a topology
A function $f : W \to \{ 0, 1 \}$ is "*non-decreasing* (on R)" when all arrows in R are "non-decreasing".

$$\mathbb{V} := (W, R) := (\{ \alpha, \beta, \gamma \}, \{ \alpha \to \gamma, \beta \to \gamma \})$$

$\mathbb{V} \quad \equiv$



$\substack{1\,0\\0}$ decreases on the arrow $\alpha \to \gamma$: $f(\alpha \to \gamma) = 1 \to 0$.
$\substack{0\,1\\0}$ decreases on $\beta \to \gamma$.
$\substack{1\,1\\0}$ decreases on both $\alpha \to \gamma$ and $\beta \to \gamma$.
The non-decreasing functions $\mathbb{V} \to \{ 0, 1 \}$ are $\substack{0\,0\\0}, \substack{0\,0\\1}, \substack{0\,1\\1}, \substack{1\,0\\1}, \substack{1\,1\\1}$.
A "*non-decreasing subset*" $W' \subseteq W$ is one whose characteristic function is non-decreasing (on R).
Definition:

$$\text{Nondecr}(W, R) := \{ W' \subseteq W \mid W' \text{ is non-decreasing on } R \}$$

For a DG $\mathbb{D} := (W, R)$ the induced topological space is:

$$(\mathbb{D}, \mathcal{O}(\mathbb{D})) := (W, \mathcal{O}(\mathbb{D})) := (W, \text{Nondecr}(W, R))$$

For the dag $\mathbb{V}$ above, this is:

$$(\mathbb{V}, \mathcal{O}(\mathbb{V})) := (\{ \alpha, \beta, \gamma \}, \{ \emptyset, \{ \gamma \}, \{ \alpha, \gamma \}, \{ \beta, \gamma \}, \{ \alpha, \beta, \gamma \} \})$$

Fact: topologies induced by DGs are closed by arbitrary intersections of open sets — not just by finite intersections.

Preamble: truth-values
Abuse of language:
We will often write subsets of W (non-decreasing or not)
as if they were the corresponding functions $W \to \{0,1\}$.
So, for example:
$\{\beta,\gamma\} \equiv \substack{0\,1\\1}$,
$\mathcal{O}(\mathbb{V}) = \mathrm{Nondecr}(\mathbb{V}) \equiv \{\substack{0\,0\\0}, \substack{0\,0\\1}, \substack{0\,1\\1}, \substack{1\,0\\1}, \substack{1\,1\\1}\}$.

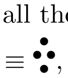
Terminology:
A function $W \to \{0,1\}$ is a "*modal truth-value*".
A non-decreasing function $W \to \{0,1\}$ is an "*intuitionistic truth-value*".
We will see later that the modal truth-values live in a category $\mathbf{Set}^W$
and that the intuitionistic truth-values live in a category $\mathbf{Set}^{\mathbb{D}}$.

Big fact: we can interpret propositional logic on modal truth-values...
just operate on each world separately, e.g.: $\substack{0\,1\\1} \wedge \substack{1\,0\\1} = \substack{0\,0\\1}$.
On modal truth-values the "logic" is boolean but not two-valued.

Bigger fact: the intuitionistic truth-values on a DAG $\mathbb{D}$ form
a "Heyting algebra", $\mathcal{O}(\mathbb{D})$ — we can interpret propositional logic there,
but it will be *intuitionistic* — we can't prove $\neg\neg P \supset P$ there because
that is *not always true*: take $P := \substack{0\,0\\1}$, then $\neg\neg P \equiv \substack{1\,1\\1}$.

Mind-blowing fact: the notion of "taking the union of all open sets in
a given cover" can be interpreted as a *new logical operation*, obeying
some axioms: namely, $\top^* = \top$, $P^{**} = P^*$, $P^* \wedge Q^* = (P \wedge Q)^*$.
This "taking the union..." operation is a particular case of something much
more general: *Lawvere-Tierney topologies*, that generalize both
sheaves and **forcing**.

We can understand sheaves through logic.

Tiny, but amazing, fact: we can understand all these ideas
from the cases of the DAGs $\mathbb{V} \equiv$ and $\mathcal{O}(\mathbb{V})^{\mathrm{op}} \equiv$ ,
and then generalizing. This tiny & amazing fact — that in a sense
is trivial, and is little more than working out in full detail
a few chosen exercises from topos theory books —
is the guiding thread for these notes.

Topological spaces
A topological space is a pair $(X, \mathcal{O}(X))$
where $\mathcal{O}(X) \subset \mathcal{P}(X)$ is a topology on the set X:
(i) $X \in \mathcal{O}(X)$, $\emptyset \in \mathcal{O}(X)$,
(ii) $\mathcal{O}(X)$ is closed by arbitrary unions,
(iii) $\mathcal{O}(X)$ is closed by finite intersections.

Sometimes an $\mathcal{O}(X)$ is closed by **arbitrary intersections**...
This happens for $(\mathbb{V}, \mathcal{O}(\mathbb{V}))$ and for $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$, but not for $(\mathbb{R}, \mathcal{O}(\mathbb{R}))$.
When this happens we say that $\mathcal{O}(X)$ is an *Alexandroff topology*,
and that $(X, \mathcal{O}(X))$ is an *Alexandroff space*.
We will refer to these things by more mnemonic names:
$\mathcal{O}(X)$ is a "topcai", $(X, \mathcal{O}(X))$ is a "topcai space".

There is an inclusion of categories - a functor:

$$\mathbf{TopCAI} \rightarrowtail \mathbf{Top}$$

and we can take a topology $\mathcal{O}(X)$ and look at the set of
arbitrary intersections of its open sets, $\bigcap_\infty \mathcal{O}(X)$ -
it turns out that $\bigcap_\infty \mathcal{O}(X)$ is closed by arbitrary unions,
and is a topology - actually a topcai.

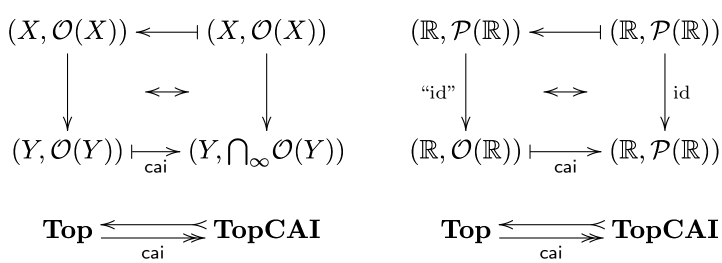
This operation - "closing by arbitrary intersections" - is a functor:

$$\begin{array}{rccc} \mathsf{cai}: & \mathbf{Top} & \to & \mathbf{TopCAI} \\ & (X, \mathcal{O}(X)) & \mapsto & (X, \bigcap_\infty \mathcal{O}(X)) \end{array}$$

and there is an adjunction $(\rightarrowtail) \dashv \mathsf{cai}$.

(**TopCAI** is a "coreflective subcategory" of **Top** -
the inclusion funtor $(\rightarrowtail)$ has a right adjoint).

Note that its counit on $(\mathbb{R}, \mathcal{O}(\mathbb{R}))$ is the
continuous map "id" : $(\mathbb{R}, \mathcal{P}(\mathbb{R})) \to (\mathbb{R}, \mathcal{O}(\mathbb{R}))$:



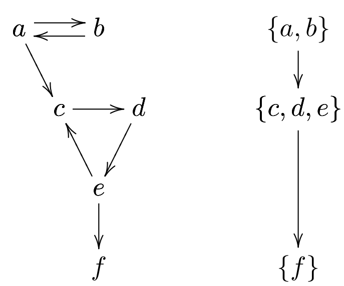
Preorders and partial orders

A *preorder* on W is a relation $(\leq) \subset W \times W$ that is:
(i) reflexive: $a \leq a$
(ii) transitive: if $a \leq b$ and $b \leq c$, then $a \leq c$.
A *partial order* is a preorder that is also:
(iii) anti-symmetric: if $a \leq b$ and $b \leq a$, then $a = b$.

A directed graph (W, R) induces a preorder $(W, \leq) := (W, R^*)$...
Mnemonic: the '$*$' is a Kleene star: if $aRa_1Ra_2Ra_3Rb$ then aR^4b, and thus aR^*b; "R^*" means "at least zero 'R's".
More formally: $R^* := R^0 \cup R^1 \cup R^2 \cup R^3 \cup \ldots$,
the reflexive/transitive closure of R.
(R^0 is the diagonal — aR^0b iff $a = b$).

Each cycle in a DG (W, R) becomes a set of
"equivalent elements" in the induced preorder
Let's consider just DAGs for a while.
DAGs induce partial orders — aciclicity leads to antisymmetry.

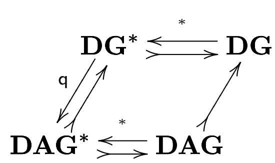
A DG can be converted to a DAG by identifying the elements
in each cycle:



$$\mathbf{DG} \xrightarrow{\mathsf{q}} \mathbf{DAG}$$

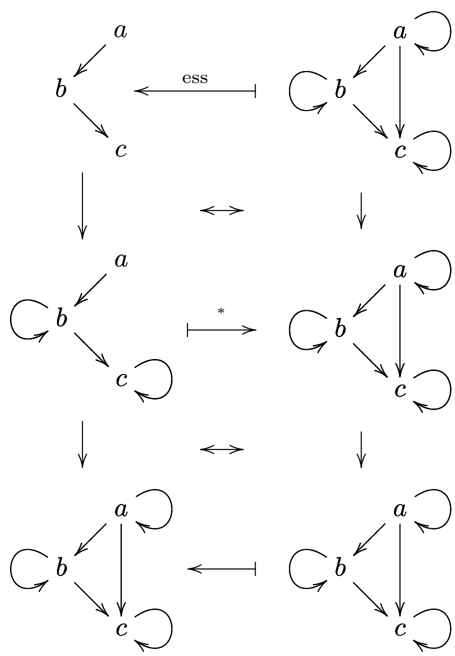
It turns out that the inclusions $\mathbf{DG}^* \rightarrowtail \mathbf{DG}$ and $\mathbf{DAG}^* \rightarrowtail \mathbf{DAG}$
have left adjoints: in both cases, $* \dashv (\rightarrowtail)$,
and the units of the adjunctions take a DG or DAG (W, R)
to its reflexive and transitive closure.

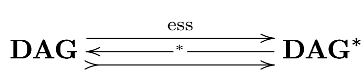
Also, the inclusion $\mathbf{DAG}^* \rightarrowtail \mathbf{DG}^*$ have a left adjoint: 'q'.



The minimal DAG for a topology
Each DG $\mathbb{D} = (W, R)$ induces a topcai: $(W, \mathrm{Nondecr}(R))$ -
but several DGs induce the same topcai.
One canonical way to represent a topcai by a DG is to pick the associated DG* - it is the maximal DG generating that topcai.

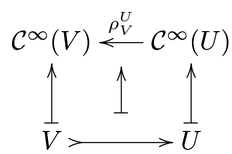
For *finite DAGs* - i.e., for finite T_0 topological spaces -
there is also a *minimal* DAG generating that topology...
The process to obtain it is to drop all the arrows that are not "essential".
Here's an example:
(by the way: $\text{ess} \dashv {}^* \dashv (\leftarrowtail)$)





(The moral is that there is something canonical about
representing topologies (T_0, and on finite sets)
by DAGs with very few arrows)

Presheaves
A *presheaf* on $(X, \mathcal{O}(X))$ is a
(contravariant) functor $\mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$.
A *sheaf* is a presheaf obeying a
"glueing condition", that we will see later.
Example: $\mathcal{C}^\infty : \mathcal{O}(\mathbb{R})^{\mathrm{op}} \to \mathbf{Set}$. If $V \subset U$, then:

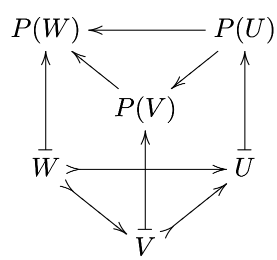


The map $\rho_V^U := \mathcal{C}^\infty(V \rightarrowtail U)$ is called the
"restriction function".

We will borrow some terminology from the case of functions
defined over open sets: for a presheaf $P : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ and
$W \subseteq V \subseteq U$,

$p_U \in P(U)$	a "function/element with support U"
$p_X \in P(X)$	a "global function/element"
$\rho_V^U : P(U) \to P(V)$	"restriction function"
$(\rho_V^U := P(V \rightarrowtail U))$	

Functoriality means two conditions on restriction maps:
$P(U \rightarrowtail U) = \mathrm{id}_{P(U)}$
$P(W \rightarrowtail V) \circ P(V \rightarrowtail U) = P(W \rightarrowtail U)$

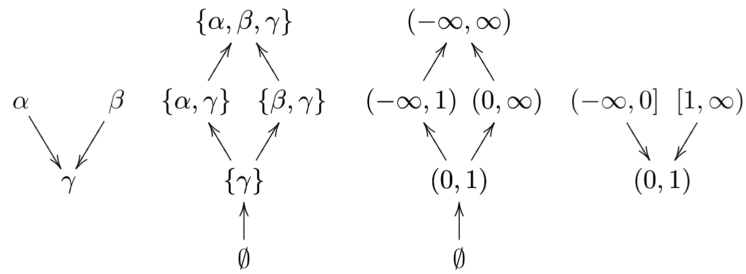


A subtopology of $\mathbb{R}$

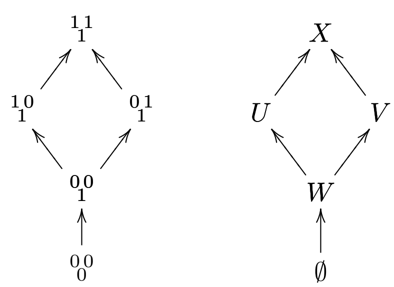
The topology on the DAG $\mathbb{V}$ can be seen as a subtopology of $\mathbb{R}$...
Consider the quotient q below, or, equivalently, q':

$$\begin{array}{l} q : \mathbb{R} \to \{(-\infty, 0],\ (0, 1),\ [1, \infty)\} \\ q' : \mathbb{R} \to \{\alpha, \gamma, \beta\} \end{array}$$

${q'}^{-1}(\mathcal{P}(\{\alpha, \gamma, \beta\})) \ \subset\ \mathcal{P}(\mathbb{R})$ is a topology on $\mathbb{R}$ with 8 open sets.
${q'}^{-1}(\mathcal{P}(\{\alpha, \gamma, \beta\})) \cap \mathcal{O}(\mathbb{R}) \ \subset\ \mathcal{O}(\mathbb{R}) \ \subset\ \mathcal{P}(\mathbb{R})$
is a topology on $\mathbb{R}$ with 5 open sets.
Compare:



We will refer to these open sets as X, U, V, W, $\emptyset$:



Note that U will sometimes mean a specific open set - ${}^{1\,0}_{\ 1}$ -,
sometimes an arbitrary open set; same for the other letters.

Coherent families
Now let $X := \mathbb{R}$, and let's consider two functions defined on subsets of X:

$$\begin{array}{rccc} x_U: & U & \to & \mathbb{R} \\ & x & \mapsto & x \\ 0_U: & U & \to & \mathbb{R} \\ & x & \mapsto & 0 \end{array}$$

(I.e., we're defining $x_X, x_U, x_V, x_W, x_\emptyset$, $0_X, 0_U, 0_V, 0_W, 0_\emptyset$).

We can also consider families of functions, whose supports are families of open sets - $\{x_U, x_V\}$ and $\{x_U, 0_V\}$ are families with support $\{U, V\}$.
Note: $\{x_U, x_V\, 0_V\}$ is *not* a family with support $\{U, V\}$ because V has two "images": x_V and 0_V.

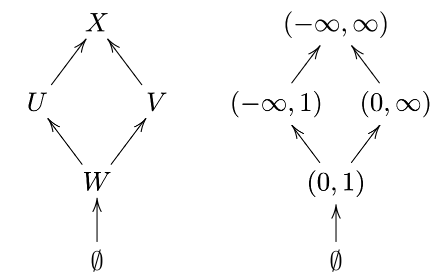
A function defined on U - say, x_U - induces a family $\{x_U\}$ defined on $\{U\}$, i.e., on $\substack{0\\1\,0\\0\\0}$ - and another family, $\{x_U, x_W, x_\emptyset,\}$, defined on all open sets under U - i.e., on the saturation of $\{U\} = \substack{0\\1\,0\\0\\0}$, which is $\substack{0\\1\,0\\1\\1}$.

When we try to extend the family $\{x_U, 0_V\}$ to the saturation of $\substack{0\\1\,1\\0\\0}$, i.e., to $\substack{0\\1\,1\\1\\1}$, we see that we get two different candidates for W - $x_W \neq 0_W$ - which is not good...

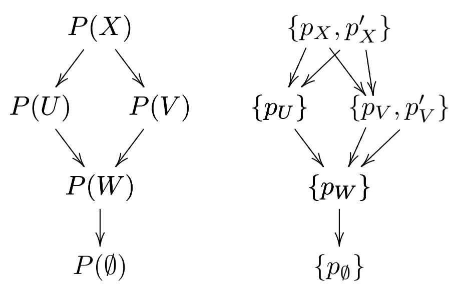
A family is said to be *coherent* when its extension to the saturation of its support is well-defined.
$\{x_U, x_V\}$ is coherent, $\{x_U, 0_V\}$ is not.
Here's a way to define formally coherence for families:
a family $a_\mathcal{U}$ is coherent iff $\forall a_U, a_V \in a_\mathcal{U}\ a_U|_{U\cap V} = a_V|_{U\cap V}$.
Note that $\{x_U, 0_V, 0_W\}$ is not coherent.

Saturation and bisaturation
Notation: the calligraphic letters $\mathcal{U}, \mathcal{V}, \mathcal{W}$
will denote families of open sets, and the annotations
'$\circ$', '$\bullet$', '$\bullet\bullet$' will indicate how saturated a family is -
$\mathcal{U}^\circ$: not necessarily saturated
$\mathcal{U}^\bullet$: saturated
$\mathcal{U}^{\bullet\bullet}$: bisaturated
We will sometimes use $\bullet$, $\bullet\bullet$ to denote *operations*:
$\bullet$ is "saturate", $\bullet\bullet$ is "bisaturate".

A (bad) presheaf on a DAG



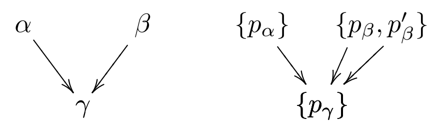
Here is a presheaf over $(X, \mathcal{O}(X))$ ("P") that is
not a sheaf - it violates the two sheaf conditions.
P is not collated - because
$\{p_U, p_{V'}\}$ is a coherent family (on $\{U, V\}$)
that cannot be collated to a global function.
P is not separated - because
there are two different collations for $\{p_U, p_V\}$.



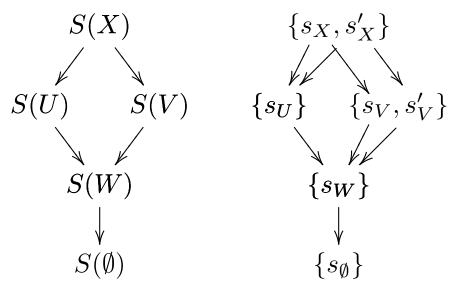
A presheaf on a DAG: its space of germs

Its space of germs is built like this:
for each point in X - i.e., for α, β, γ; let's look at α - look at all open sets containing α (namely: $U = \{\alpha, \gamma\}, X = \{\alpha, \beta, \gamma\}$) and take the colimit of P on these open sets as they get smaller and smaller.
As there is a smallest open set containing α - and β, and γ - these colimits/germs are very easy to calculate:

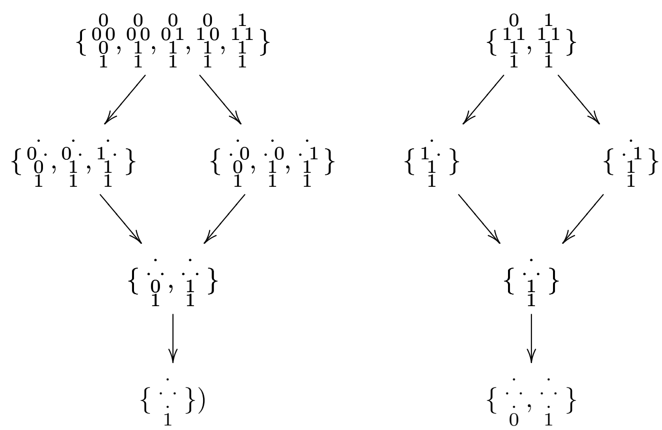
$$\begin{aligned} p_\alpha &:= p_U \\ p_\beta &:= p_V \\ p'_\beta &:= p'_V \\ p_\gamma &:= p_W \end{aligned}$$



The projection map $E \to X$ is the obvious one.
We need to put a topology to E; it turns out (why?) that the right topology is the one induced by the obvious graph.
Now this induces a sheaf of sections...
(I am skipping some steps -)



Dense and stable truth-values
At the left below we see the representation as a presheaf of the "••-stable truth-values", $\{\,\omega \mid \omega^{\bullet\bullet} = \omega\,\} \subset \Omega$;
It is a sheaf, and it can be recovered from its "global elements".



At the right above we see the representation as a presheaf of the "••-dense truth-values", $\Omega_{\bullet\bullet} := \{\,\omega \mid \omega^{\bullet\bullet} = \top\,\} \subset \Omega$.
It is not a sheaf, it can't be recovered from its "global elements" as $\{\,\omega \mid \omega^{\bullet\bullet} = \omega\,\}$ can; yet - and I have to admit that I found that *very* surprising - we can recover the modality from the subobject $\Omega_{\bullet\bullet} \rightarrowtail \Omega$, by:
$\omega^{\bullet\bullet} := \omega \in \Omega_{\bullet\bullet}$

Substitution principles for '$\Leftrightarrow$'
We will also use the following "substitution principles":
if P, Q, Q', R, R' are formulas, and R' is obtained from R by replacing some occurrences of Q in it by Q', then

$$\frac{P \vdash Q \Leftrightarrow Q'}{P \vdash R \Leftrightarrow R'} \qquad \frac{P \vdash Q \Leftrightarrow Q' \quad P \vdash R}{P \vdash R'}$$

The "theorems" above - and the ones in the following slides - can be proved using just the sequent calculus rules for intuitionistic propositional logic augmented with the three axioms for '$*$'.

To make the proofs more manageable we will often make use of the "'$\Leftrightarrow$' trick": starting from $P \vdash Q \Leftrightarrow Q'$ and a proof of $P \vdash R$ we can produce a proof of $P \vdash R'$, where R' is R with some occurrences of 'Q' replaced by 'Q''s.

Example:
(...)
To prove these first theorems —
and the ones in the next slides —
we will need some facts about the biconditional, '$\Leftrightarrow$', that is defined as:

$$P \Leftrightarrow Q := (P \supset Q) \wedge (Q \supset P)$$

Lawvere-Tierney Modalities
A *(Lawvere-Tierney) modality* is an operation '$*$' on intuitionistic truth-values obeing the following three axioms:

$$\frac{}{\vdash \top^*} \qquad \frac{P \vdash Q}{P^* \vdash Q^*} \qquad \frac{}{P^{**} \vdash P^*}$$

The supersaturation operation, $P^* := P^{\bullet\bullet}$, is an example of an LT-modality — but there are others:

$P^* := P^{\neg\neg} := \neg\neg P$
$P^* := P^{(\alpha\vee)} := \alpha \vee P$
$P^* := P^{(\beta\supset)} := \beta \supset P$

First theorems:

$P \vdash P^*$

$(P \wedge Q)^* \vdash P^* \wedge Q^*$

$P \wedge Q^* \vdash (P \wedge Q)^*$
$P^* \wedge Q^* \vdash (P \wedge Q)^*$

From what we already have,
we can prove that $P \Leftrightarrow Q$ implies
$P^* \Leftrightarrow Q^*$ in a weak sense:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash P \Leftrightarrow Q}{\vdash P \supset Q}}{P \vdash Q}}{P^* \vdash Q^*}}{\vdash P^* \supset Q^*} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash P \Leftrightarrow Q}{\vdash Q \supset P}}{Q \vdash P}}{Q^* \vdash P^*}}{\vdash Q^* \supset P^*}}{\vdash P^* \Leftrightarrow Q^*}$$

But there isn't much that we can do when $P \Leftrightarrow Q$ is weaker than $\top$...
For example, if $P := {}^{0\,1}_{\;1}$ and $Q := {}^{1\,0}_{\;1}$, then $P \Leftrightarrow Q = {}^{0\,0}_{\;1}$.

We will treat this as an axiom:

$$\frac{}{P \Leftrightarrow Q \vdash P^* \Leftrightarrow Q^*}$$

(Actually this is true for any unary operation
on truth-values of a Heyting algebra).

LT-modalities and 'and'

Theorems:

$P \vdash P^*$

$(P \wedge Q)^* \vdash P^* \wedge Q^*$

$P \wedge Q^* \vdash (P \wedge Q)^*$

$P^* \wedge Q^* \vdash (P \wedge Q)^*$

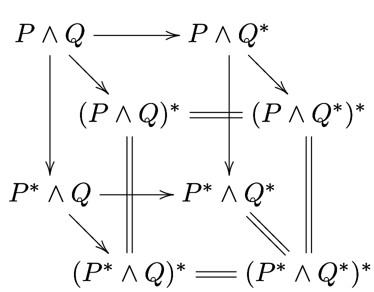
Proofs:

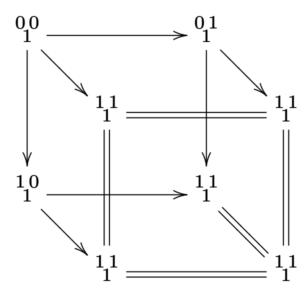
$$\dfrac{\dfrac{\dfrac{}{\vdash \top \Leftrightarrow \top^*} \qquad \dfrac{\dfrac{}{P \vdash \top \Leftrightarrow P}}{P \vdash \top^* \Leftrightarrow P^*}}{P \vdash \top \Leftrightarrow P^*}}{P \vdash P^*}$$

$$\dfrac{\dfrac{\dfrac{}{P \wedge Q \vdash P}}{(P \wedge Q)^* \vdash P^*} \qquad \dfrac{\dfrac{}{P \wedge Q \vdash Q}}{(P \wedge Q)^* \vdash Q^*}}{(P \wedge Q)^* \vdash P^* \wedge Q^*} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash Q \Leftrightarrow (P \wedge Q)}}{P \vdash Q^* \Leftrightarrow (P \wedge Q)^*}}{P \vdash Q^* \supset (P \wedge Q)^*}}{P \wedge Q^* \vdash (P \wedge Q)^*}$$

$$\dfrac{\dfrac{}{P^* \wedge Q^* \vdash (P^* \wedge Q)^*} \qquad \dfrac{\dfrac{}{P^* \wedge Q \vdash (P \wedge Q)^*}}{(P^* \wedge Q)^* \vdash (P \wedge Q)^{**}} \qquad \dfrac{}{(P \wedge Q)^{**} \vdash (P \wedge Q)^*}}{P^* \wedge Q^* \vdash (P \wedge Q)^*}$$

The cube of modalities for '$\wedge$' has only four different truth-values (the case $P^* := P^{\neg\neg}$, $P = \substack{0\,1\\1}$, $Q = \substack{1\,0\\1}$ shows that they are all distinct):

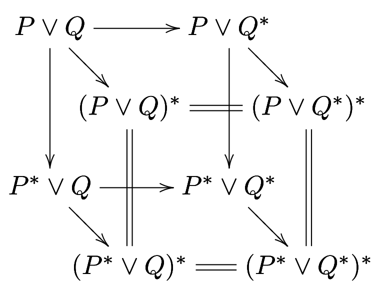


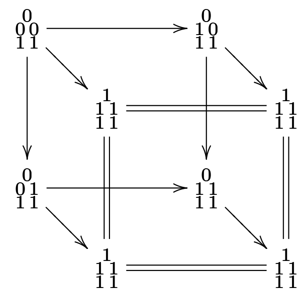


LT-modalities and 'or'
Theorems:

$$\frac{\dfrac{\overline{P \vdash P \vee Q}}{P^* \vdash (P \vee Q)^*} \qquad \dfrac{\overline{Q \vdash P \vee Q}}{Q^* \vdash (P \vee Q)^*}}{P^* \vee Q^* \vdash (P \vee Q)^*} \qquad\qquad \frac{\dfrac{\overline{P^* \vee Q^* \vdash (P \vee Q)^*}}{(P^* \vee Q^*)^* \vdash (P \vee Q)^{**}}}{(P^* \vee Q^*)^* \vdash (P \vee Q)^*}$$

The cube of modalities for '$\vee$' has only five different truth-values (the case $P^* := P^{\neg\neg}$, $P = \substack{0\\0\,0\\0\,1}$, $Q = \substack{0\\0\,0\\1\,0}$ shows that they are all distinct):



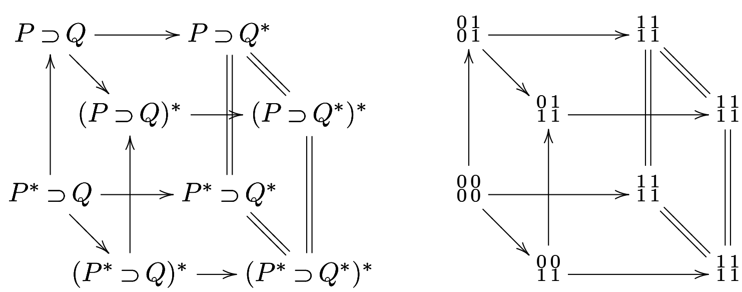


LT-modalities and implication
Theorems:

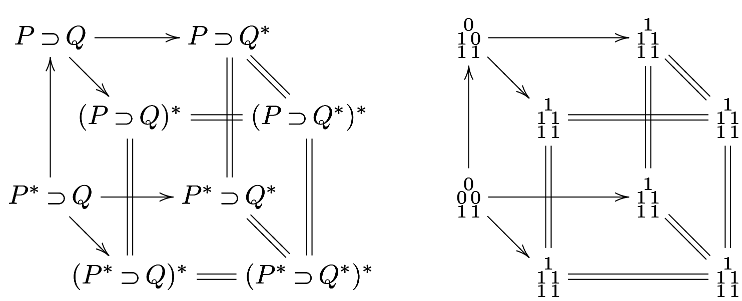
$$\dfrac{\dfrac{\dfrac{\overline{(P \supset Q) \land P \vdash Q}}{((P \supset Q) \land P)^* \vdash Q^*}}{(P \supset Q)^* \land P^* \vdash Q^*}}{(P \supset Q)^* \vdash P^* \supset Q^*} \qquad \dfrac{\overline{P \supset Q \vdash (P \supset Q)^*} \quad \overline{(P \supset Q)^* \vdash P^* \supset Q^*}}{P \supset Q \vdash P^* \supset Q^*}$$

$$\dfrac{\overline{(P \supset Q)^* \vdash P^* \supset Q^*}}{P^* \land (P \supset Q)^* \vdash Q^*} \qquad \dfrac{\dfrac{\overline{P^* \land (P \supset Q^*)^* \vdash Q^{**}} \quad \overline{Q^{**} \vdash Q^*}}{P^* \land (P \supset Q^*)^* \vdash Q^*}}{(P \supset Q^*)^* \vdash P^* \supset Q^*}$$

The cube of modalities for '$\supset$' has only five different truth-values.
The case $P^* := {\substack{0\,0\\1\,1}} \lor P$, $P = {\substack{0\,0\\1\,0}}$, $Q = {\substack{0\,0\\0\,0}}$ distinguishes them all:



When the modality is $P^* := \neg\neg P$ we can't distinguish the four truth-values in the front face of the cube (the '$(P^? \supset Q^?)^*$'s)...
The best that we can do is this. For $P^* := \neg\neg P$, $P = {\substack{0\\0\,0\\0\,1}}$, $Q = {\substack{0\\0\,0\\1\,0}}$,

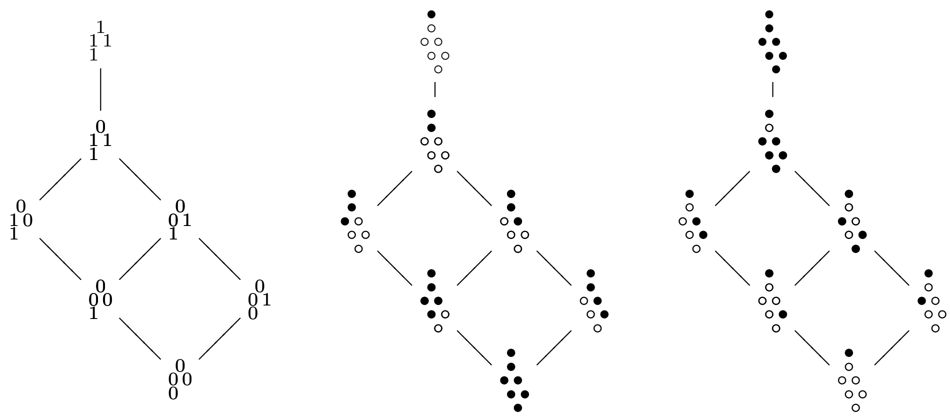


The topologies for 'or' and 'implies'

$\begin{smallmatrix}&0&\\0&&1\\&1&\end{smallmatrix} \vee \begin{smallmatrix}&?&\\?&&?\\&?&\end{smallmatrix} = \begin{smallmatrix}&?&\\?&&1\\&1&\end{smallmatrix}$

$\begin{smallmatrix}&0&\\0&&1\\&1&\end{smallmatrix} \supset \begin{smallmatrix}&?&\\?&&?\\&?&\end{smallmatrix} = (\begin{smallmatrix}&0&\\0&&1\\&1&\end{smallmatrix} \supset_c \begin{smallmatrix}&?&\\?&&?\\&?&\end{smallmatrix})^o = (\neg_c \begin{smallmatrix}&0&\\0&&1\\&1&\end{smallmatrix} \vee_c \begin{smallmatrix}&?&\\?&&?\\&?&\end{smallmatrix})^o = (\begin{smallmatrix}&1&\\1&&0\\&0&\end{smallmatrix} \vee_c \begin{smallmatrix}&?&\\?&&?\\&?&\end{smallmatrix})^o = (\begin{smallmatrix}&1&\\1&&?\\&?&\end{smallmatrix})^o$

The image of an idempotent operator is its fixset.

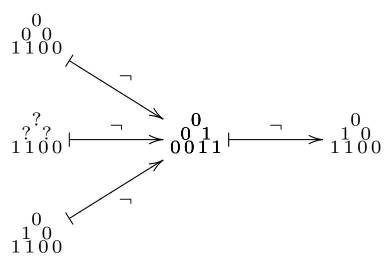


More about double negation

The value of $\neg\neg P$ depends only on the values of P on the terminal worlds (the "leaves").

How to see this:

$\neg P$ is the opposite of P at the leaf-worlds and is the biggest possible (i.e., as 1-ish as possible) in the other worlds;
$\neg\neg P$ coincides with P at the leaf-worlds and is the biggest possible in the other worlds.



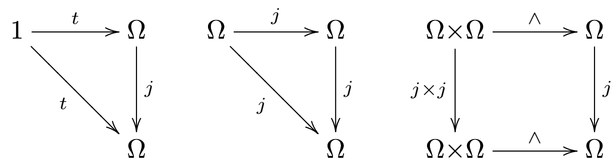
Let's write (temporarily) '$^?$' for "apply '*' or not".
$P^? \wedge Q^?$ stands for: $P \wedge Q$, $P \wedge Q^*$, $P^* \wedge Q$, $P^* \wedge Q^*$ — four truth-values.

Fact: when $^* = \neg\neg$,
$(P^? \wedge Q^?)^*$ is well-defined,
$(P^? \vee Q^?)^*$ is well-defined,
$(P^? \supset Q^?)^*$ is well-defined,
$(\neg P)^*$ is well-defined —
the outer '*' dominates everything and makes all inner applications of '*'s irrelevant.

In the cubes from the previous slides,
"the outer '*' dominates" means:
"the four truth-values in the front face —
the '$(P^? \text{ op } Q^?)^*$'s — are all equivalent".
This is not true for the modality $P^* = P^{(\alpha\vee)} = \alpha \vee P$!

Modalities: alternative axioms

A *Lawvere-Tierney topology* is usually defined as an arrow $j : \Omega \to \Omega$ such that these three diagrams commute:



Which means:

$$\omega[\top] = \omega[\top^*] \qquad \omega[P^*] = \omega[P^{**}] \qquad \omega[P^* \wedge Q^*] = \omega[(P \wedge Q)^*]$$

$$\dfrac{}{\top \vdash \top^*} \qquad \dfrac{}{P^* \vdash P^{**}} \qquad \dfrac{}{P^{**} \vdash P^*}$$

$$\dfrac{}{P^* \wedge Q^* \vdash (P \wedge Q)^*} \qquad \dfrac{}{(P \wedge Q)^* \vdash P^* \wedge Q^*}$$

It is not clear that these axioms ("LT axioms")
are equivalent to the three axioms ("LT-modality axioms")
that we were using before...
We know that the modality axioms imply all the LT axioms,
but it is not obvious that the modality axioms $\top \vdash \top^*$
and $\dfrac{P \vdash Q}{P^* \vdash Q^*}$
can be proved from the LT axioms...

Here are the proofs:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash \top \Leftrightarrow P}}{P \vdash \top^* \Leftrightarrow P^*}}{P \vdash \top \Leftrightarrow P^*}}{P \vdash P^*} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \vdash Q}{\top \vdash P \supset Q}}{\top \vdash P \Leftrightarrow (P \wedge Q)}}{\top \vdash P^* \Leftrightarrow (P^* \wedge Q^*)}}{\top \vdash P^* \supset Q^*}}{P^* \vdash Q^*}$$

LT-modalities and the quantifiers
Quantifiers:

$$\dfrac{\dfrac{\dfrac{}{P \vdash \exists x.P}}{P^* \vdash (\exists x.P)^*}}{\exists x.P^* \vdash (\exists x.P)^*} \qquad \dfrac{\dfrac{\dfrac{}{\exists x.P^* \vdash (\exists x.P)^*}}{(\exists x.P^*)^* \vdash (\exists x.P)^{**}}}{(\exists x.P^*)^* \vdash (\exists x.P)^*}$$

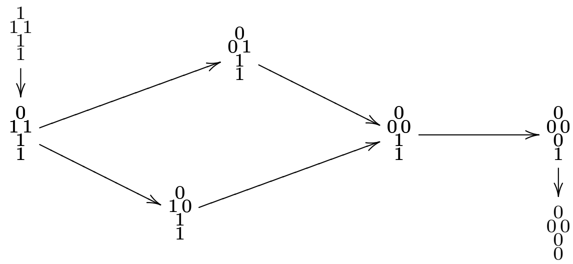
$$\dfrac{\dfrac{\dfrac{}{\forall x.P \vdash P}}{(\forall x.P)^* \vdash P^*}}{(\forall x.P)^* \vdash \forall x.P^*} \qquad \dfrac{\dfrac{}{(\forall x.P^*)^* \vdash \forall x.P^{**}}}{(\forall x.P^*)^* \vdash \forall x.P^*}$$

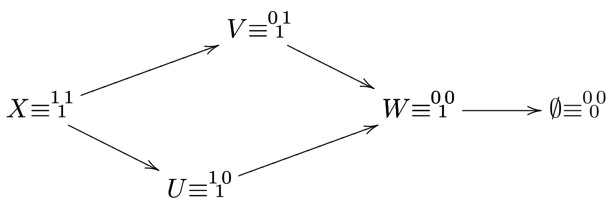
$$\begin{array}{ccc} \exists x.P & \longrightarrow & \exists x.P^* \\ \downarrow & & \downarrow \\ (\exists x.P)^* & = & (\exists x.P^*)^* \end{array} \qquad \begin{array}{ccc} \forall x.P & \longrightarrow & \forall x.P^* \\ \downarrow & & \| \\ (\forall x.P)^* & \longrightarrow & (\forall x.P^*)^* \end{array}$$

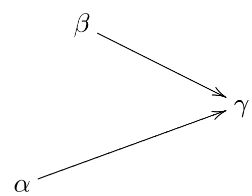
The fibration of saturated covers

For a DAG $\mathbb{D}$, define $\mathbb{D}' := \mathcal{O}(\mathbb{D})^{\mathrm{op}} \equiv (\mathcal{O}(\mathbb{D}), \supseteq)$.

For example, when $\mathbb{V} := \raisebox{0pt}{$\bullet\!\bullet$}$, we have:



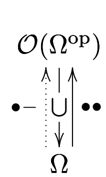


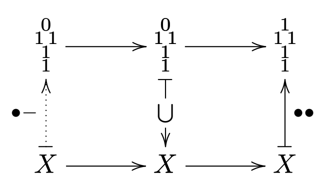


The projection $\bigcup : \mathcal{O}(\Omega^{\mathrm{op}}) \to \Omega$ respects $\wedge$ and $\vee$, i.e.,
if $\bigcup \mathcal{U}^\bullet = U$ and $\bigcup \mathcal{V}^\bullet = V$ then
$\bigcup(\mathcal{U}^\bullet \vee \mathcal{V}^\bullet) = U \vee V$ (this is easy to see), and also
$\bigcup(\mathcal{U}^\bullet \wedge \mathcal{V}^\bullet) = U \wedge V$ (look at each $w \in \bigcup(\mathcal{U}^\bullet \wedge \mathcal{V}^\bullet)$).

Each fiber $\bigcup^{-1} U$ is a lattice with top element $\mathcal{U}^{\bullet\bullet}$.
When Ω comes from a finite topology we can take
the intersection of all saturated covers of U,
and this gives a minimal saturated cover for U,
that we will call $\mathcal{U}^{\bullet-}$.

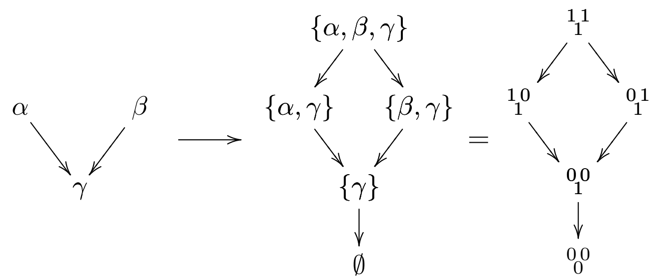
Fact: $\bullet- \dashv \bigcup \dashv \bullet\bullet$.





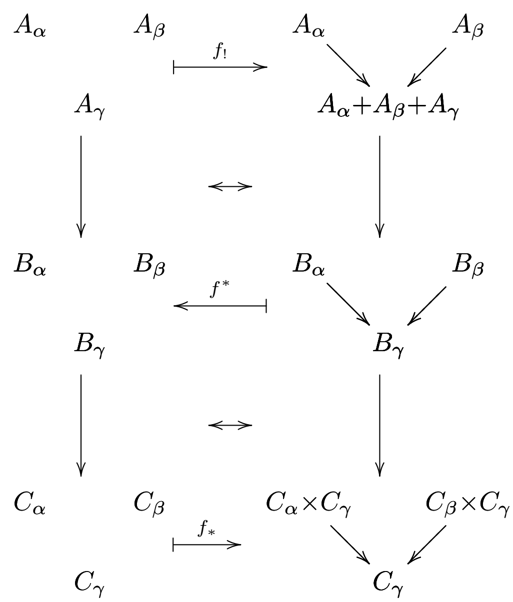
Embedding
A topology is a DAG in a natural way:
if $V, U \in \mathcal{O}(X)$, then $V \to U$ iff $V \subseteq U$.
We will prefer $\mathcal{O}(X)^{\mathrm{op}}$ rather than $\mathcal{O}(X)$, for two
reasons: one is because then we will have a monotonic function

$$\begin{array}{rccc} \downarrow: & \mathbb{D} & \to & \mathcal{O}(\mathbb{D})^{\mathrm{op}} \\ & \alpha & \mapsto & \downarrow \alpha \end{array}$$



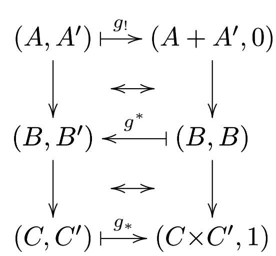
Geometric morphisms

The obvious continuous function $f : 3 \to \mathbb{V}$
induces a geometric morphism, $(f^* \dashv f_*) : \mathbf{Set}^3 \to \mathbf{Set}^V$.
It is essential: $f^! \dashv f^* \dashv f_*$.



$$\mathbf{Set}^3 \;\underset{f_*}{\overset{f_!}{\rightrightarrows}}\; \xleftarrow{f^*}\; \mathbf{Set}^b bV$$

A simpler example:



$$\mathbf{Set}^2 \;\underset{g_*}{\overset{g_!}{\rightrightarrows}}\; \xleftarrow{g^*}\; \mathbf{Set}^2$$