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### Hyperdoctrines: substitution, quantifiers, reflexivity

The substitution rule:

$$\begin{array}{ccc}
 \begin{array}{c} \left( \begin{array}{c} P \\ a, c \end{array} \right) \\ \downarrow a, c; P \vdash Q \end{array} & \begin{array}{c} \xrightarrow{\square} \\ \xleftarrow{\square} \\ \xrightarrow{\square} \\ \xleftarrow{\square} \end{array} & \begin{array}{c} \left( \begin{array}{c} P \\ b, c \end{array} \right) \\ \downarrow b, c; P \vdash Q \end{array} \\
 \begin{array}{c} \left( \begin{array}{c} Q \\ b, c \end{array} \right) \\ \downarrow a, c; P \vdash Q \end{array} & \begin{array}{c} \xrightarrow{\square} \\ \xleftarrow{\square} \\ \xrightarrow{\square} \\ \xleftarrow{\square} \end{array} & \begin{array}{c} \left( \begin{array}{c} Q \\ b, c \end{array} \right) \\ \downarrow b, c; P \vdash Q \end{array} \\
 a, c \vdash & \longrightarrow & b, c \\
 \swarrow a & & \searrow b \\
 a \vdash & \xrightarrow{a \vdash b} & b
 \end{array}$$

The ‘ $(\forall I)$ ’ and ‘ $(\forall E)$ ’ rules:

$$\begin{array}{ccc}
 \begin{array}{c} \left( \begin{array}{c} P \\ a, c \end{array} \right) \\ \downarrow a, b; P \vdash Q \end{array} & \begin{array}{c} \xrightarrow{\square} \\ \xleftarrow{\square} \\ \xrightarrow{\square} \\ \xleftarrow{\square} \end{array} & \begin{array}{c} \left( \begin{array}{c} P \\ b, c \end{array} \right) \\ \downarrow a; P \vdash \forall b. Q \end{array} \\
 \left( \begin{array}{c} Q \\ b, c \end{array} \right) & \Longrightarrow & \left( \begin{array}{c} \forall b. Q \\ b, c \end{array} \right) \\
 a, b \vdash & \longrightarrow & a
 \end{array}
 \qquad
 \begin{array}{ccc}
 \begin{array}{c} \left( \begin{array}{c} P \\ a \end{array} \right) \\ \downarrow a; Q \vdash R \end{array} & \begin{array}{c} \xrightarrow{\square} \\ \xleftarrow{\square} \\ \xrightarrow{\square} \\ \xleftarrow{\square} \end{array} & \begin{array}{c} \left( \begin{array}{c} P \\ a, b \end{array} \right) \\ \downarrow a; Q \vdash \forall b. R \end{array} \\
 \left( \begin{array}{c} Q \\ a \end{array} \right) & \begin{array}{c} \xrightarrow{\square} \\ \xleftarrow{\square} \\ \xrightarrow{\square} \\ \xleftarrow{\square} \end{array} & \left( \begin{array}{c} Q \\ a, b \end{array} \right) \Longrightarrow \left( \begin{array}{c} \forall b. Q \\ a \end{array} \right) \\
 a \vdash & \xrightarrow{a \vdash b} & a, b \vdash \longrightarrow a
 \end{array}$$

The ‘ $(\exists E)$ ’ and ‘ $(\exists I)$ ’ rules:

$$\begin{array}{ccc}
 \begin{array}{c} \left( \begin{array}{c} P \\ a, b \end{array} \right) \\ \downarrow a, b; P \vdash Q \end{array} & \begin{array}{c} \xrightarrow{\text{co}\square} \\ \xleftarrow{\text{co}\square} \\ \xrightarrow{\text{co}\square} \\ \xleftarrow{\text{co}\square} \end{array} & \begin{array}{c} \left( \begin{array}{c} \exists b. P \\ a \end{array} \right) \\ \downarrow a; \exists b. P \vdash Q \end{array} \\
 \left( \begin{array}{c} Q \\ b, c \end{array} \right) & \begin{array}{c} \xrightarrow{\square} \\ \xleftarrow{\square} \\ \xrightarrow{\square} \\ \xleftarrow{\square} \end{array} & \left( \begin{array}{c} Q \\ a, b \end{array} \right) \\
 a, b \vdash & \longrightarrow & a
 \end{array}
 \qquad
 \begin{array}{ccc}
 \begin{array}{c} \left( \begin{array}{c} P \\ a \end{array} \right) \\ \downarrow a; P \vdash \exists b. P \end{array} & \begin{array}{c} \xrightarrow{\square} \\ \xleftarrow{\square} \\ \xrightarrow{\square} \\ \xleftarrow{\square} \end{array} & \begin{array}{c} \left( \begin{array}{c} P \\ a, b \end{array} \right) \\ \downarrow a; P \vdash \exists b. P \end{array} \\
 \left( \begin{array}{c} \exists b. P \\ a \end{array} \right) & \begin{array}{c} \xrightarrow{\square} \\ \xleftarrow{\square} \\ \xrightarrow{\square} \\ \xleftarrow{\square} \end{array} & \left( \begin{array}{c} \exists b. P \\ a, b \end{array} \right) \Longrightarrow \left( \begin{array}{c} \exists b. P \\ a \end{array} \right) \\
 a \vdash & \xrightarrow{a \vdash b} & a, b \vdash \longrightarrow a
 \end{array}$$

The “reflexivity” rule for equality (on variables):

$$\begin{array}{ccc}
 \begin{array}{c} \left( \begin{array}{c} \top \\ a, b \end{array} \right) \\ \downarrow a, b; \top \vdash (b=b) \end{array} & \begin{array}{c} \xrightarrow{\text{co}\square} \\ \xleftarrow{\text{co}\square} \\ \xrightarrow{\text{co}\square} \\ \xleftarrow{\text{co}\square} \end{array} & \begin{array}{c} \left( \begin{array}{c} b=b' \\ a, b, b' \end{array} \right) \\ \downarrow \text{id} \end{array} \\
 \left( \begin{array}{c} b=b \\ a, b \end{array} \right) & \begin{array}{c} \xrightarrow{\square} \\ \xleftarrow{\square} \\ \xrightarrow{\square} \\ \xleftarrow{\square} \end{array} & \left( \begin{array}{c} b=b' \\ a, b, b' \end{array} \right) \\
 a, b \vdash & \xrightarrow{b' := b} & a, b, b'
 \end{array}$$

### Hyperdoctrines: symmetry, transitivity

The “symmetry” rule for equality (on variables):

$$\begin{array}{ccc}
 \left( \begin{array}{c} \top \\ a, b \end{array} \right) & \xRightarrow{\text{co}\square} & \left( \begin{array}{c} b=b' \\ a, b, b' \end{array} \right) \\
 \text{refl} \downarrow & \vdash & \downarrow a, b, b'; (b=b') \vdash (b'=b) \\
 \left( \begin{array}{c} b=b \\ a, b \end{array} \right) & \xRightarrow{\square} & \left( \begin{array}{c} b'=b \\ a, b, b' \end{array} \right) \\
 & & \swarrow \square \\
 & & \left( \begin{array}{c} \top \\ a, b' \end{array} \right) \xRightarrow{\text{co}\square} \left( \begin{array}{c} b'=b \\ a, b', b \end{array} \right) \\
 & & \swarrow \\
 a, b \vdash & \xrightarrow{b' := b} & a, b, b' \\
 & & \searrow \\
 & & a, b' \vdash \xrightarrow{b' := b} a, b', b
 \end{array}$$

The “transitivity” rule for equality (on variables):

$$\begin{array}{ccc}
 \left( \begin{array}{c} b=b' \\ a, b, b' \end{array} \right) & \xRightarrow{\text{co}\square} & \left( \begin{array}{c} b'=b'' \& b=b' \\ a, b, b', b'' \end{array} \right) \\
 \text{id} \downarrow & \vdash & \downarrow a, b, b', b''; (b'=b'') \& (b=b') \vdash (b=b'') \\
 \left( \begin{array}{c} b=b' \\ a, b, b' \end{array} \right) & \xRightarrow{\square} & \left( \begin{array}{c} b=b'' \\ a, b, b', b'' \end{array} \right) \\
 & & \swarrow \square \\
 & & \left( \begin{array}{c} \top \\ a, b \end{array} \right) \xRightarrow{\text{co}\square} \left( \begin{array}{c} b=b'' \\ a, b', b'' \end{array} \right) \\
 & & \swarrow \\
 a, b, b' \vdash & \xrightarrow{b'' := b'} & a, b, b', b'' \\
 & & \searrow \\
 & & a, b \vdash \xrightarrow{b'' := b} a, b', b''
 \end{array}$$

### Weak and strong exists-elim

There are two versions of  $(\exists E)$ ,  $(\exists E^-)$  and  $(\exists E^+)$ .

They are “equivalent enough”, and  $(\exists E^-)$  is much simpler categorically.

The rules, in natural deduction and sequent calculus forms:

$$\frac{[P(a,b)]^1 \quad Q(a)}{\exists b.P(a,b) \quad R(a)} 1; (\exists E^+) \quad \frac{a, b; P(a,b), Q(a) \vdash R(a)}{a; \exists b.P(a,b), Q(a) \vdash R(a)} (\exists E^+)$$

$$\frac{[P(a,b)]^1 \quad Q(a)}{Q(a)} 1; (\exists E^-) \quad \frac{a, b; P(a,b) \vdash Q(a)}{a; \exists b.P(a,b) \vdash Q(a)} (\exists E^-)$$

$(\exists E^-)$  is a particular case of  $(\exists E^+)$  — take  $Q(a) := \top$  in  $(\exists E^+)$ .

In the presence of  $(\supset)$  the rule  $(\exists E^-)$  implies  $(\exists E^+)$ :

$$\frac{[P(a,b)]^2 \quad [Q(a)]^1 \quad \frac{\exists b.P(a,b) \quad \frac{R(a)}{Q(a) \supset R(a)} 1; (\supset I)}{Q(a) \supset R(a)} 2; (\exists E^+)}{R(a)} (\supset E)$$

$(\exists E^-)$ , categorically:

$$\begin{array}{ccc} P(a,b) \Rightarrow \exists b.P(a,b) & & \\ \downarrow \quad \mapsto \quad \downarrow & & \\ Q(a) \Leftarrow Q(a) & & \\ & & a, b \vdash \longrightarrow a \end{array}$$

$(\exists E^+)$ , categorically:

$$\begin{array}{ccccccc} P(a,b) \& Q(a) \Leftarrow P(a,b) \Longrightarrow \exists b.P(a,b) \Longrightarrow (\exists b.P(a,b)) \& Q(a) & & & & \\ \downarrow \quad \mapsto \quad \downarrow \quad \mapsto \quad \downarrow \quad \mapsto \quad \downarrow & & & & & & \\ R(a) \Longrightarrow Q(a) \supset R(a) \Leftarrow Q(a) \supset R(a) \Leftarrow R(a) & & & & & & \\ & & & & & & a, b \Longrightarrow a, b \vdash \longrightarrow a \Longrightarrow a \end{array}$$

Note that  $\mathbf{O}\left[\begin{array}{c} Q(a) \supset R(a) \\ a, b \end{array}\right]$  has two constructions, with a hidden iso between them.

### Rules in ND and sequent calculus

The rules, in natural deduction form (( $\forall E$ ) is wrong):

$\frac{\begin{array}{c} \vdots \\ \vdots \\ Q \quad R \\ \hline Q \& R \end{array}}{(\&I)}$	$P \quad \frac{Q \& R}{Q} (\&E_1) \quad P \quad \frac{Q \& R}{R} (\&E_2)$ $\begin{array}{c} \vdots \\ \vdots \\ S \end{array} \quad \begin{array}{c} \vdots \\ \vdots \\ S \end{array}$
$P \quad [Q]^1$ $\frac{\begin{array}{c} \vdots \\ R \end{array}}{Q \supset R} 1; (\supset I)$	$P \quad P$ $\frac{\begin{array}{c} \vdots \\ Q \end{array} \quad \begin{array}{c} \vdots \\ Q \supset R \end{array}}{R} (\supset E)$
$P(a)$ $\frac{\begin{array}{c} \vdots \\ Q(a, b) \end{array}}{\forall b. Q(a, b)} (\forall I)$	$Q(a)$ $\frac{\begin{array}{c} \vdots \\ f(a) \quad \forall b. R(a, b) \end{array}}{R(a, f(a))} (\forall E)$
$Q(a)$ $\frac{\begin{array}{c} \vdots \\ P(a, f(a)) \end{array}}{\exists b. P(a, b)} (\exists I)$	$[P(a, b)]^1 \quad Q(a)$ $\frac{\exists b. P(a, b) \quad \begin{array}{c} \vdots \\ R(a) \end{array}}{R(a)} (\exists E)$

The rules, in sequent calculus form (( $\forall E$ ) is wrong):

$\frac{P \vdash Q \quad P \vdash R}{P \vdash Q \& R} (\&I)$	$\frac{P, Q \vdash S}{P, Q \& R \vdash S} (\&E_1) \quad \frac{P, R \vdash S}{P, Q \& R \vdash S} (\&E_2)$
$\frac{P, Q \vdash R}{P \vdash Q \supset R} (\supset I)$	$\frac{P \vdash Q \quad P \vdash Q \supset R}{P \vdash R} (\supset E)$
$\frac{a, b; P(a) \vdash Q(a, b)}{a; P(a) \vdash \forall b. Q(a, b)} (\forall I)$	$\frac{a \vdash f(a) \quad a; Q(a) \vdash \forall b. R(a, b)}{a; Q(a) \vdash R(a, f(a))} (\forall E)$
$\frac{a; Q(a) \vdash P(a, f(a))}{a; Q(a) \vdash \exists b. P(a, b)} (\exists I)$	$\frac{a, b; P(a, b), Q(a) \vdash R(a)}{a; \exists b. P(a, b), Q(a) \vdash R(a)} (\exists E)$

**Rules, categorically**

The rules, categorically ((&E) needs explanations, ( $\forall E$ ) is wrong):

$  \begin{array}{c}  P \\  \swarrow \quad \downarrow \quad \searrow \\  Q \xleftarrow{\pi} Q \& R \xrightarrow{\pi'} R  \end{array}  $	$  \begin{array}{ccccc}  Q & \xleftarrow{\pi} & Q \& R & \xrightarrow{\pi'} & R \\  \Downarrow & \downarrow & \Downarrow & \downarrow & \Downarrow \\  P \& Q & \xleftarrow{\pi} & P \& Q \& R & \xrightarrow{\pi'} & P \& R  \end{array}  $
$  \begin{array}{c}  P \& Q \Leftarrow P \\  \downarrow \quad \dashv \quad \downarrow \\  R \Rightarrow Q \supset R  \end{array}  $	$  \begin{array}{c}  P \\  \swarrow \quad \downarrow \quad \searrow \\  Q \xleftarrow{\pi'} (Q \supset R) \& Q \xrightarrow{\pi} Q \supset R \\  \downarrow \quad \dashv \quad \downarrow \text{id} \\  R \Longrightarrow Q \supset R  \end{array}  $
$  \begin{array}{c}  Q(a) \Leftarrow Q(a) \\  \downarrow \quad \dashv \quad \downarrow \\  R(a, b) \Rightarrow \forall b. R(a, b) \\  a, b \dashv \rightarrow a  \end{array}  $	$  \begin{array}{c}  Q(a) \Leftarrow Q(a) \Leftarrow Q(a) \\  \downarrow \quad \dashv \quad \downarrow \quad \dashv \quad \downarrow \\  R(a, f(a)) \Rightarrow R(a, b) \Rightarrow \forall b. R(a, b) \\  a \dashv \xrightarrow{b:=f(a)} a, b \dashv \rightarrow a  \end{array}  $
$  \begin{array}{c}  P(a, f(a)) \Leftarrow P(a, b) \Rightarrow \exists b. P(a, b) \\  \downarrow \quad \dashv \quad \downarrow \quad \dashv \quad \downarrow \\  \exists b. P(a, b) \Leftarrow \exists b. P(a, b) \Leftarrow \exists b. P(a, b) \\  a \dashv \xrightarrow{b:=f(a)} a, b \dashv \rightarrow a  \end{array}  $	(see next page)

### First preservations

Change-of-base preserves terminals and binary products:

$$\begin{array}{ccccc}
 & & \begin{array}{c} (P) \\ \left( \begin{array}{c} P \\ a \end{array} \right) \end{array} & \longleftarrow & \begin{array}{c} (P) \\ \left( \begin{array}{c} P \\ b \end{array} \right) \end{array} \\
 & & \uparrow \pi & \swarrow & \uparrow \pi \\
 \begin{array}{c} (\top) \\ \left( \begin{array}{c} \top \\ a \end{array} \right) \end{array} & \longleftarrow & \begin{array}{c} (P \& Q)' \\ \left( \begin{array}{c} P \& Q \\ a \end{array} \right) \end{array} & \longleftarrow & \begin{array}{c} (P \& Q) \\ \left( \begin{array}{c} P \& Q \\ a \end{array} \right) \end{array} & \longleftarrow & \begin{array}{c} (P \& Q) \\ \left( \begin{array}{c} P \& Q \\ b \end{array} \right) \end{array} \\
 \downarrow \natural & & \downarrow \pi' & \swarrow & \downarrow \pi' \\
 \begin{array}{c} (\top)' \\ \left( \begin{array}{c} \top \\ a \end{array} \right)' \end{array} & & \begin{array}{c} (Q) \\ \left( \begin{array}{c} Q \\ a \end{array} \right) \end{array} & \longleftarrow & \begin{array}{c} (Q) \\ \left( \begin{array}{c} Q \\ b \end{array} \right) \end{array} \\
 a \dashv \rightarrow b & & a \dashv \rightarrow b & & a \dashv \rightarrow b
 \end{array}$$

Change-of-base preserves exponentials:

$$\begin{array}{ccccc}
 & & \begin{array}{c} ((Q \supset R) \& Q) \\ \left( \begin{array}{c} (Q \supset R) \& Q \\ a \end{array} \right) \end{array} & \longleftarrow & \begin{array}{c} ((Q \supset R) \& Q) \\ \left( \begin{array}{c} (Q \supset R) \& Q \\ b \end{array} \right) \end{array} \\
 & & \downarrow & \swarrow & \downarrow \\
 \begin{array}{c} (R) \\ \left( \begin{array}{c} R \\ a \end{array} \right) \end{array} & \longleftarrow & \begin{array}{c} (R) \\ \left( \begin{array}{c} R \\ b \end{array} \right) \end{array} & & \begin{array}{c} (R) \\ \left( \begin{array}{c} R \\ b \end{array} \right) \end{array} \\
 \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\
 \begin{array}{c} (Q \supset R) \\ \left( \begin{array}{c} Q \supset R \\ a \end{array} \right) \end{array} & \longleftarrow & \begin{array}{c} (Q \supset R) \\ \left( \begin{array}{c} Q \supset R \\ b \end{array} \right) \end{array} & & \begin{array}{c} (Q \supset R) \\ \left( \begin{array}{c} Q \supset R \\ b \end{array} \right) \end{array} \\
 \downarrow \natural & & \downarrow \text{id} & & \downarrow \text{id} \\
 \begin{array}{c} (Q \supset R)' \\ \left( \begin{array}{c} Q \supset R \\ a \end{array} \right)' \end{array} & & \begin{array}{c} (Q \supset R) \\ \left( \begin{array}{c} Q \supset R \\ b \end{array} \right) \end{array} \\
 a \dashv \rightarrow b & & a \dashv \rightarrow b & & a \dashv \rightarrow b
 \end{array}$$

In all the diagrams above the natural morphisms marked 'natural' are required to be isos.

Note that in the diagram for preservation of exponentials the object at the upper left has two constructions; in fact it is two objects, and we are hiding the iso between them.

### First preservations (2)

A category with the structure above is a **pre-hyperdoctrine**.

A **hyperdoctrine** is a pre-hyperdoctrine that obeys certain extra equations, like this one:

In a pre-hyperdoctrine given  $a \xrightarrow{f} b$  and two propositions  $P(b)$  and  $Q(b)$  over the space of ‘ $b$ ’s we have two different constructions for  $P(f(a)) \& Q(f(a))$  over the space of ‘ $a$ ’s, and a natural morphism from one construction to the other one:

$$\begin{aligned} \binom{P \& Q}{b} &:= (P(b) \& Q(b)) [b := f(a)] \\ \binom{P \& Q}{b}' &:= P(b) [b := f(a)] \& Q(b) [b := f(a)] \end{aligned}$$

$$\begin{array}{ccccc} \binom{P}{a} & \xleftarrow{\quad\quad\quad} & \binom{P}{b} & & \\ \uparrow \pi & \swarrow f^* \pi & \uparrow \pi & & \\ \binom{P \& Q}{a}' & \xleftarrow{\natural} & \binom{P \& Q}{a} & \xleftarrow{\quad\quad\quad} & \binom{P \& Q}{b} \\ \downarrow \pi' & \swarrow f^* \pi' & \downarrow \pi' & & \\ \binom{Q}{a} & \xleftarrow{\quad\quad\quad} & \binom{Q}{b} & & \\ a \xrightarrow{f} & & b & & \end{array}$$

we want these two constructions to be equivalent, so in the definition of hyperdoctrine we include a condition that says that the morphism ‘ $\natural$ ’ is an iso, i.e., that it has an inverse.