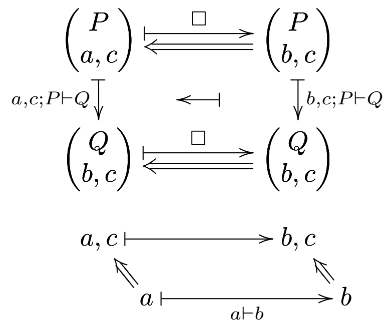
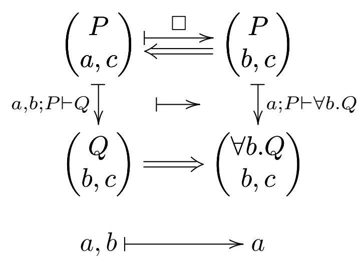
Index of the slides:

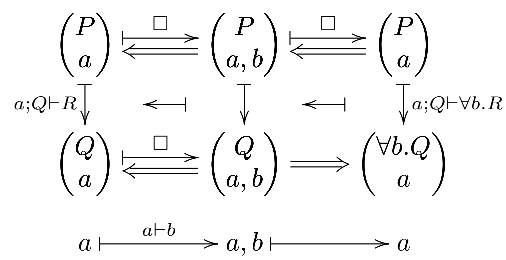
Hyperdoctrines: substitution, quantifiers, reflexivity

The substitution rule:

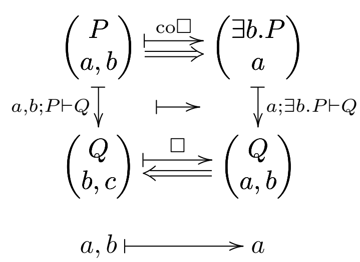


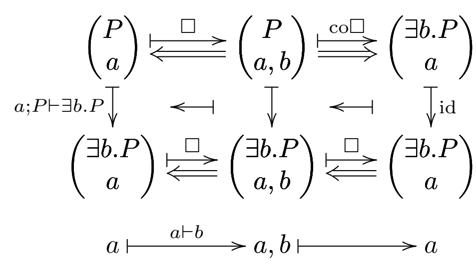
The ‘(∀I)’ and ‘(∀E)’ rules:



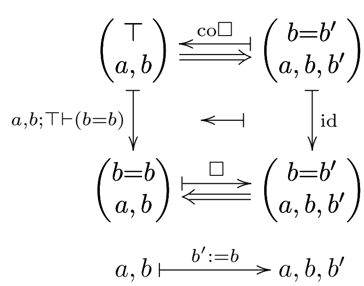


The ‘(∃E)’ and ‘(∃I)’ rules:



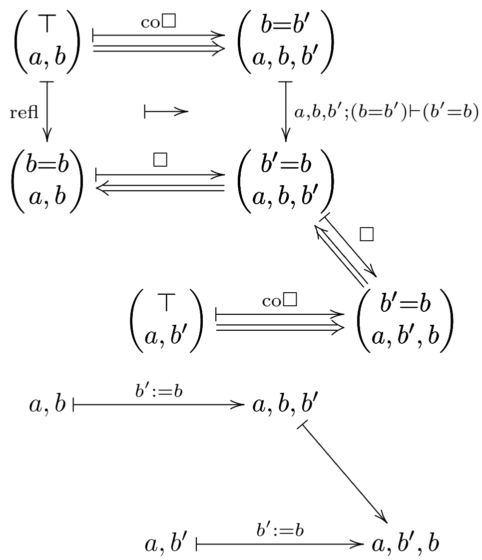


The “reflexivity” rule for equality (on variables):

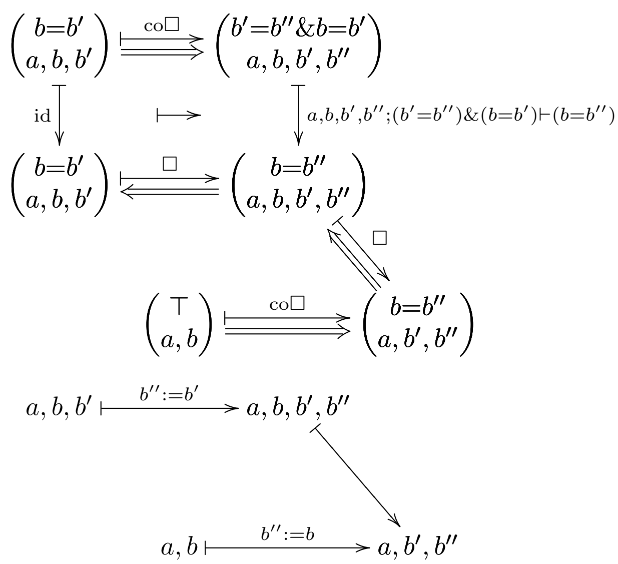


Hyperdoctrines: symmetry, transitivity

The “symmetry” rule for equality (on variables):



The “transitivity” rule for equality (on variables):



Weak and strong exists-elim

There are two versions of $(\exists E)$, $(\exists E^-)$ and $(\exists E^+)$.
They are "equivalent enough", and $(\exists E^-)$ is much simpler categorically.
The rules, in natural deduction and sequent calculus forms:

$$\dfrac{\exists b.P(a,b) \qquad \begin{matrix}[P(a,b)]^1 \quad Q(a)\\ \vdots \\ R(a)\end{matrix}}{R(a)}\;1;(\exists E^+) \qquad \dfrac{a,b;P(a,b),Q(a)\vdash R(a)}{a;\exists b.P(a,b),Q(a)\vdash R(a)}\;(\exists E^+)$$

$$\dfrac{\exists b.P(a,b) \qquad \begin{matrix}[P(a,b)]^1\\ \vdots \\ Q(a)\end{matrix}}{Q(a)}\;1;(\exists E^-) \qquad \dfrac{a,b;P(a,b)\vdash Q(a)}{a;\exists b.P(a,b)\vdash Q(a)}\;(\exists E^-)$$

$(\exists E^-)$ is a particular case of $(\exists E^+)$ — take $Q(a) := \top$ in $(\exists E^+)$.

In the presence of $(\supset)$ the rule $(\exists E^-)$ implies $(\exists E^+)$:

$$\dfrac{Q(a) \qquad \dfrac{\exists b.P(a,b) \qquad \dfrac{\begin{matrix}[P(a,b)]^2 \quad [Q(a)]^1\\ \vdots \\ R(a)\end{matrix}}{Q(a)\supset R(a)}\;1;(\supset I)}{Q(a)\supset R(a)}\;2;(\exists E^+)}{R(a)}\;(\supset E)$$

$(\exists E^-)$, categorically:

$$\begin{array}{ccc}
P(a,b) & \Rightarrow & \exists b.P(a,b) \\
\downarrow & \mapsto & \downarrow \\
Q(a) & \Leftarrow & Q(a) \\
\\
a,b & \longmapsto & a
\end{array}$$

$(\exists E^+)$, categorically:

$$\begin{array}{ccccccc}
P(a,b)\&Q(a) & \Leftarrow & P(a,b) & \Longrightarrow & \exists b.P(a,b) & \Longrightarrow & (\exists b.P(a,b))\&Q(a) \\
\downarrow & \mapsto & \downarrow & \mapsto & \downarrow & \mapsto & \downarrow \\
R(a) & \Longrightarrow & Q(a)\supset R(a) & \Leftarrow & Q(a)\supset R(a) & \Longleftarrow & R(a) \\
\\
a,b & = & a,b & \longmapsto & a & = & a
\end{array}$$

Note that $\mathbf{O}[\begin{pmatrix}Q(a)\supset R(a)\\ a,b\end{pmatrix}]$ has two constructions,
with a hidden iso between them.

Rules in ND and sequent calculus

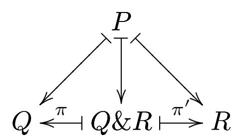
The rules, in natural deduction form (($\forall E$) is wrong):

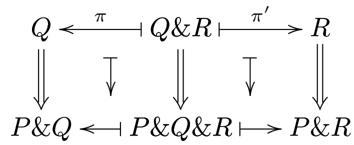
$\dfrac{\begin{matrix}\vdots & \vdots \\ Q & R\end{matrix}}{Q\&R}\ (\&I)$	$\begin{matrix} P & \dfrac{Q\&R}{Q}\ (\&E_1) \\ & \vdots \\ & S\end{matrix} \qquad \begin{matrix} P & \dfrac{Q\&R}{R}\ (\&E_2) \\ & \vdots \\ & S\end{matrix}$
$\begin{matrix} P \quad [Q]^1 \\ \vdots \\ \dfrac{R}{Q \supset R}\ 1; (\supset I)\end{matrix}$	$\dfrac{\begin{matrix} P & P \\ \vdots & \vdots \\ Q & Q \supset R\end{matrix}}{R}\ (\supset E)$
$\begin{matrix} P(a) \\ \vdots \\ \dfrac{Q(a,b)}{\forall b.Q(a,b)}\ (\forall I)\end{matrix}$	$\dfrac{\begin{matrix} & Q(a) \\ \vdots & \vdots \\ f(a) & \forall b.R(a,b)\end{matrix}}{R(a,f(a))}\ (\forall E)$
$\begin{matrix} Q(a) \\ \vdots \\ \dfrac{P(a,f(a))}{\exists b.P(a,b)}\ (\exists I)\end{matrix}$	$\dfrac{\begin{matrix} & [P(a,b)]^1 \quad Q(a) \\ & \vdots \\ \exists b.P(a,b) & R(a)\end{matrix}}{R(a)}\ (\exists E)$

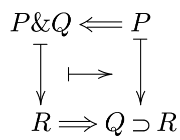
The rules, in sequent calculus form (($\forall E$) is wrong):

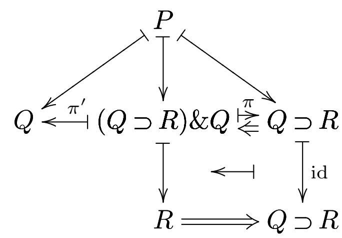
$\dfrac{P \vdash Q \quad P \vdash R}{P \vdash Q\&R}\ (\&I)$	$\dfrac{P, Q \vdash S}{P, Q\&R \vdash S}\ (\&E_1) \qquad \dfrac{P, R \vdash S}{P, Q\&R \vdash S}\ (\&E_2)$
$\dfrac{P, Q \vdash R}{P \vdash Q \supset R}\ (\supset I)$	$\dfrac{P \vdash Q \quad P \vdash Q \supset R}{P \vdash R}\ (\supset E)$
$\dfrac{a, b; P(a) \vdash Q(a,b)}{a; P(a) \vdash \forall b.Q(a,b)}\ (\forall I)$	$\dfrac{a \vdash f(a) \quad a; Q(a) \vdash \forall b.R(a,b)}{a; Q(a) \vdash R(a, f(a))}\ (\forall E)$
$\dfrac{a; Q(a) \vdash P(a, f(a))}{a; Q(a) \vdash \exists b.P(a,b)}\ (\exists I)$	$\dfrac{a, b; P(a,b), Q(a) \vdash R(a)}{a; \exists b.P(a,b), Q(a) \vdash R(a)}\ (\exists E)$

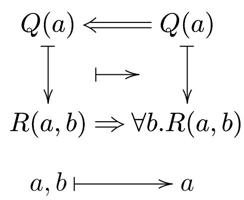
Rules, categorically
The rules, categorically (($(\&E)$ needs explanations, $(\forall E)$ is wrong):

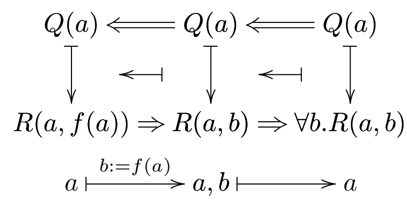


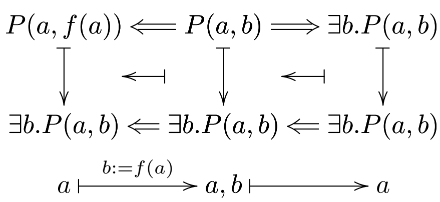








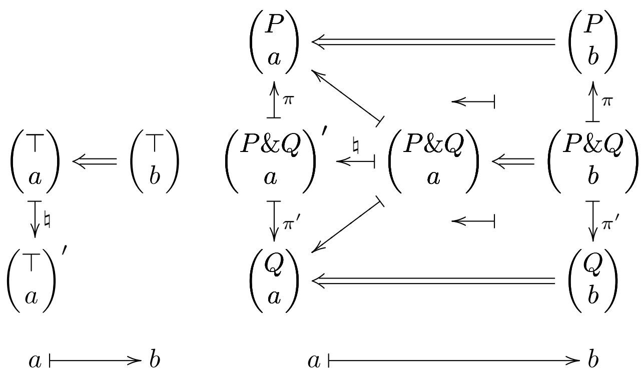




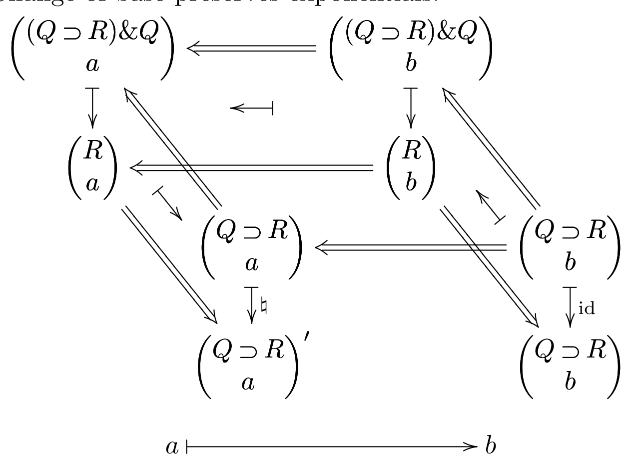
(see next page)

First preservations

Change-of-base preserves terminals and binary products:



Change-of-base preserves exponentials:



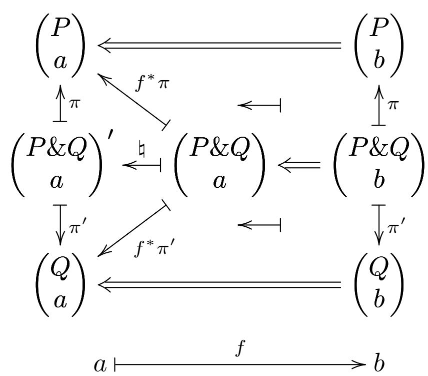
In all the diagrams above the natural morphisms marked '♮' are required to be isos.

Note that in the diagram for preservation of exponentials the object at the upper left has two constructions; in fact it is two objects, and we are hiding the iso between them.

First preservations (2)
A category with the structure above is a **pre-hyperdoctrine**.
A **hyperdoctrine** is a pre-hyperdoctrine that obeys certain extra equations, like this one:

In a pre-hyperdoctrine given $a \overset{f}{\mapsto} b$ and two propositions $P(b)$ and $Q(b)$ over the space of 'b's we have two different constructions for $P(f(a))\&Q(f(a))$ over the space of 'a's, and a natural morphism from one construction to the other one:

$$\begin{pmatrix} P\&Q \\ b \end{pmatrix} := (P(b)\&Q(b))[b := f(a)]$$
$$\begin{pmatrix} P\&Q \\ b \end{pmatrix}' := P(b)[b := f(a)] \,\&\, Q(b)[b := f(a)]$$



we want these two constructions to be equivalent, so in the definition of hyperdoctrine we include a condition that says that the morphism '♮' is an iso, i.e., that it has an inverse.