Index of the slides:

E-mail

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Tenho boas notícias sim... vou dar um resumo.

O Petrúcio tava tentando fazer todas as provas numa linguagem - que eu
chamo de "L1" - que não tem conjuntos, interseções, uniões, etc. Eu
tava usando uma linguagem - "L2" - que permite tudo isso e algumas
coisas mais, e que pra mim era óbvio que tudo que eu fizesse em L2
podia ser traduzido para L1...

Bom, eu formalizei essa L2 - não totalmente ainda, mas acho que
suficiente bem - e a tradução dela para L1, e tenho um monte de lemas
interessantes cujas provas em L2 são curtíssimas e bem intuitivas... e
isto inclui vários pedaços da demonstração de que aquela construção do
Petrúcio prova Ind |- Zer \/ Inj.

Essa outra abordagem daqui me pareceu ainda mais natural. Como sempre,
tudo fica mais fácil com um exemplo ("dever de casa pro leitor:
generalize" - só que como eu não tou dando detalhes suficientes esse
dever de casa não é pra ser levado a sério). Imaginem que o nosso N é
isso aqui:

  0->1->2->3->4->5->6->4

eu encontrei um modo de caracterizar os elementos do loop (i.e., {4,
5, 6}) e o primeiro cara de fora do loop que aponta pro loop (o 3),
usando L2 - e portanto, com a tradução, tenho um modo de fazer tudo
isso em L1... daí dá pra provar que o 4 tem dois antecessores. Ainda
não chequei todos os detalhes do que eu vou dizer agora, mas lá vai...

Eu chamo de "L" a construção que me dá esse loop, de "[a,\infty)" a
construção que me dá todos os sucessores de a, e de "\notS A" a
construção que me retorna todos os elementos de A cujo sucessor não
está em A. Então nesse caso

  L(0) = {4, 5, 6},

  [0,\infty) \ L(0) = {1, 2, 3},

  \notS([0,\infty) \ L(0)) = {3},

e o sucessor do 3 é o único cara de [0,\infty) que pode ter mais de um
predecessor em [0,\infty) - um dentro do loop e um fora. No caso em
que eu tenho um w com w->0 (por ~Zer) e vale Inj, então todo mundo
pertence a [0,\infty), inclusive o w; aí eu vou ter L(0) = todo mundo,
[0,\infty) \ L(0) = vazio, e não vou poder ter um cara com dois
predecessores... ou seja, Inj (o "Inj de verdade", sobre o N todo),
vai valer.

Depois vou escrever os detalhes direito.

  [[]]s, té quinta,
    E.
```

Increasing and decreasing

We will say that a predicate P on N is "non-decreasing" when $\forall x.Px \supset PSx$, and "non-increasing" when $\forall x.PSx \supset Px$; obviously, the idea is that the characteristic function of a predicate on N may be non-decreasing or non-increasing (for each arrow $x \to Sx$), and we are making these terms apply to predicates too.

Notation:

$$\begin{array}{rcll} \nearrow P & := & \forall x.Px \supset PSx & \text{(non-decreasing)}, \\ \searrow P & := & \forall x.PSx \supset Px & \text{(non-increasing)}. \end{array}$$

Note that $\nearrow P \Leftrightarrow \searrow(\neg P)$.

We can define the set of successors of a and the set of predecessors of b as:

$$\begin{array}{rcl} x \in [a, \infty) & := & \forall A.Aa \wedge \nearrow A \supset Ax \\ x \in (-\infty, b] & := & \forall B.Bb \wedge \searrow A \supset Bx \end{array}$$

It is easy to see that $\nearrow(\in [a, \infty))$ and that $\searrow(\in (-\infty, b])$.

Lemma: $b \in [a, \infty) \Leftrightarrow a \in (-\infty, b]$.

Proof:

$$\begin{array}{rcl} b \in [a, \infty) & \equiv & \forall A.Aa \wedge \nearrow A \supset Ab \\ a \in (-\infty, b] & \equiv & \forall B.Bb \wedge \searrow A \supset Ba \\ & \equiv & \forall A.\neg Ab \wedge \nearrow A \supset \neg Aa \\ & \equiv & \forall A.Aa \wedge \nearrow A \supset Ab. \end{array}$$

Now let's define the "closed interval" $[a, b]$. This will be like $[a, \infty)$, except that the characteristic function will be allowed to decrease at the arrow $b \to Sb$; when $b \notin [a, \infty)$ we will have $[a, b] = [a, \infty)$.

$$x \in [a, b] \quad := \quad \forall A.Aa \wedge (\forall x \neq b.Px \supset PSx) \supset Ax$$

Lemma: if $Sb \in [0, b]$ then $[0, b] = [0, \infty)$.

Proof: straightforward (look at the arrow $b \to Sb$).

Lemma: if $x \in [a, b]$ and $Sx \notin [a, b]$ then $x = b$.

Proof: straightforward.

L1, L2, Types

Conventions:

x, y	:	N
P, Q	:	Ω
A, B	:	$\mathcal{P}(N)$ (i.e., $N \to \Omega$)
$\mathcal{A}, \mathcal{B}$	:	$\mathcal{P}(\mathcal{P}(N))$ (i.e., $(N \to \Omega) \to \Omega$)

L1:

$x, 0, Sx$	:	N
$\top, \bot, \neg P, P \land Q, P \supset Q, P \lor Q$	:	Ω
$Ax, x = y$	:	Ω
$\forall x.P, \exists x.P$	:	Ω
$\forall A.P, \exists A.P$	:	Ω

L1":

$\forall x \in N.P, \exists x \in N.P$	:	Ω
$\forall A \subseteq N.P, \exists A \subseteq N.P$	:	Ω

L2:

$\emptyset, N, A \cap B, A \cup B, A \backslash B$	:	$\mathcal{P}(N)$
$\{a\}, \{a, b\}, \ldots$	:	$\mathcal{P}(N)$
$\emptyset, \mathcal{P}(A), \mathcal{A} \cap \mathcal{B}, \mathcal{A} \cup \mathcal{B}, \mathcal{A} \backslash \mathcal{B}$	:	$\mathcal{P}(\mathcal{P}(N))$
$\{\, a \in N \mid P \,\}, \bigcap \mathcal{A}, \bigcup \mathcal{B}$	:	$\mathcal{P}(N)$
$\forall A \subseteq N.P, \exists A \subseteq N.P$	:	Ω
$\{\, A \subseteq N \mid P \,\}$	:	$\mathcal{P}(\mathcal{P}(N))$
$x \in A$	:	Ω
$A \in \mathcal{A}$	:	Ω

Extras:

$[a, \infty)$, $(-\infty, b]$, $[a, b]$	:	$\mathcal{P}(N)$
$\nearrow A, \searrow A$	:	Ω
$\not{S} A$	:	$\mathcal{P}(A)$
$La, a \to b, a \twoheadrightarrow b,$	:	Ω
$a \leq b, a < b, a \sim b$	:	Ω
$L^9 a, L' a, L^\circ a$	:	$\mathcal{P}(N)$
$\mathsf{Ind}, \mathsf{Inj}, \mathsf{Zer}$	:	Ω

Definitions

Definitions:

$a \in A$	$:=$	Aa
$A \in \mathcal{A}$	$:=$	$\mathcal{A}A$
$\{\, x \mid P \,\}$	$:=$	$\lambda x{:}N.P$
N	$:=$	$\lambda x{:}N.\top$
$\emptyset$	$:=$	$\lambda x{:}N.\bot$
$\{\, a, b \,\}$	$:=$	$\{\, x \mid x = a \vee x = b \,\}$
$A \cap B$	$:=$	$\{\, x \mid x \in A \wedge x \in B \,\}$
$A \cup B$	$:=$	$\{\, x \mid x \in A \vee x \in B \,\}$
$A \backslash B$	$:=$	$\{\, x \mid x \in A \wedge \neg x \in B \,\}$
$\{\, A \mid P \,\}$	$:=$	$\lambda A.P$
$\bigcup \mathcal{A}$	$:=$	$\{\, x \mid \exists A \in \mathcal{A}.x \in A \,\}$
$\bigcap \mathcal{A}$	$:=$	$\{\, x \mid \forall A \in \mathcal{A}.x \in A \,\}$
$\nearrow A$	$:=$	$\forall x.x \in A \supset Sx \in A$
$\searrow A$	$:=$	$\forall x.Sx \in A \supset x \in A$
$[a, \infty)$	$:=$	$\bigcap \{\, A \mid a \in A \wedge \nearrow A \,\}$
$[-\infty, b)$	$:=$	$\bigcap \{\, B \mid b \in B \wedge \searrow B \,\}$
$\not\$A$	$:=$	$\{\, a \in A \mid Sa \notin A \,\}$
La	$:=$	$a \in [Sa, \infty)$
$L^\circ a$	$:=$	$\{\, b \in [a, \infty) \mid Lb \,\}$
$L'a$	$:=$	$\{\, b \in [a, \infty) \mid \neg Lb \,\}$
$a \rightarrow b$	$:=$	$Sa = b$
$a \twoheadrightarrow b$	$:=$	$[a, \infty) \ni b$
$\mathcal{A}$ is disjoint	$:=$	$\forall A, B \in \mathcal{A}.A \neq B \Rightarrow A \cup B = \emptyset$
$\mathcal{A}$ covers $a \rightarrow Sa$	$:=$	$\exists A \in \mathcal{A}.\{a, Sa\} \subseteq A$
$[a, b]$	$:=$	$\bigcap \{\, A \mid a \in A \wedge \forall x \in A \backslash \{b\}.Sx \in A \,\}$
$[a, bc]$	$:=$	$\bigcap \{\, A \mid a \in A \wedge \forall x \in A \backslash \{b, c\}.Sx \in A \,\}$

Cases 0, 1 and 9

We say that b *is in a loop* — and we write this as Lb — when $Sb \twoheadrightarrow b$, i.e., when $b \in [Sb, \infty)$.
With the predicate L we can split $[a, \infty)$ in two disjoint parts: a "linear part", $L'a$, and a "loop part", $L^\circ a$, that are defined as:

$$\begin{array}{rcl} L'a & := & \{\, b \in [a, \infty) \mid \neg Lb \,\} \\ L^\circ a & := & \{\, b \in [a, \infty) \mid Lb \,\} \end{array}$$

Theorem: $[a, \infty)$ is either a loop, an infinite straight line, or a "nine". We will call these cases "0", "1" and "9", according to the shapes of the numbers.

```
    . <- ... <- .
 <-               <-
w                    .
 ->                ->
   a -> . -> . -> .

      Case 0
```

```
   a -> . -> . -> ...

      Case 1
```

```
                          ...
                       <-     <-
                      c           .
                       ->      ->
   a -> . -> ... -> b -> d -> .
   \________  ______/ \____  ____/
            \/             \/
           L'a            L°a

      Case 9
```

Cases 0, 1 and 9 (2)

If $a = 0$ and Ind holds, then $[a, \infty)$ is the whole universe (i.e., N).
We want to show that $\neg\mathsf{Zer}$ only holds in the case "0",
and that $\neg\mathsf{Inj}$ only holds in the case "9".

Lemmas:
$L^\circ a$ is a single loop: for $b, c \in L^\circ a$ we have $b \twoheadrightarrow c$ and $c \twoheadrightarrow b$.
A loop is a sink: there may be arrows from $L'a$ to $L^\circ a$,
but no arrows leave $L^\circ a$ back to $L'a$.
$L'a$ is linearly ordered: $b, c \in L'a$ either $b \twoheadrightarrow c$ and $c \twoheadrightarrow b$ hold,
and if both $b \twoheadrightarrow c$ and $c \twoheadrightarrow b$ hold then $b = c$.
Each element $\neq a$ of $L'a$ has exactly one predecessor (in $L'a$).

↑ *[I have not proved this yet!]*

$L'a$ has a last element if and only if we are in the case "9";
and this last element ('b' in the figure), when it exists, is unique.
The successor of the last element of $L'a$ is in the loop part
(the arrow $b \to d$ in the figure for the case "9").
Every element in a loop has a predecessor in the loop
(arrows $w \to a$ in the figure "0", $c \to d$ in the figure "9");
this shows that Inj fails in the case "9".

*[I have not shown yet that the predecessor-in-the-loop is unique...
I will need that to show that* Inj *holds in "0"]*

In the cases "1" and "9" the element a (i.e., 0) has no
predecessor — i.e., Zer holds.

Lemmas

(1) $\nearrow[a,\infty)$
(2) $\searrow(-\infty,b]$
(3) $\nearrow A \Leftrightarrow \searrow(N\backslash A)$
(4) $b\in[a,\infty)\Leftrightarrow a\in(-\infty,b]$

(5) $[a,\infty)=\{a\}\cup[Sa,\infty)$
(6) $a\twoheadrightarrow b\Leftrightarrow[a,\infty)\supseteq[b,\infty)$
(7) $a\twoheadrightarrow b, b\twoheadrightarrow a\Leftrightarrow[a,\infty)=[b,\infty)$
(8) $a\twoheadrightarrow b\Rightarrow Sa\twoheadrightarrow Sb$
(9) $La\Leftrightarrow[a,\infty)=[Sa,\infty)$
(10) $La\Rightarrow LSa$
(11) $\nearrow(-\infty,b]\Leftrightarrow Lb$

(12) $a\neq b, a\twoheadrightarrow b\Rightarrow Sa\twoheadrightarrow b$
(13) $\not\$(-\infty,b]\subseteq\{b\}$
(14) $La\Rightarrow[a,\infty)\subseteq(-\infty,a]$
(15) $La, a\twoheadrightarrow b\Rightarrow b\twoheadrightarrow a$
(15′) $La, a\twoheadrightarrow b\Rightarrow Lb$
(15″) $La, a\twoheadrightarrow b, a\twoheadrightarrow c\Rightarrow b\twoheadrightarrow c\wedge c\twoheadrightarrow b$
(15‴) $La, a\twoheadrightarrow b\Rightarrow[a,\infty)=[b,\infty)$

(16) $c\not\twoheadrightarrow d, d\not\twoheadrightarrow c\Rightarrow\nearrow\{\,b\mid b\twoheadrightarrow c\wedge b\twoheadrightarrow d\,\}$
(17) $a\twoheadrightarrow c, a\twoheadrightarrow d, c\not\twoheadrightarrow d, d\not\twoheadrightarrow c\Rightarrow\bot$
(18) $a\twoheadrightarrow c, a\twoheadrightarrow d\Rightarrow c\twoheadrightarrow d\vee d\twoheadrightarrow c$
(19) $a\twoheadrightarrow c, a\twoheadrightarrow d, Lc, Ld\Rightarrow c\twoheadrightarrow d\wedge d\twoheadrightarrow c$

(20) $L'a\neq\emptyset, L^\circ a\neq\emptyset\Rightarrow\not\$L'a\neq\emptyset$
(21) $b,c\in\not\$L'a, b\neq c, b\twoheadrightarrow c\Rightarrow\bot$
(22) $b,c\in\not\$L'a,\Rightarrow b=c$
(23) $Ld, Sd\neq d\Rightarrow\not\$([d,\infty)\backslash\{d\})\neq\emptyset$

Proofs (5 to 11)

$$\dfrac{a \in (\{a\} \cup [Sa, \infty)) \quad \nearrow(\{a\} \cup [Sa, \infty))}{[a, \infty) \subseteq \{a\} \cup [Sa, \infty)} \qquad \dfrac{a \in [a, \infty) \quad \dfrac{Sa \in [a, \infty) \quad \nearrow[a, \infty)}{[Sa, \infty) \subseteq [a, \infty)}}{\{a\} \cup [Sa, \infty) \subseteq [a, \infty)}$$

$$\dfrac{\dfrac{a \twoheadrightarrow b}{b \in [a, \infty)} \quad \dfrac{}{\nearrow[a, \infty)}}{[b, \infty) \subseteq [a, \infty)} \qquad \dfrac{\dfrac{[a, \infty) \supseteq [b, \infty)}{[a, \infty) \ni b}}{a \twoheadrightarrow b}$$

$$\dfrac{\dfrac{a \twoheadrightarrow b}{[a, \infty) \supseteq [b, \infty)} \quad \dfrac{b \twoheadrightarrow a}{[b, \infty) \supseteq [a, \infty)}}{[a, \infty) = [b, \infty)} \qquad \dfrac{\dfrac{[a, \infty) = [b, \infty)}{[a, \infty) \supseteq [b, \infty)}}{a \twoheadrightarrow b} \qquad \dfrac{\dfrac{[a, \infty) = [b, \infty)}{[a, \infty) \subseteq [b, \infty)}}{b \twoheadrightarrow a}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{a \twoheadrightarrow b}{b \in [a, \infty)}}{b \in \{a\} \cup [Sa, \infty)} \quad \dfrac{\dfrac{\dfrac{[b \in \{a\}]^1}{b = a}}{Sb = Sa}}{Sb \in [Sa, \infty)} \quad \dfrac{[b \in [Sa, \infty)]^1 \quad \dfrac{}{\nearrow[Sa, \infty)}}{Sb \in [Sa, \infty)}}{Sb \in [Sa, \infty)}\,1 \quad \dfrac{}{\nearrow[Sa, \infty)}}{\dfrac{[Sb, \infty) \subseteq [Sa, \infty)}{Sa \twoheadrightarrow Sb}}$$

$$\dfrac{\dfrac{\dfrac{La}{Sa \twoheadrightarrow a}}{[Sa, \infty) \supseteq [a, \infty)} \quad \dfrac{\dfrac{}{a \twoheadrightarrow Sa}}{[a, \infty) \supseteq [Sa, \infty)}}{[a, \infty) = [Sa, \infty)} \qquad \dfrac{\dfrac{\dfrac{[a, \infty) = [Sa, \infty)}{a \in [Sa, \infty)}}{Sa \twoheadrightarrow a}}{La}$$

$$\dfrac{\dfrac{\dfrac{La}{Sa \twoheadrightarrow a}}{SSa \twoheadrightarrow Sa}}{LSa}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[b \in (-\infty, a]]^1}{b \twoheadrightarrow a}}{Sb \twoheadrightarrow Sa} \quad \dfrac{La}{Sa \twoheadrightarrow a}}{Sb \twoheadrightarrow a}}{Sb \in (-\infty, a]}}{\nearrow(-\infty, a]}\,1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{a \in (-\infty, a]} \quad \nearrow(-\infty, a]}{Sa \in (-\infty, a]}}{Sa \twoheadrightarrow a}}{La}$$

Proofs (12 to 15‴)

$$\dfrac{\dfrac{\dfrac{\dfrac{[a \in (-\infty, b]\backslash\{b\}]^1}{a \neq b} \qquad \dfrac{\dfrac{[a \in (-\infty, b]\backslash\{b\}]^1}{a \in (-\infty, b]}}{a \twoheadrightarrow b}}{Sa \twoheadrightarrow b}}{Sa \in (-\infty, b]}}{\not{S}(-\infty, b] \subseteq \{b\}}\ 1$$

$$\dfrac{\dfrac{}{a \in (-\infty, a]} \qquad \dfrac{La}{\nearrow(-\infty, a]}}{[a, \infty) \subseteq (-\infty, a]}$$

$$\dfrac{\dfrac{\dfrac{La}{[a, \infty) \subseteq (-\infty, a]}}{b \in [a, \infty) \Rightarrow b \in (-\infty, a]}}{a \twoheadrightarrow b \Rightarrow b \twoheadrightarrow a}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{La \quad a \twoheadrightarrow b}{b \twoheadrightarrow a}}{Sb \twoheadrightarrow Sa} \qquad \dfrac{La}{Sa \twoheadrightarrow a} \quad a \twoheadrightarrow b}{Sb \twoheadrightarrow b}}{Lb}$$

$$\dfrac{\dfrac{\dfrac{La \quad a \twoheadrightarrow b}{b \twoheadrightarrow a} \qquad a \twoheadrightarrow c}{b \twoheadrightarrow c} \qquad \dfrac{\dfrac{La \quad a \twoheadrightarrow c}{c \twoheadrightarrow a} \qquad a \twoheadrightarrow b}{c \twoheadrightarrow b}}{b \twoheadrightarrow c \wedge c \twoheadrightarrow b}$$

$$\dfrac{a \twoheadrightarrow b \qquad \dfrac{La \quad a \twoheadrightarrow b}{b \twoheadrightarrow a}}{[a, \infty) = [b, \infty)}$$

Proofs (16 to 19)

$$\dfrac{\dfrac{\dfrac{[b \twoheadrightarrow d]^1 \quad c \not\twoheadrightarrow d}{b \neq c} \quad [b \twoheadrightarrow c]^1}{Sb \twoheadrightarrow c} \qquad \dfrac{\dfrac{[b \twoheadrightarrow c]^1 \quad d \not\twoheadrightarrow c}{b \neq d} \quad [b \twoheadrightarrow d]^1}{Sb \twoheadrightarrow d}}{\nearrow\{\, b \mid b \twoheadrightarrow c \wedge b \twoheadrightarrow d \,\}}\, 1$$

$$\dfrac{\dfrac{\dfrac{a \twoheadrightarrow c \quad a \twoheadrightarrow d}{c, d \in [a, \infty)} \quad \dfrac{\dfrac{a \twoheadrightarrow c \quad a \twoheadrightarrow d}{a \in \{\, b \mid b \twoheadrightarrow cd \,\}} \quad \dfrac{c \not\twoheadrightarrow d \quad d \not\twoheadrightarrow c}{\nearrow\{\, b \mid b \twoheadrightarrow cd \,\}}}{[a, \infty) \subseteq \{\, b \mid b \twoheadrightarrow cd \,\}}}{c, d \in \{\, b \mid b \twoheadrightarrow cd \,\}} \qquad \dfrac{c \not\twoheadrightarrow d \quad d \not\twoheadrightarrow c}{c, d \notin \{\, b \mid b \twoheadrightarrow cd \,\}}}{\bot}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \twoheadrightarrow c \quad a \twoheadrightarrow d}{c \not\twoheadrightarrow d \wedge d \not\twoheadrightarrow c \Rightarrow \bot}}{\neg(c \not\twoheadrightarrow d \wedge d \not\twoheadrightarrow c)}}{\neg(\neg c \twoheadrightarrow d \wedge \neg d \twoheadrightarrow c)}}{\neg\neg(c \twoheadrightarrow d \vee d \twoheadrightarrow c)}}{c \twoheadrightarrow d \vee d \twoheadrightarrow c}$$

$$\dfrac{\dfrac{a \twoheadrightarrow c \quad a \twoheadrightarrow d}{c \twoheadrightarrow d \vee d \twoheadrightarrow c} \quad \dfrac{[c \twoheadrightarrow d]^1 \quad \dfrac{Lc \quad [c \twoheadrightarrow d]^1}{d \twoheadrightarrow c}}{c \twoheadrightarrow d \wedge d \twoheadrightarrow c} \quad \dfrac{\dfrac{Ld \quad [d \twoheadrightarrow c]^1}{c \twoheadrightarrow d} \quad [d \twoheadrightarrow c]^1}{c \twoheadrightarrow d \wedge d \twoheadrightarrow c}}{c \twoheadrightarrow d \wedge d \twoheadrightarrow c}\, 1$$

Proofs (20 to 23)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{L'a\neq\emptyset}{a\in L'a}\quad \dfrac{[\not\$L'a=\emptyset]^1}{\nearrow L'a}}{[a,\infty)\subseteq L'a}\quad \dfrac{}{L'a\subseteq[a,\infty)}}{L'a=[a,\infty)}}{L^\circ a=\emptyset}\quad L^\circ a\neq\emptyset}{\bot}}{\not\$L'a\neq\emptyset}\;1$$

$$\dfrac{\dfrac{\dfrac{b\in\not\$L'a}{a\twoheadrightarrow Sb}\quad \dfrac{b\neq c\quad b\twoheadrightarrow c}{Sb\twoheadrightarrow c}\quad \dfrac{c\in\not\$L'a}{c\in L'a}}{Sb\in L'a}\quad \dfrac{b\in\not\$L'a}{Sb\notin L'a}}{\bot}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{b,c\in\not\$L'a}{b,c\in L'a}}{b,c\in[a,\infty)}}{b\twoheadrightarrow c\lor c\twoheadrightarrow b}\quad \dfrac{b,c\in\not\$L'a\quad [b\neq c]^2\quad [b\twoheadrightarrow c]^1}{\bot}\quad \dfrac{b,c\in\not\$L'a\quad [b\neq c]^2\quad [b\twoheadrightarrow c]^1}{\bot}}{\bot}\;1}{b=c}\;2$$

$$\dfrac{\dfrac{\dfrac{\dfrac{Ld}{[d,\infty)=[Sd,\infty)}\quad \dfrac{\dfrac{Sd\neq d}{Sd\in([d,\infty)\backslash\{d\})}\quad \dfrac{[\not\$([d,\infty)\backslash\{d\})=\emptyset]^1}{\nearrow([d,\infty)\backslash\{d\})}}{[Sd,\infty)\subseteq[d,\infty)\backslash\{d\}}}{[d,\infty)\subseteq[d,\infty)\backslash\{d\}}}{\bot}}{\not\$([d,\infty)\backslash\{d\})\neq\emptyset}\;1$$

Interval lemmas

Definitions:

$$
\begin{array}{lll}
S(A) & := & \{\, x \mid \exists a \in A. Sa = x \,\} \\
\llbracket A, \_ \rrbracket & := & \{\, X \mid A \subseteq X \,\} \\
\llbracket \_, B \rrbracket & := & \{\, X \mid S(X \backslash B) \subseteq X \,\} \\
\llbracket A, B \rrbracket & := & \llbracket A, \_ \rrbracket \cap \llbracket \_, B \rrbracket \\
[A, B] & := & \bigcap \llbracket A, B \rrbracket \\
[a, b] & := & \bigcap \llbracket \{a\}, \{b\} \rrbracket \\
[ab, cd] & := & \bigcap \llbracket \{a, b\}, \{c, d\} \rrbracket
\end{array}
$$

Elementary lemmas:

$$
\begin{array}{ll}
(1) & A \subseteq [A, B] \\
(2) & \not{S}([A, B]) \subseteq B \\
(3) & [a, B] \subseteq [a, \infty) = [a, \emptyset] \\
(4) & \llbracket A \cup A', \_ \rrbracket = \llbracket A, \_ \rrbracket \cap \llbracket A', \_ \rrbracket \\
(5) & \llbracket A \cap A', \_ \rrbracket = \llbracket A, \_ \rrbracket \cup \llbracket A', \_ \rrbracket \\
(6) & \llbracket \_, B \cup B' \rrbracket = \llbracket \_, B \rrbracket \cup \llbracket \_, B' \rrbracket \\
(7) & \llbracket \_, B \cap B' \rrbracket = \llbracket \_, B \rrbracket \cap \llbracket \_, B' \rrbracket \\
(8) & [A \cup A', B] \\
 & = \bigcap(\llbracket A, \_ \rrbracket \cap \llbracket A', \_ \rrbracket \cap \llbracket \_, B \rrbracket) \\
 & = [A, B] \cup [A', B] \\
(9) & [A, B \cap B'] \\
 & = \bigcap(\llbracket A, \_ \rrbracket \cap \llbracket \_, B \rrbracket \cap \llbracket \_, B' \rrbracket) \\
 & = [A, B] \cup [A, B']
\end{array}
$$

Order lemmas:

$$
\begin{array}{ll}
(1) & [A \cup S(B), B \cup B'] = [A \cup S(B), B'] \\
(2) & b \notin [A, C] \Rightarrow [A, C] = [A, \{b\} \cup C] \\
(3) & B \cap [A, C] = \emptyset \Rightarrow [A, C] = [A, B \cup C] \\
(4) & b, b' \in \not{S}([a, B]) \Rightarrow b = b' \\
(5) & [a, B \cup C] = [a, B] \text{ or } [a, B \cup C] = [a, C] \\
(6) & b \in [a, c] \Rightarrow [a, b] \subseteq [a, c] \\
(7) & b \in [a, c] \Rightarrow [b, c] \subseteq [a, c] \\
(8) & b, Sb \in [a, c] \qquad\qquad \Rightarrow [a, c] = [a, b] \cup [Sb, c] \\
(9) & b, Sb \in [a, c],\ b \neq Sb \Rightarrow [a, c] = [a, b] \sqcup [Sb, c]
\end{array}
$$

Induction in intervals

Induction lemmas:

(1) $A \subseteq X,\ S(X \backslash B) \subseteq X \Rightarrow [A, B] \subseteq X$

(2) $A \subseteq X \subseteq [A, B],\ S(X \backslash B) \subseteq X \Rightarrow [A, B] = X$

Small lemmas: one whose proof I have not finished yet,

(1) $b \in [a, c] \Rightarrow [a, b] = [a, bc] \subseteq [a, c]$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[\![\_, bc]\!] \supseteq [\![\_, b]\!]}}{[\![a, \_]\!] \cap [\![\_, bc]\!] \supseteq [\![a, \_]\!] \cap [\![\_, b]\!]}}{[\![a, bc]\!] \supseteq [\![a, b]\!]}}{\bigcap [\![a, bc]\!] \subseteq \bigcap [\![a, b]\!]}}{[a, bc] \subseteq [a, b]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a \in [a, b]} \qquad \dfrac{\dfrac{c \notin [a, b] \vee b = c}{[a, b] \backslash \{b, c\} = [a, b] \backslash \{b\}}}{S([a, b] \backslash \{b, c\}) \subseteq [a, b]}}{[a, b] \in [\![a, \_]\!] \cap [\![\_, bc]\!]}}{[a, b] \in [\![a, bc]\!]}}{[a, b] \supseteq \bigcap [\![a, bc]\!]}}{[a, b] \supseteq [a, bc]}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\![\_, bc]\!] \subseteq [\![\_, c]\!]}{[\![a, \_]\!] \cap [\![\_, bc]\!] \subseteq [\![a, \_]\!] \cap [\![\_, c]\!]}}{\bigcap ([\![a, \_]\!] \cap [\![\_, bc]\!]) \supseteq \bigcap ([\![a, \_]\!] \cap [\![\_, c]\!])}}{\bigcap [\![a, bc]\!] \supseteq \bigcap [\![a, c]\!]}}{[a, bc] \subseteq [a, c]}$$

Induction in intervals (2)

...and another small lemma (whose proof is complete):

(2) $b \in [a, c] \Rightarrow [b, c] \subseteq [ab, c] = [a, c]$

$$\begin{array}{c} \hline [\![b, \_]\!] \supseteq [\![ab, \_]\!] \\ \hline [\![b, \_]\!] \cap [\![\_, c]\!] \supseteq [\![ab, \_]\!] \cap [\![\_, c]\!] \\ \hline \bigcap([\![b, \_]\!] \cap [\![\_, c]\!]) \subseteq \bigcap([\![ab, \_]\!] \cap [\![\_, c]\!]) \\ \hline [b, c] \subseteq [ab, c] \end{array} \qquad \begin{array}{c} b \in [a, c] \\ \hline b \in \bigcap [\![a, c]\!] \\ \hline [\![b, \_]\!] \supseteq [\![a, c]\!] \\ \hline [\![b, \_]\!] \cap [\![a, c]\!] = [\![a, c]\!] \\ \hline [\![b, \_]\!] \cap [\![a, \_]\!] \cap [\![\_, c]\!] = [\![a, c]\!] \\ \hline [\![ab, \_]\!] \cap [\![\_, c]\!] = [\![a, c]\!] \\ \hline [\![ab, c]\!] = [\![a, c]\!] \\ \hline [ab, c] = [a, c] \end{array}$$

Big lemmas (not proved yet):

(1) $b, c \in [a, c], b \neq c \Rightarrow [a, Sb] = [a, b] \sqcup \{Sb\}$
(2) $b, c \in [a, c], b \neq c \Rightarrow [b, c] = \{b\} \sqcup [Sb, c]$
(3) $b, c \in [a, c], b \neq c \Rightarrow [a, c] = [a, b] \sqcup [Sb, c]$

Intervals
[This is old! Cannibalize and delete it.]
Convention: whenever we write $[a,b]$ it is implicit that $a \twoheadrightarrow b$.
We will write $A \sqcup B \sqcup C$ to indicate a disjoint union —
i.e., $A \cup B \cup C$, but it is implicit that $\{A, B, C\}$ is disjoint.

Lemmas:
(1) $\not S[a,b] \subseteq \{b\}$
(2) $a \neq b \Rightarrow [a,b] = \{a\} \sqcup [Sa, b]$
(3) a, b, c different, $b \in [a,c] \Rightarrow [a,b] \sqcup [Sb, c]$
(4) $b, c \in [a,b] \Rightarrow [a,bc] = [a,b] \vee [a,bc] = [a,c]$
(5) a, b, c, d different, $b, c \in [a,d]$, $b \in [a,bc] \Rightarrow$
$Sb, c \notin [a,bc]$
(6) a, b, c, d different, $b, c \in [a,d] \Rightarrow$
$[a,d] = \{a\} \sqcup [Sa, bc] \sqcup [Sb, cd] \sqcup [Sc, bd]$
(7) a, b, c, d different, $b, c \in [a,d] \Rightarrow$
$\{\{a\}, [Sa,bc], [Sb,cd], [Sc,bd]\}$ does not cover
$a \to Sa$, $b \to Sb$, or $c \to Sc$
(8) $La \Rightarrow [Sa, a] = [Sa, \infty) = [a, \infty)$

Now suppose that a is in a loop, that $b, c \in [Sa, a]$,
that Sa, b, c, a are different, and that $Sb = Sc$.
This implies that

$$[Sa, a] = \{Sa\} \sqcup [SSa, bc] \sqcup [Sb, ca] \sqcup [Sc, ba],$$

and this should be absurd (check).

This implies that each element in a loop
has at most one predecessor in the loop.
We already knew that each element in a loop
has a predecessor in the loop, so this shows that

$$La \Rightarrow \exists! p \in [a, \infty). Sp = a.$$