Index of the slides:

Quantifiers in Pavlovic's thesis

The rules $\mathrm{I}\Pi, \mathrm{E}\Pi, \mathrm{I}\Sigma, \mathrm{E}\Sigma$, as they appear in Pavlovič:

$$\dfrac{\dfrac{[X : P : \Delta']}{q : Q : \Delta''}}{\lambda X.q : \Pi X{:}P.Q : \Delta''}\ \mathrm{I}\Pi\Delta'\Delta''$$

$$\dfrac{p : P : \Delta' \quad r : \Pi X{:}P.Q(X) : \Delta''}{rp : Q[X := p] : \Delta''}\ \mathrm{E}\Pi\Delta'\Delta''$$

$$\dfrac{p : P : \Delta' \qquad \dfrac{[X : P : \Delta']}{Q : \Delta'} \qquad q : Q(p) : \Delta''}{\langle p, q\rangle : \Sigma X{:}P.Q : \Delta'}\ \mathrm{I}\Sigma\Delta'\Delta''$$

$$\dfrac{r : \Sigma X{:}P.Q : \Delta'' \qquad \dfrac{\dfrac{[X : P : \Delta']}{[Y : Q(X) : \Delta'']}}{s(X,Y) : S(\langle X, Y\rangle) : \Delta}}{\nu(r, (X,Y).s) : S(r) : \Delta}\ \mathrm{E}\Sigma\Delta'\Delta''$$

The same rules, but with a DNC-ish choice of letters:

$$\dfrac{\dfrac{[b : B : \Theta']}{c : C : \Theta''}}{\lambda b.c : \Pi b{:}B.C : \Theta''}\ \mathrm{I}\Pi\Theta'\Theta''$$

$$\dfrac{b' : B : \Theta' \quad f : \Pi b{:}B.C(b) : \Theta''}{fb : C[b := b'] : \Theta''}\ \mathrm{E}\Pi\Theta'\Theta''$$

$$\dfrac{b' : B : \Theta' \qquad \dfrac{[b : B : \Theta']}{C : \Theta'} \qquad c : C(b') : \Theta''}{\langle b', c\rangle : \Sigma b{:}B.C : \Theta'}\ \mathrm{I}\Sigma\Theta'\Theta''$$

$$\dfrac{p : \Sigma b{:}B.C : \Theta'' \qquad \dfrac{\dfrac{[b : B : \Theta']}{[c : C(b) : \Theta'']}}{d(b,c) : D(\langle b, c\rangle) : \Theta'''}}{\nu(p, (b,c).d) : D(p) : \Theta'''}\ \mathrm{E}\Sigma\Theta'\Theta''$$

A derivation from Lambek/Scott
Lambek/Scott, p.131:

$$\dfrac{\dfrac{\dfrac{\forall_{x\in A}\varphi(x) \vdash \forall_{x\in A}\varphi(x)}{\forall_{x\in A}\varphi(x) \vdash_x \varphi(x)}\;(2.4) \qquad \dfrac{\exists_{x\in A}\varphi(x) \vdash \exists_{x\in A}\varphi(x)}{\varphi(x) \vdash_x \exists_{x\in A}\varphi(x)}\;(2.4')}{\forall_{x\in A}\varphi(x) \vdash_x \exists_{x\in A}\varphi(x)}\;(1.2)}{\forall_{x\in A}\varphi(x) \vdash \exists_{x\in A}\varphi(x)}\;(1.4)$$

$$\dfrac{a \vdash b \qquad \dfrac{\dfrac{a;\forall b.P \vdash \forall b.P}{a,b;\forall b.P \vdash P}\;(\forall\mathrm{E}) \qquad \dfrac{a;\exists b.P \vdash \exists b.P}{a,b;P \vdash \exists b.P}\;(\exists\mathrm{I})}{a,b;\forall b.P \vdash \exists b.P}}{a;\forall b.P \vdash \exists b.P}\;(\mathrm{s})$$

Seely's PLC paper
(1.1.1. *Orders*) 1 and Ω are orders; if A and B are orders, then $A \times B$ and Ω^A are also orders.
(1.1.2. *Operators*) In the following, "$\sigma \in A$" means σ is an operator of order A; the rest of the arity is left unspecified for simplicity.
For each order, there is a countable set of variable operators (called "indeterminates").
$* \in 1$. $\top \in \Omega$.
If $\sigma, \tau \in \Omega$, then $\sigma \wedge \tau$ and $\sigma \supset \tau \in \Omega$.
If $\sigma \in \Omega$ and α is an indeterminate of order A, then $\Sigma\alpha \in A \cdot \sigma$ and $\Pi\alpha \in A \cdot \sigma \in \Omega$.
$(\times I)$ If $\sigma \in A$, $\tau \in B$, then $\langle \sigma, \tau \rangle \in A \times B$.
$(\times E)$ If $\sigma \in A \times B$, then $\pi_1 \sigma \in A$, $\pi_2 \sigma \in B$.
(PI) If α is an indeterminate of order A and $\sigma \in \Omega$, then $[\alpha \in A : \sigma] \in \Omega^A$.
(PE) If $\tau \in A$, $\sigma \in \Omega^A$, then $\sigma(\tau) \in \Omega$.
DEFINITION 1.1.3. A type is an operator of order Ω.
(1.1.4. *Terms*) In the following, "$a \in \tau$" means a is a term of type τ; the rest of the arity is left unspecified for simplicity.
For each type, there is a countable set of variable terms (called "variables").
$(\top I)$ $* \in \top$.
$(\supset I)$ If $a \in \tau$, x a variable of type σ, then $\lambda x \in \sigma \cdot a \in \sigma \supset \tau$.
$(\supset E)$ If $a \in \sigma \supset \tau$, $b \in \sigma$, then $a(b) \in \tau$.
$(\wedge I)$ If $a \in \sigma$, $b \in \tau$, then $\langle a, b \rangle \in \sigma \wedge \tau$.
$(\wedge E)$ If $a \in \sigma \wedge \tau$, then $\pi_1 a \in \sigma$ $\pi_2 a \in \tau$.
(ΣI) If $\alpha$is an indeterminate of order A, $\sigma \in \Omega$, $\tau \in A$, then $I_{\Sigma\alpha \cdot \sigma, \tau} \in \sigma[\tau/\alpha] \supset \Sigma\alpha \in A \cdot \sigma$. When clear from the context, we shall denote this term by I_τ, or even by I; in particular, if $b \in \sigma[\tau/\alpha]$, then $I(b) \in \Sigma\alpha \in A \cdot \sigma$.
(ΣE) In $a \in \sigma \supset \rho$, α an indeterminate of order A which is not free in ρ nor in the type of any free variable in a, then $\mathbf{V}\alpha \in A \cdot a \in (\Sigma\alpha \in A \cdot \sigma) \supset \rho$.
(ΠI) If $a \in \sigma$, α an indeterminate of order A which is not free in the type of any free variable in a, then $\Lambda\alpha \in A \cdot a \in \Pi\alpha \in A \cdot \sigma$.
(ΠE) If $a \in \Pi\alpha \in A \cdot \alpha$, $\tau \in A$, then $a\{\tau\} \in \sigma[\tau/\alpha]$, where $\sigma[\tau/\alpha]$ is σ with τ replacing α.

Seely's PLC paper: trees

$$\frac{}{\Omega : \Theta}$$

$$\frac{}{1 : \Theta} \qquad \frac{}{* : 1}$$

$$\frac{}{\top : \Omega} \qquad \frac{}{* : \top}$$

$$\frac{A : \Theta \quad B : \Theta}{A \times B : \Theta} \qquad \frac{\sigma : A \quad \tau : B}{\langle \sigma, \tau \rangle : A \times B} \qquad \frac{\sigma : A \times B}{\pi_1 \sigma : A} \qquad \frac{\sigma : A \times B}{\pi_2 \sigma : B}$$

$$\frac{A : \Theta}{A \to \Omega : \Theta} \qquad \frac{\sigma : \Omega}{[\alpha \in A : \sigma] : A \to \Omega} \qquad \frac{\tau : A \quad \sigma : A \to \Omega}{\sigma(\tau) : \Omega}$$

$$\frac{\sigma : \Omega \quad \tau : \Omega}{\sigma \wedge \tau : \Omega} \qquad \frac{a : \sigma \quad b : \tau}{\langle a, b \rangle : \sigma \wedge \tau} \qquad \frac{a : \sigma \wedge \tau}{\pi_1 a : \sigma} \qquad \frac{a : \sigma \wedge \tau}{\pi_2 a : \tau}$$

$$\frac{\sigma : \Omega \quad \tau : \Omega}{\sigma \supset \tau : \Omega} \qquad \frac{a : \tau}{\lambda x \in \sigma \cdot a : \sigma \supset \tau} \qquad \frac{b : \sigma \quad a : \sigma \supset \tau}{a(b) : \tau}$$

$$\frac{\sigma : \Omega}{\Sigma \alpha \in A \cdot \sigma : \Omega} \qquad \frac{\sigma : \Omega \quad \tau : A}{I : \sigma[\tau/\alpha] \supset \Sigma \alpha \in A \cdot \sigma} \qquad \frac{a : \sigma \supset \rho}{\mathbf{V} \alpha \in A \cdot a : (\Sigma \alpha \in A \cdot \sigma) \supset \rho}$$

$$\frac{\sigma : \Omega}{\Pi \alpha \in A \cdot \sigma : \Omega} \qquad \frac{a : \sigma}{\Lambda \alpha \in A \cdot a : \Pi \alpha \in A \cdot \sigma} \qquad \frac{\tau : A \quad a : \Pi \alpha \in A \cdot \alpha}{a\{\tau\} : \sigma[\tau/\alpha]}$$

Seely's PLC paper: trees, in DNC

$$\frac{}{\Omega : \Theta}$$

$$\frac{}{1 : \Theta} \qquad \frac{}{* : 1}$$

$$\frac{}{\omega[\top] : \Omega} \qquad \frac{}{\top : \omega[T]\top}$$

$$\frac{A \quad B}{A \times B} \qquad \frac{a \quad b}{a, b} \qquad \frac{a, b}{a} \qquad \frac{a, b}{b}$$

$$\frac{A}{A \to \Omega} \qquad \frac{\omega[P]}{a \mapsto \omega[P]} \qquad \frac{a \quad a \mapsto \omega[P]}{\omega[P]}$$

$$\frac{\omega[P] \quad \omega[Q]}{\omega[P\&Q]} \qquad \frac{P \quad Q}{P\&Q} \qquad \frac{P\&Q}{P} \qquad \frac{P\&Q}{Q}$$

$$\frac{\omega[P] \quad \omega[Q]}{\omega[P \supset Q]} \qquad \frac{Q}{P \supset Q} \qquad \frac{P \quad P \supset Q}{Q}$$

$$\frac{\omega[P]}{\omega[\exists b.P]} \qquad \frac{a, b \vdash \omega[P] \quad a \vdash b}{a \vdash P \supset (\exists b.P)} \qquad \frac{a \vdash \omega[Q] \quad a, b; \top \vdash P \supset Q}{a; \top \vdash (\exists b.P) \supset Q}$$

$$\frac{\omega[P]}{\omega[\forall b.P]} \qquad \frac{a \vdash \omega[P] \quad a, b; P \vdash Q}{a; P \vdash \forall b.Q} \qquad \frac{b \quad \forall b.P}{b}$$

Local set theories

```
(T1)  |-*

(T2)  a|-a

      a|-b  b|->c
(T3)  -----------
        a|-c

      a|-b_1 ... a|-b_n
(T4)  -----------------
      a|-(b_1,...,b_n)

      a|-(b_1,...,b_n)
(T5)  -----------------
          a|-b_i

      a,b|-ω[P]
(T6)  ---------
      a|-{b|P}

      a|-b  a|-b'
(T7)  -----------
      a|-ω[b=b']

      a|-b  a|-{b|P}
(T8)  --------------
       a|-ω[b∈{b|P}]


(L1)  P<=>Q := ω[P]=ω[Q]
(L2)      ⊤ := *=*
(L3)    P∧Q := (ω[P],ω[Q])=(ω[⊤],ω[⊤])
(L4)    P⊐Q := (P∧Q)<=>P
(L5)   ∀b.P := {b|P}={b|⊤}
(L6)      ⊥ := ∀ω[P].P
(L7)     ¬P := P⊐⊥
(L8)    P∨Q := ∀ω[R].(((P⊐R)∧(Q⊐R))⊐R)
(L9)   ∃b.P := ∀ω[Q]. ...
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Local set theories (1)

```
(Tautology)     P|-P
(Unity)         |-*'=*
(Equality)      b'=b''|-c[b:=b']=c[b:=b'']
(Products)      |-π<b,c>=b
                |-π'<b,c>=c
                |-<π(b,c),π'(b,c)>=(b,c)
(Comprehension) |-b∈{b|P}<=>P


                  P|-R
(Thinning)        ------
                  P,Q|-R

                  a;P|-Q   a;P,Q|-R
(Cut)             -----------------
                       a;P|-R

                    a,b;P|-Q   a|-b'
(Subst)           --------------------
                  a;P[b:=b']|-Q[b:=b']

                  a,b;P|-b∈{b|Q}<=>b∈{b|R}
(Extensionality)  ------------------------
                    a;P|-{b|Q}<=>{b|R}

                  P,Q|-R   P,R|-Q
(Equivalence)     ---------------
                     P|-Q<=>R
```

Notes on reading SeelyHyp

SeelyHyp, §4:

(5') (i) For $t : X \to Y$, φ over X, we define

$\Sigma_t \varphi \stackrel{\text{def}}{=} \exists \xi (t\xi = y \wedge \varphi(\xi))$,

$\Pi_t \varphi \stackrel{\text{def}}{=} \forall \xi (t\xi = y \supset \varphi(\xi))$,

$$\dfrac{\exists\xi(t\xi = y \wedge \gamma(\xi)) \qquad \dfrac{\dfrac{\dfrac{\dfrac{tx = y \wedge \gamma(x)}{[\gamma(x)]}\,(\wedge E) \\ \vdots\; P \\ \varphi(x) \qquad \dfrac{tx = y \wedge \gamma(x)}{tx = y}\,(\wedge E)}{tx = y \wedge \varphi(x)}\,(\wedge I)}{\exists\xi(t\xi = y \wedge \varphi(\xi))}\,(\exists I)}{}}{\exists\xi(t\xi = y \wedge \varphi(\xi))}\,(\exists E)$$

For $f : A \to B$, define:

$\Sigma_f \{\, a \,||\, P(a) \,\} \quad \stackrel{\text{def}}{=} \quad \{\, b \,||\, \exists a.(f(a) = b \;\&\; P(a)) \,\}$

$\Pi_f \{\, a \,||\, P(a) \,\} \quad \stackrel{\text{def}}{=} \quad \{\, b \,||\, \forall a.(f(a) = b \;\supset\; P(a)) \,\}$

$$\dfrac{\exists a.(f(a) = b \wedge P(a)) \qquad \dfrac{\dfrac{\dfrac{[f(a) = b \wedge P(a)]^1}{f(a) = b}\,(\wedge E) \qquad \dfrac{[f(a) = b \wedge P(a)]^1}{P(a)}\,(\wedge E) \\ \vdots \\ Q(a)}{f(a) = b \wedge Q(a)}\,(\wedge I)}{\exists a.(f(a) = b \wedge Q(a))}\,(\exists I)}{\exists a.(f(a) = b \wedge Q(a))}\,(\exists E)$$