Ring objects

$(\mathbb{R}, 0, 1, +, \cdot, -)$ can be seen as a "ring object" in **Set**,
that is, as five functions from powers of $\mathbb{R}$ to $\mathbb{R}$,
one for each operation:

$$1 \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{R} \underset{\cdot}{\overset{+}{\leftleftarrows}} \mathbb{R}^2$$

$$\begin{array}{rcl} * \longmapsto 0 & & \\ * \longmapsto 1 & & \\ a+b & \longleftarrow\!\shortmid & a,b \\ ab & \longleftarrow\!\shortmid & a,b \end{array}$$

(we will never draw the subtraction $- : \mathbb{R} \to \mathbb{R}$).

These arrows must obey some equations —
for example, $(a+b)c = ac + bc$, that becomes:

$$\dfrac{\dfrac{\dfrac{a,b,c}{a} \quad \dfrac{a,b,c}{b}}{a+b} \quad \dfrac{a,b,c}{c}}{(a+b)c} \quad = \quad \dfrac{\dfrac{\dfrac{a,b,c}{a} \quad \dfrac{a,b,c}{c}}{ac} \quad \dfrac{\dfrac{a,b,c}{b} \quad \dfrac{a,b,c}{c}}{bc}}{ac+bc}$$

$$\dfrac{\dfrac{\dfrac{\mathrm{id}}{\pi_1} \quad \dfrac{\mathrm{id}}{\pi_2}}{\langle \pi_1, \pi_2 \rangle; +} \quad \dfrac{\mathrm{id}}{\pi_3}}{\langle (\langle \pi_1, \pi_2 \rangle; +), \pi_3 \rangle; \cdot} \quad = \quad \dfrac{\dfrac{\dfrac{\mathrm{id}}{\pi_1} \quad \dfrac{\mathrm{id}}{\pi_3}}{\langle \pi_1, \pi_3 \rangle; \cdot} \quad \dfrac{\dfrac{\mathrm{id}}{\pi_2} \quad \dfrac{\mathrm{id}}{\pi_3}}{\langle \pi_2, \pi_3 \rangle; \cdot}}{\langle (\langle \pi_1, \pi_3 \rangle; \cdot), (\langle \pi_2, \pi_3 \rangle; \cdot) \rangle; +}$$

As **Set** has finite products every
$(\mathbb{R}, 0, 1, +, \cdot, -)$-polynomial in n variables
can be represented as a morphism $\mathbb{R}^n \to \mathbb{R}$;
each of the ring axioms becomes the statement
that two "$(\mathbb{R}, 0, 1, +, \cdot, -)$-polynomials" are equal.

A ring object: the tangent space

The tangent space of $\mathbb{R}$, $T\mathbb{R}$, has the same points as $\mathbb{R}^2$, and a ring structure, with special definitions for '1' and '$\cdot$'. We will denote its points as $(a, a_x), (b, b_x), \ldots$

Here is its ring structure:

$$1 \overset{0}{\underset{1}{\rightrightarrows}} T\mathbb{R} \overset{+}{\underset{\cdot}{\leftleftarrows}} (T\mathbb{R})^2$$

$$\begin{array}{rcl} * & \mapsto & (0,0) \\ * & \mapsto & (1,0) \\ (a+b, a_x+b_x) & \leftarrow\!\shortmid & (a,a_x),(b,b_x) \\ (ab, a_xb+b_xa) & \leftarrow\!\shortmid & (a,a_x),(b,b_x) \end{array}$$

Another ring object: a ring of functions
For any set S the space of functions $S \to \mathbb{R}$ (a.k.a. "$\mathbb{R}^S$")
has a natural ring structure:

$$1 \overset{0}{\underset{1}{\rightrightarrows}} (S \to \mathbb{R}) \overset{+}{\underset{\cdot}{\leftleftarrows}} (S \to \mathbb{R})^2$$

$$\begin{array}{rcl} * & \mapsto & (s \mapsto 0) \\ * & \mapsto & (s \mapsto 1) \\ (s \mapsto a[s] + b[s]) & \leftarrow\!\shortmid & (s \mapsto a[s], (s \mapsto b[s]) \\ (s \mapsto a[s]b[s]) & \leftarrow\!\shortmid & (s \mapsto a[s], (s \mapsto b[s]) \end{array}$$

If $S \subseteq \mathbb{R}$ then some functions $S \to \mathbb{R}$ are "affine linear",
in the sense that they can be characterized by two reals —
a "constant part" ('a') and a "slope" ('a_x').

Let's write these functions as $s \mapsto a + a_x s$.

Then the set of affine linear functions in $S \to \mathbb{R}$ is almost
closed by the ring operations — the only problem is the
second-order term in the result of the multiplication
(underlined below):

$$\begin{array}{rcl} * & \mapsto & (s \mapsto 0) \\ * & \mapsto & (s \mapsto 1) \\ (s \mapsto a + b + (a_x + b_x)s) & \leftarrow\!\shortmid & (\ldots), (\ldots) \\ (s \mapsto ab + (a_x b + a b_x)s + \underline{a_x b_x s^2}) & \leftarrow\!\shortmid & (\ldots), (\ldots) \end{array}$$

However, if $S \subseteq \{\, x \in \mathbb{R} \mid x^2 = 0 \,\}$ then the second-order term
disappears, and the set of affine linear functions

$$\mathrm{AffLin}(S \to R) := \{\, f : S \to \mathbb{R} \mid f \text{ is affine linear} \,\} \subseteq (S \to \mathbb{R})$$

is a subring of $S \to \mathbb{R}$, and, furthermore, there is a ring
homeomorphism $\varphi : T\mathbb{R} \to (S \to \mathbb{R})$...

A homomorphism between ring objects
"$\varphi : T\mathbb{R} \to (S \to \mathbb{R})$ is a ring homomorphism" means that for each of the five operations, $0, 1, +, \cdot, -$, a certain square commutes...

$$\begin{array}{ccccc}
1 & \overset{0}{\underset{1}{\rightrightarrows}} & T\mathbb{R} & \overset{+}{\underset{\cdot}{\leftleftarrows}} & T\mathbb{R} \times T\mathbb{R} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi\times\varphi} \\
1 & \overset{0}{\underset{1}{\rightrightarrows}} & (S \to \mathbb{R}) & \overset{+}{\underset{\cdot}{\leftleftarrows}} & (S \to \mathbb{R}) \times (S \to \mathbb{R})
\end{array}$$

(We do not draw the '$-$' arrows).
The less trivial case is the square for '$\cdot$':

$$\begin{array}{ccc}
(ab, a_x b + ab_x) & \longleftarrow\!\!\shortmid & (a, a_x), (b, b_x) \\
\downarrow & & \downarrow \\
(s \mapsto ab + (a_x b + ab_x)s) & & \\
(s \mapsto ab + (a_x b + ab_x)s + \underline{a_x b_x s^2}) & \longleftarrow\!\!\shortmid & (s \mapsto a + a_x s), (s \mapsto b + b_x s)
\end{array}$$

As we are supposing that $S \subseteq \{ x \in \mathbb{R} \mid x^2 = 0 \}$,
the term $a_x b_x s^2$ is zero, and that square commutes.

In $\mathbb{R}$ the set of square-zero elements, $\{ x \in \mathbb{R} \mid x^2 = 0 \}$, is too small for this to be interesting — *but the same constructions work for any ring R.*

Example: $R := \mathbb{R}[X, Y]/\langle X^2, Y^2 \rangle$ — the ring of polynomials on two variables, 'X' and 'Y', with coefficients on $\mathbb{R}$, divided by an ideal to force $X^2 = 0$ and $Y^2 = 0$.

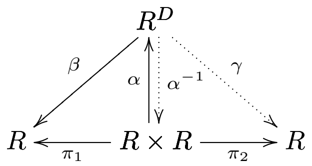
Notational convention: $\epsilon^2 = 0$ and $\delta^2 = 0$.
Then, using 'ϵ' and 'δ' as variables, we can write just "$\mathbb{R}[\epsilon, \delta]$" instead of "$\mathbb{R}[\epsilon, \delta]/\langle \epsilon^2, \delta^2 \rangle$".

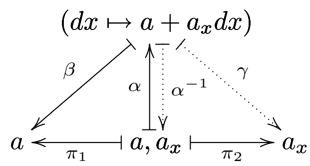
Note that $(\epsilon + \delta)^2 = \epsilon^2 + 2\epsilon\delta + \delta^2 = 2\epsilon\delta \neq 0$ — so $\epsilon + \delta$ is not a square-zero element in $\mathbb{R}[\epsilon, \delta]$.

Ring objects of line type

Fact (a.k.a. "Main Theorem", proved in the next slides):
When the arrow α below is invertible we can use the composite $\gamma := (\alpha^{-1}; \pi_2)$ to define, for any $f : R \to R$, its derivative $f' : R \to R$, and these derivatives behave as expected:

$$\begin{aligned}(kf)' &= kf' \\ (f+g)' &= f' + g', \\ (fg)' &= f'g + fg', \\ (f \circ g)' &= (f' \circ g)\, g'.\end{aligned}$$

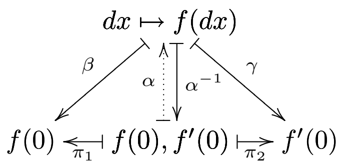




The hypotheses are just these:
C is a category with finite limits,
$(R, 0, 1, +, \cdot, -)$ is a ring object in **C**,
and $D := \{ dx \in R \mid dx^2 = 0 \}$
(that is definable as an equalizer)
is exponentiable.

(*Stronger hypotheses, simpler to understand:*
C is cartesian closed and has pullbacks,
$(R, 0, 1, +, \cdot, -)$ is a ring object in **C**.)

Then if the (definable) map $\alpha : R \times R \to R^D$ is invertible, we have a notion of "derivative" for functions $R \to R$, that behaves as expected.



A ring $(R, 0, 1, +, \cdot, -)$ for which $\alpha : R \times R \to R^D$ is invertible is said to be "of line type".

Lemma: even when α^{-1} does not exist β is known...
More precisely: *define* β as "evaluate $dx \mapsto a + a_x dx$ at $dx := 0$"; then $(\alpha; \beta) = \pi_1$.
If α^{-1} exists then $(\alpha; \beta) = \pi_1$ iff $\beta = (\alpha^{-1}; \pi_1)$.