

If L and R are (proto-)functors going in opposite directions between two (proto-)categories, say,

$$\mathbf{B} \begin{array}{c} \xleftarrow{L} \\ \xrightarrow{R} \end{array} \mathbf{A}$$

then a *proto-adjunction*, $L \dashv R$, is an 8-uple,

$$(\mathbf{A}, \mathbf{B}, L, R, b, \sharp, \eta, \epsilon)$$

that we draw as:

$$\begin{array}{ccccc} LRB & LA \longleftarrow A & A & & \\ \varepsilon_B \downarrow & f_g \downarrow \longleftarrow f_{g^\sharp} \downarrow & \eta_A \downarrow & & \\ B & B \longmapsto RB & RLA & & \\ & & & & \\ & \mathbf{B} \begin{array}{c} \xleftarrow{L} \\ \xrightarrow{R} \end{array} \mathbf{A} & & & \end{array}$$

There is some redundancy in this definition, as we may reconstruct some of the entities $b, \sharp, \eta, \epsilon$ in terms of the other ones:

$$\begin{array}{ccccc} & LA \longleftarrow A & & & \\ & \downarrow Lf \longleftarrow f \downarrow & & & \\ & LRB \longmapsto RB & & & \\ f^b := & \downarrow \varepsilon_B & & & \\ Lf; \varepsilon_B & B & A & & \\ & \downarrow \varepsilon_B & \eta_A \downarrow & & \\ & B \longmapsto RB & LA \longleftarrow A & & \\ \varepsilon_B := & \downarrow \varepsilon_B & \downarrow \text{id}_{LA} \longleftarrow \downarrow \mu_A := & & \\ \text{id}_{RB} b & \downarrow \varepsilon_B & \downarrow \text{id}_{LA} \longleftarrow \downarrow \text{id}_{LA} \sharp & & \\ & B \longmapsto RB & LA \longmapsto RLA & & \\ & & \downarrow g \longleftarrow \downarrow Rg & & \\ & & B \longmapsto RB & & \\ & & \downarrow \varepsilon_B & & \end{array}$$

A *protomonad* for a proto-endofunctor $T : \mathbf{A} \rightarrow \mathbf{A}$ is a 4-uple:

$$(\mathbf{A}, T, \eta, \mu)$$

that we draw as:

$$A \xrightarrow{\eta_A} TA \xleftarrow{\mu_A} TTA$$

A *proto-comonad* for a proto-endofunctor $S : \mathbf{B} \rightarrow \mathbf{B}$ is a 4-uple:

$$(\mathbf{B}, S, \varepsilon, \delta)$$

that we draw as:

$$B \xleftarrow{\varepsilon_B} SB \xrightarrow{\delta_B} SSB$$

Each proto-adjunction induces both a proto-monad and a proto-comonad. We draw all these together as:

$$\begin{array}{ccc} & & \begin{array}{c} LA \longleftarrow A \\ \downarrow f_g^b \quad \longleftarrow \quad \downarrow f_g^\sharp \\ B \longrightarrow RB \end{array} \\ \begin{array}{c} LRLRB \\ \uparrow \delta_B \\ LRB \\ \downarrow \varepsilon_B \\ B \end{array} & & \begin{array}{c} A \\ \downarrow \eta^A \\ RLA \\ \uparrow \mu^A \\ RLRLA \end{array} \\ & & \begin{array}{c} \mathbf{B} \xrightleftharpoons[L]{L} \mathbf{A} \end{array} \end{array}$$

We define $\mu_A := R(\text{id}_{RLA}^b)$ and $\delta_B := L(\text{id}_{LRB}^\sharp)$:

$$\frac{\frac{\text{id}_{RLA} : RLA \rightarrow RLA}{\text{id}_{RLA}^b : LRLA \rightarrow LA}}{\mu_A := R(\text{id}_{RLA}^b) : RLRLA \rightarrow RLA} \quad \frac{\frac{\text{id}_{LRB} : LRB \rightarrow LRB}{\text{id}_{LRB}^\sharp : RB \rightarrow RLRB}}{\delta_B := L(\text{id}_{LRB}^\sharp) : LRB \rightarrow LRLRB}$$

We have seen how a proto-adjunction induces a proto-monad; now we will see how a proto-monad induces *two* proto-adjunctions.

The Kleisli proto-adjunction

The *Kleisli proto-category* of a proto-monad $(\mathbf{A}, T, \eta, \mu)$ is the proto-category:

$$\mathbf{A}_T := ((\mathbf{A}_T)_0, \text{Hom}_{\mathbf{A}_T}, \text{id}_{\mathbf{A}_T}, \circ_{\mathbf{A}_T})$$

where $(\mathbf{A}_T)_0$ is equal to \mathbf{A}_0 , but we write the objects of $(\mathbf{A}_T)_0$ in a funny way: an object $A \in \mathbf{A}$ becomes

$$[A - \succ TA]$$

when we regard it as an object of $(\mathbf{A}_T)_0$.

A morphism in $\text{Hom}_{\mathbf{A}_T}([A - \succ TA], [C - \succ TC])$ is just a map $f : A \rightarrow TC$ in $\text{Hom}_{\mathbf{A}}(A, TC)$. We write it as $[f] : \text{Hom}_{\mathbf{A}_T}([A - \succ TA], [C - \succ TC])$ to stress that its (formal) type is different from f .

The identity operation, $\text{id}_{\mathbf{A}_T}$, is the η (the “unit”) of the monad in disguise:

$$\text{id}_{\mathbf{A}_T}([A - \succ TA]) := [\eta_A]$$

Note that:

$$\frac{\frac{\frac{A : \mathbf{A}_0}{\eta_A : A \rightarrow TA}}{[\eta_A] : [A - \triangleright TA] \rightarrow [A - \triangleright TA]}}{\text{id}_{\mathbf{A}_T}([A - \triangleright TA]) : \text{Hom}_{\mathbf{A}_T}([A - \triangleright TA], [A - \triangleright TA])} \text{ren}$$

The composition, $\circ_{\mathbf{A}_T}$, needs a trick: if $f : A \rightarrow TC$ and $g : C \rightarrow TE$ then $[f]; [g] := [f; Tg; \mu_E]$. In diagrams:

$$\begin{array}{ccc} [A - \triangleright TA] & & A \\ \downarrow \text{id}=[\eta_A] & & \searrow \eta_A \\ [A - \triangleright TA] & & TA \end{array} \qquad \begin{array}{ccc} A & & \\ \searrow f & & \\ C - - \triangleright TC & & \\ \searrow g & \searrow Tg & \\ E - - \triangleright TE & \longleftarrow TTE & \\ & \mu_E & \end{array}$$

$$\begin{array}{ccc} [A - \triangleright TA] & & \\ \downarrow [f] & & \\ [C - \triangleright TC] & \xrightarrow{[f]; [g] := [f; Tg; \mu_E]} & \\ \downarrow [g] & & \\ [E - \triangleright TE] & & \end{array}$$

The dashed arrow in, say, $[A - \triangleright TA]$, is to suggest three things:
 that morphisms in \mathbf{A}_T follow the direction of the ‘ $- \triangleright$ ’,
 that a morphism $A \rightarrow TA$ is not part of the definition of an object $[A - \triangleright TA]$,
 that the ‘ $- \triangleright$ ’ is the ghost of the unit of the monad — the unit would go from A to TA , but it is not used in the definitions; nevertheless, its memory remains.

We can draw the Kleisli (proto-)adjunction as:

$$\begin{array}{ccccc}
[TTC \multimap TTTC] & & & & \\
\uparrow \mu^? & & & & \\
[TC \multimap TTC] & [A \multimap TA] \leftarrow A & & A & \\
\downarrow \gamma & \downarrow f^b := \begin{smallmatrix} [f] \\ [g] \end{smallmatrix} & \rightleftarrows & \downarrow f & \downarrow \eta_A \\
[C \multimap TC] & [C \multimap TC] \multimap TC & & \downarrow [g]^\sharp := g & TA \\
& & & & \uparrow \mu_A \\
& & & & TTA \\
& & \mathbf{A}_T \xrightleftharpoons[R_T]{L_T} \mathbf{A} & &
\end{array}$$

The Eilenberg-Moore proto-adjunction

The *Eilenberg-Moore proto-category* for a proto-monad $(\mathbf{A}, T, \eta, \mu)$ is:

$$\mathbf{A}^T := ((\mathbf{A}^T)_0, \text{Hom}_{\mathbf{A}^T}, \text{id}_{\mathbf{A}^T}, \circ_{\mathbf{A}^T})$$

where an object of $(\mathbf{A}^T)_0$ is a pair (A, α) (a ‘‘proto-algebra’’), that we write as:

$$[A \xleftarrow{\alpha} TA]$$

We use a non-dashed arrow, ‘ $\xleftarrow{\alpha}$ ’, to stress that the map $\alpha : \text{Hom}_{\mathbf{A}}(TA, A)$ is part of the definition of the object.

A (proto-)morphism $f : [A \xleftarrow{\alpha} TA] \rightarrow [C \xleftarrow{\gamma} TC]$ is just a morphism $f : \text{Hom}_{\mathbf{A}}(A, C)$. The identity $\text{id}_{\mathbf{A}^T}$ and the composition $\circ_{\mathbf{A}^T}$ are defined in the obvious way (inherited from \mathbf{A}).

The Eilenberg-Moore adjunction can be drawn as:

$$\begin{array}{ccccc}
[TTC \xleftarrow{TT\gamma} TTTC] & & & & \\
\uparrow \mu^? & & & & \\
[TC \xleftarrow{T\gamma} TTC] & [TA \xleftarrow{\mu_A} TTA] \leftarrow A & & A & \\
\downarrow \gamma & \downarrow f^b := \begin{smallmatrix} Tf; \gamma \\ g \end{smallmatrix} & \rightleftarrows & \downarrow f & \downarrow \eta_A \\
[C \xleftarrow{\gamma} TC] & [C \xleftarrow{\gamma} TC] \multimap TC & & \downarrow [g]^\sharp := \eta_A; g & TA \\
& & & & \uparrow \mu_A \\
& & \mathbf{A}^T \xrightleftharpoons[R^T]{L^T} \mathbf{A} & & TTA
\end{array}$$

where [two triangles showing the transpositions]: