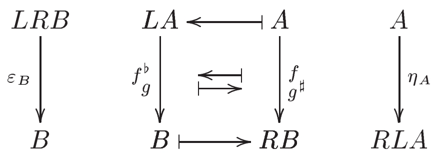
If L and R are (proto-)functors going in opposite directions between two (proto-)categories, say,

$$\mathbf{B} \underset{R}{\overset{L}{\leftrightarrows}} \mathbf{A}$$

then a *proto-adjunction*, $L \dashv R$, is an 8-uple,

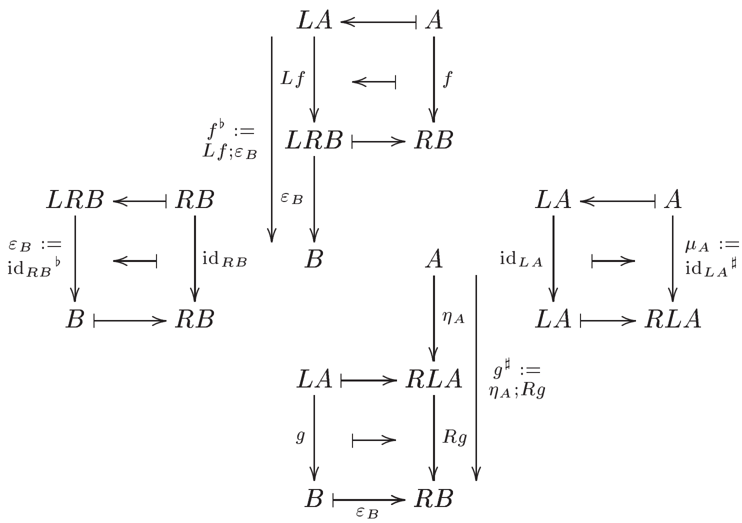
$$(\mathbf{A}, \mathbf{B}, L, R, \flat, \sharp, \eta, \epsilon)$$

that we draw as:



$$\mathbf{B} \underset{R}{\overset{L}{\leftrightarrows}} \mathbf{A}$$

There is some redundancy in this definition, as we may reconstruct some of the entities $\flat$, $\sharp$, η, ϵ in terms of the other ones:



A *protomonad* for a proto-endofunctor $T : \mathbf{A} \to \mathbf{A}$ is a 4-uple:

$$(\mathbf{A}, T, \eta, \mu)$$

that we draw as:

$$A \xrightarrow{\eta_A} TA \xleftarrow{\mu_A} TTA$$

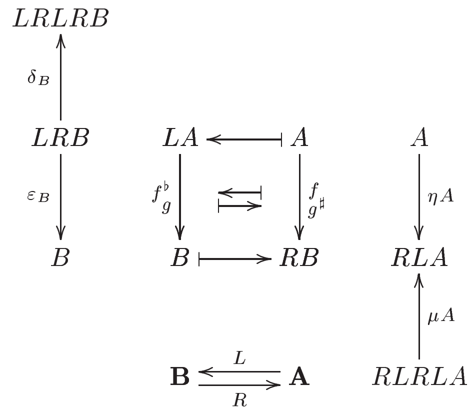
A *proto-comonad* for a proto-endofunctor $S : \mathbf{B} \to \mathbf{B}$ is a 4-uple:

$$(\mathbf{B}, S, \varepsilon, \delta)$$

that we draw as:

$$B \xleftarrow{\varepsilon_B} SB \xrightarrow{\delta_B} SSB$$

Each proto-adjunction induces both a proto-monad and a proto-comonad. We draw all these together as:



We define $\mu_A := R(\mathrm{id}_{RLA}{}^\flat)$ and $\delta_B := L(\mathrm{id}_{LRB}{}^\sharp)$:

$$\dfrac{\dfrac{\mathrm{id}_{RLA} : RLA \to RLA}{\mathrm{id}_{RLA}{}^\flat : LRLA \to LA}}{\mu_A := R(\mathrm{id}_{RLA}{}^\flat) : RLRLA \to RLA} \qquad \dfrac{\dfrac{\mathrm{id}_{LRB} : LRB \to LRB}{\mathrm{id}_{LRB}{}^\sharp : RB \to RLRB}}{\delta_B := L(\mathrm{id}_{LRB}{}^\sharp) : LRB \to LRLRB}$$

We have seen how a proto-adjunction induces a proto-monad; now we will see how a proto-monad induces *two* proto-adjunctions.

The Kleisli proto-adjunction

The *Kleisli proto-category* of a proto-monad $(\mathbf{A}, T, \eta, \mu)$ is the proto-category:

$$\mathbf{A}_T := ((\mathbf{A}_T)_0, \mathrm{Hom}_{\mathbf{A}_T}, \mathrm{id}_{\mathbf{A}_T}, \circ_{\mathbf{A}_T})$$

where $(\mathbf{A}_T)_0$ is equal to $\mathbf{A}_0$, ut we write the objects of $(\mathbf{A}_T)_0$ in a funny way: an object $A \in \mathbf{A}$ becomes

$$[A - \!\!\!> TA]$$

when we regard it as an object of $(\mathbf{A}_T)_0$.

A morphism in $\mathrm{Hom}_{\mathbf{A}_T}([A - \!\!\!> TA], [C - \!\!\!> TC])$ is just a map $f : A \to TC$ in $\mathrm{Hom}_{\mathbf{A}}(A, TC)$. We write it as $[f] : \mathrm{Hom}_{\mathbf{A}_T}([A - \!\!\!> TA], [C - \!\!\!> TC])$ to stress that its (formal) type is different from f.

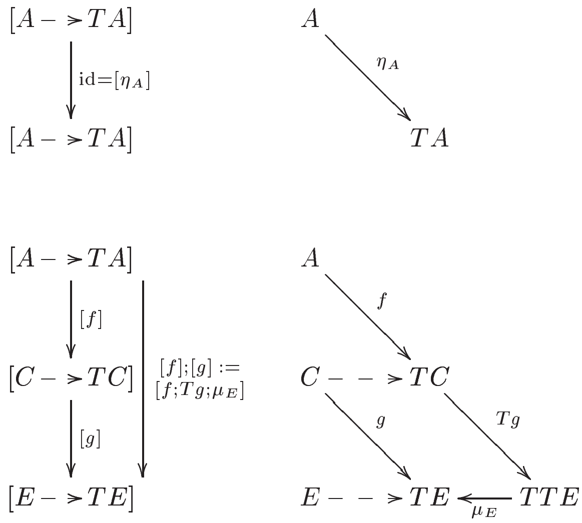
The identity operation, $\mathrm{id}_{\mathbf{A}_T}$, is the η (the "unit") of the monad in disguise:

$$\mathrm{id}_{\mathbf{A}_T}([A - \!\!\!> TA]) := [\eta_{\mathbf{A}}]$$

Note that:

$$\dfrac{\dfrac{\dfrac{A : \mathbf{A}_0}{\eta_A : A \to TA}}{[\eta_A] : [A - \succ TA] \to [A - \succ TA]}}{\mathrm{id}_{\mathbf{A}_T}([A - \succ TA]) : \mathrm{Hom}_{\mathbf{A}_T}([A - \succ TA], [A - \succ TA])}\ \mathsf{ren}$$

The composition, $\circ_{\mathbf{A}_T}$, needs a trick: if $f : A \to TC$ and $g : C \to TE$ then $[f];[g] := [f;Tg;\mu_E]$. In diagrams:



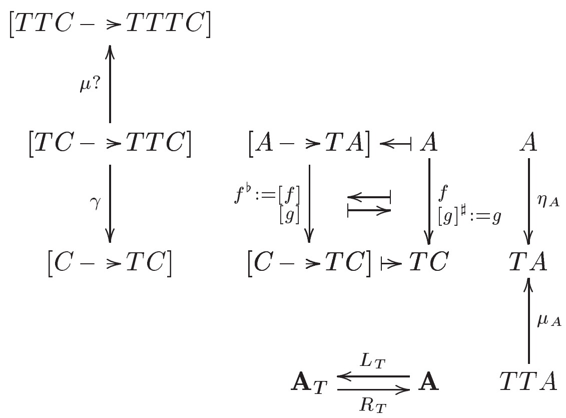
The dashed arrow in, say, $[A - \succ TA]$, is to suggest three things:

that morphisms in $\mathbf{A}_T$ follow the dirrection of the '$- \succ$',

that a morphism $A \to TA$ is not part of the definition of an object $[A - \succ TA]$,

that the '$- \succ$' is the ghost of the unit of the monad — the unit would go from A to TA, but it is not used in the definitions; nevertheless, its memory remains.

We can draw the Kleisli (proto-)adjunction as:



The Eilenberg-Moore proto-adjunction

The *Eilenberg-Moore proto-category* for a proto-monad $(\mathbf{A}, T, \eta, \mu)$ is:

$$\mathbf{A}^T := ((\mathbf{A}^T)_0, \mathrm{Hom}_{\mathbf{A}^T}, \mathrm{id}_{\mathbf{A}^T}, \circ_{\mathbf{A}^T})$$

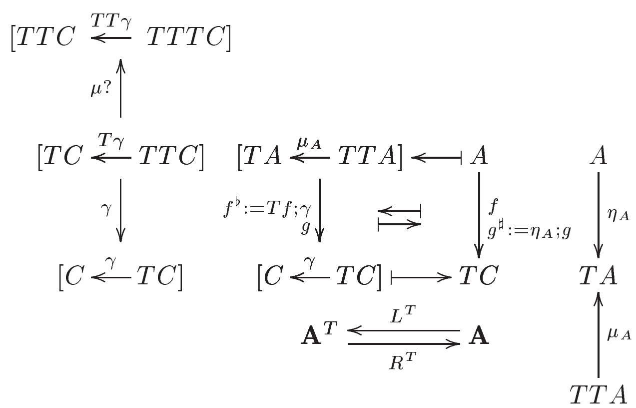
where an object of $(\mathbf{A}^T)_0$ is a pair (A, α) (a "proto-algebra"), that we write as:

$$[A \xleftarrow{\alpha} TA]$$

We use a non-dashed arrow, '$\longleftarrow$', to stress that the map $\alpha : \mathrm{Hom}_{\mathbf{A}}(TA, A)$ is part of the definition of the object.

A (proto-)morphism $f : [A \xleftarrow{\alpha} TA] \to [C \xleftarrow{\gamma} TC]$ is just a morphism $f : \mathrm{Hom}_{\mathbf{A}}(A, C)$. The identity $\mathrm{id}_{\mathbf{A}^T}$ and the composition $\circ_{\mathbf{A}^T}$ are defined in the obvious way (inherited from $\mathbf{A}$).

The Eilenberg-Moore adjunction can be drawn as:



where [two triangles showing the transpositions]: