Notes (very preliminary!) on downcasing:
Kock, Anders: A simple axiomatics for differentiation.
Math. Scand. 40 (1977), no. 2, 183-193.
http://www.mscand.dk/
http://www.mscand.dk/article.php?id=2356

The idea of "downcasing" is detailed here:
http://angg.twu.net/math-b.html#internal-diags-in-ct
http://angg.twu.net/LATEX/2010diags.pdf
Its section 17 is about "ring objects of line type".

Diagrams for the definition of the map α:

$$\begin{array}{ccc} A \times A \times A & & b, b', c \\ \Big\downarrow{\scriptstyle \lambda(a_1,a_2,a_3).\, a_1+(a_2\cdot a_3)} & & \Big\downarrow \\ A & & b + b'c \end{array}$$

$$\begin{array}{ccccccc} A \times A \times D & \longleftarrow & A \times A & \qquad & b, b_a, \mathsf{d}a & \Longleftarrow & b, b_a \\ \Big\downarrow{\scriptstyle \check{\alpha}} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \check{\alpha}} & & \Big\downarrow{\scriptstyle \alpha} \\ A & \longmapsto & A^D & & b + b_a \mathsf{d}a & \Longrightarrow & \mathsf{d}a \mapsto (b + b_a \mathsf{d}a) \end{array}$$

K77's Proposition 1: $\alpha : A \times A \to A^D$ is a morphims of ring objects.

$$1 \underset{1}{\overset{0}{\rightrightarrows}} T\mathbb{R} \underset{\cdot}{\overset{+}{\leftleftarrows}} (T\mathbb{R})^2$$

$$1 \underset{1}{\overset{0}{\rightrightarrows}} (A \times A) \underset{\cdot}{\overset{+}{\leftleftarrows}} (A \times A) \times (A \times A)$$

$$\begin{array}{l} * \longmapsto (0,0) \\ * \longmapsto (1,0) \\ (b+c, b_a + c_a) \longleftarrow (b, b_a), (c, c_a) \\ (bc, b_a c + bc_a) \longleftarrow (b, b_a), (c, c_a) \end{array}$$

$$\begin{array}{ccc} (A \times A) & \overset{*}{\longleftarrow} & (A \times A) \times (A \times A) \\ & & \Big\downarrow{\scriptstyle \alpha\times\alpha} \\ \Big\downarrow{\scriptstyle \alpha} & & A^D \times A^D \\ & & \Big\updownarrow{\scriptstyle \cong} \\ A^D & \underset{m^D}{\longleftarrow} & (A \times A)^D \end{array}$$

$$\begin{array}{ccc} (bc, b_a c{+}bc_a) & \overset{*}{\longleftarrow} & (b, b_a), (c, c_a) \\ \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle \alpha\times\alpha} \\ & & (da \mapsto b{+}b_a da), (da \mapsto c{+}c_a da) \\ da \mapsto bc{+}(b_a c{+}bc_a)da & & \Big\updownarrow{\scriptstyle \cong} \\ da \mapsto (b{+}b_a da)(c{+}c_a da) & \underset{m^D}{\longleftarrow} & da \mapsto (b{+}b_a da, c{+}c_a da) \end{array}$$

The translation to λ-calculus:

Let $\ulcorner 0 \urcorner^T \ := \ \lambda * .(0,0)$.
Let $\ulcorner 1 \urcorner^T \ := \ \lambda * .(1,0)$.
Let $+^T \ := \ \lambda((b,b_a),(c,c_a)).(b+c, b_a+c_a)$.
Let $\cdot^T \ := \ \lambda((b,b_a),(c,c_a)).(bc, b_a c + b c_a)$.
Then $(\ulcorner 0 \urcorner^T, \ulcorner 1 \urcorner^T, +^T, \cdot^T)$ is a ring object.

Let $\ulcorner 0 \urcorner^D \ := \ \lambda * .\lambda da.0$.
Let $\ulcorner 1 \urcorner^D \ := \ \lambda * .\lambda da.1$.
Let $+^D \ := \ \lambda(f_\Delta, g_\Delta), \lambda da.(f(da)+g(da))$.
Let $\cdot^D \ := \ \lambda(f_\Delta, g_\Delta), \lambda da.(f(da)g(da))$.
Then $(\ulcorner 0 \urcorner^D, \ulcorner 1 \urcorner^D, +^D, \cdot^D)$ is a ring object.

Let $\breve{\alpha} \ := \ \lambda(b, b_a, da).(b + b_a da)$.
Let $\alpha \ := \ \lambda(b, b_a).\lambda da.(b + b_a da)$.
Then α is a ring homomorphism.

Let $\hat{+} \ := \ \lambda a.\lambda da.(a+da)$.
Let $\tau \ := \ \lambda a.\langle a, 1 \rangle$.
Then $\tau; \alpha = \hat{+}$.

Let $\beta^\natural \ := \ \lambda f_\Delta.f_\Delta(0)$.
Then $\alpha; \beta^\natural = \pi$.

From now on let's suppose that α is an iso.
Let $\beta \ := \ \alpha^{-1}; \pi$.
Let $\gamma \ := \ \alpha^{-1}; \pi'$.
Then $\beta = \beta^\natural$.

Let's now define the derivative of a function $f : A \to A$.
Let $f' \ := \ \lambda a.\gamma(\lambda da.f(a+da))$.

First Taylor lemma: $\lambda(a,da).f(a+da) = \lambda(a,da).f(a) + f'(a)da$.
Abbreviated form: $f(a+da) = f(a) + f'(a)da$.

Let $(f+g) \ := \ \lambda a.f(a)+g(a)$.
Let $(fg) \ := \ \lambda a.f(a)g(a)$.
Let $(f \circ g) \ := \ \lambda a.f(g(a))$.

Product rule:

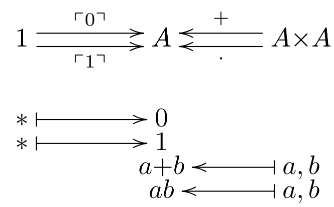
$$\begin{aligned}
(fg)(a+da) &= f(a+da)g(a+da) \\
&= (f(a)+f'(a)da)(g(a)+g'(a)da) \\
&= f(a)g(a) + (f'(a)g(a)+f(a)g'(a))da + f'(a)g'(a)da^2 \\
&= f(a)g(a) + (f'(a)g(a)+f(a)g'(a))da \\
&= (fg)(a) + (f'g+fg')(a)da
\end{aligned}$$

Chain rule:

$$\begin{aligned}
(f \circ g)(a+da) &= f(g(a+da)) \\
&= f(g(a)+g'(a)da) \\
&= f(g(a)) + f'(g(a))g'(a)da \\
&= (f \circ g)(a) + ((f' \circ g)g')(a)da
\end{aligned}$$

(Section 17 of the "Internal Diagrams" paper:)

Let $(R, \ulcorner 0 \urcorner, \ulcorner 1 \urcorner, +, \cdot)$ be a commutative ring in a CCC. That means: we have a diagram



and the morphisms $\ulcorner 0 \urcorner$, $\ulcorner 1 \urcorner$, $+$, $\cdot$ behave as expected.

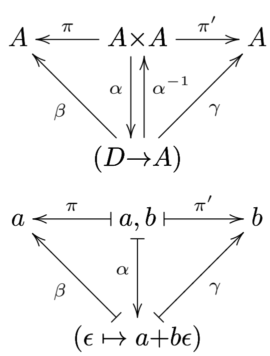
Let D be the set of zero-square infinitesimals of A, i.e., $\{\, \epsilon \in A \mid \epsilon^2 = 0 \,\}$; D can be defined categorically as an equalizer. If we take $A := \mathbb{R}$, then $D = \{0\}$; but if we let A be a ring with nilpotent infinitesimals, then $\{0\} \subsetneq A$.

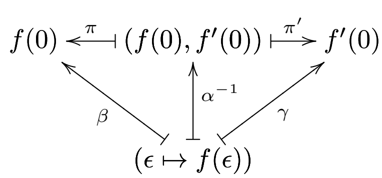
The main theorem of [Kock77] says that if the map

$$\begin{array}{rrcl} \alpha : & A \times A & \to & (D{\to}A) \\ & (a, b) & \mapsto & \lambda\epsilon{:}D.(a + b\epsilon) \end{array}$$

is invertible, then we can use α and α^{-1} to *define* the derivative of maps from A to A — *every* morphism $f : A \to A$ in the category $\mathbf{C}$ will be "differentiable" —, and the resulting differentiation operation $f \mapsto f'$ behaves as expected: we have, for example, $(fg)' = f'g + fg'$ and $(f \circ g)' = (f' \circ g)g'$.

Commutative rings with the property that their map α is invertible are called *ring objects of line type*. ROLTs are hard to construct, so most of the proofs about them have to be done in a very abstract setting. However, if we can use the following downcasings for α and α^{-1} — note that $\beta = (\alpha^{-1}; \pi)$, that $\gamma = (\alpha^{-1}; \pi')$, and that these notations do not make immediately obvious that α and α^{-1} are inverses —,





and then all the proofs in the first two sections of [Kock77] can be reconstructed from half-diagrammatic, half-λ-calculus-style proofs, done in the archetypal language, where the intuitive content is clear. This will be shown in a sequel to [OchsHyp].