Sheaves over finite DAGs may be archetypal

(Or: "Sheaves for non-categorists". Work in progress)

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http://angg.twu.net/math-b.html#sheaves-on-zdags

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These slides will probably be updated soon
to make them more self-contained.

Index of the slides:

3. Let's mystify the audience with technical terms

Modal logic:
S4 has the finite model property.
We have Gödel's translation: intuitionistic logic $\to$ S4
So: as $\neg\neg P \supset P$ is not a theorem of intutionistic logic
$\Rightarrow$ there is a finite model (with two worlds)
in which $\neg\neg P \supset P$ is not true.
These finite counter-models are good for developing intuition about intuitionistic logic.

Category Theory:
Let $\mathbb{W}$ be a finite poset.
($\mathbb{W}$ is a system of possible worlds for S4, viewed as a category).
Then $\mathbf{Set}^{\mathbb{W}}$ is a topos of presheaves.
The logic of toposes is intuitionistic,
and in $\mathbf{Set}^{\mathbb{W}} = \mathbf{Set}^{\bullet\to\bullet}$ we can falsify $\neg\neg P \supset P$.

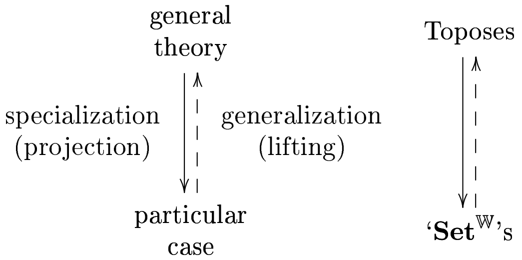
Claim:
Toposes of the form $\mathbf{Set}^{\mathbb{W}}$ are good for developing intuition about Topos Theory (and CT in general).

4. Let's mystify the audience a bit more

Sheaves are very important in Topos Theory.
Category Theory is hard (too abstract).
Even basic sheaf theory is too hard.
Idea: Let's use toposes of the form $\mathbf{Set}^{\mathbb{W}}$
to learn about sheaves!

In "Internal Diagrams in Category Theory" (2010)
I "defined" (loosely) a way of thinking
diagrammatically, and a notion of how much
"mental space" each idea takes.

Specializations behave like projections,
Generalizations behave like liftings:



Can we learn/define/understand sheaves
in toposes of the form $\mathbf{Set}^{\mathbb{W}}$
and then lift the theory to the general case?

5. Well-positioned subsets of Z^2 (and ZSets)

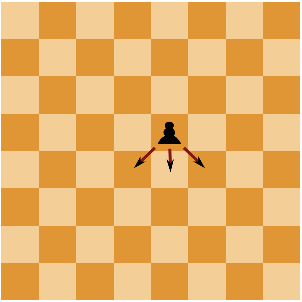
Def: a subset $D = \{(x_1, y_1), ..., (x_n, y_n)\} \subset \mathbb{Z}^2$
is **well-positioned** when $\inf_i x_i = 0$ and $\inf_i y_i = 0$.

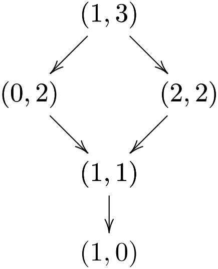
Def: **ZSet** is a finite well-positioned subset of $\mathbb{Z}^2$.

Examples:
$Y = \{(0, 2), (2, 2), (1, 1), (1, 0)\}$
$K = \{(1, 3),\ (0, 2), (2, 2),\ (1, 1),\ (1, 0)\}$

They will usually be named according to their shapes
('K' is for 'Kite').

6. Black pawn's moves (and ZDags)



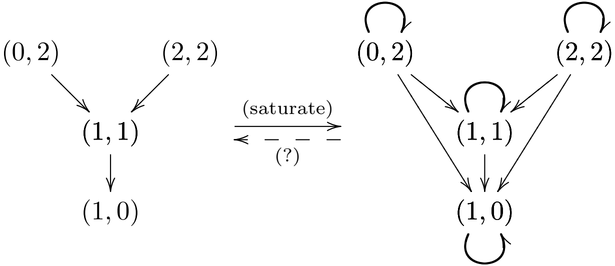


Example:
Let $K = \{(1,3),\ (0,2), (2,2),\ (1,1),\ (1,0)\}$.
Then the set of **black pawn's moves** on K, BPM_K,
is the set of 5 arrows at the right.
Let $\mathbb{K} = (K, \mathsf{BPM}_K)$ ← this a DAG.

Every ZSet D induces a DAG
$\mathbb{D} = (D, \mathsf{BPM}_D)$ ← this a **ZDag**.

7. Partial orders

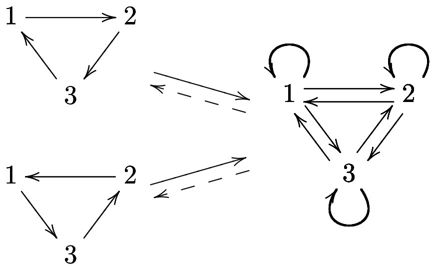
We are interested in S4 and categories, so
we like relations that are reflexive and transitive.
It is clumsy to draw (Y, BPM^*_Y) (at the right),
so we'd like to make (Y, BPM_Y) (at the left)
stand for (Y, BPM^*_Y).



Let's say that two relations, R and S,
are **equivalent** if $R^* = S^*$.
The class $[R] = \{ S \mid S^* = R^* \}$ has a top element,
R^*, obtained by a kind of saturation process
(transitive-reflexive closure).

8. Cycles are evil

Let $T = (\{1,2,3\}, \{1,2,3\}^2)$ be the complete graph on $\{1,2,3\}$. Then $[T]$ has two different minimal elements:



If we want to represent partial orders by minimal graphs we will need to avoid these...
"Reflexive" arrows, i.e., those of the form $\alpha \to \alpha$ are (sort of) irrelevant, so let's ignore them:
Def: R^{refl} is R plus all reflexive arrows.
Def: R^{irr} is R minus all reflexive arrows.
Def: R is acyclic when R^{irr} has no cycles. ← not standard!
Then in each class $[R]$ either all elements are acyclic or all are cyclic.

9. DAGs are good

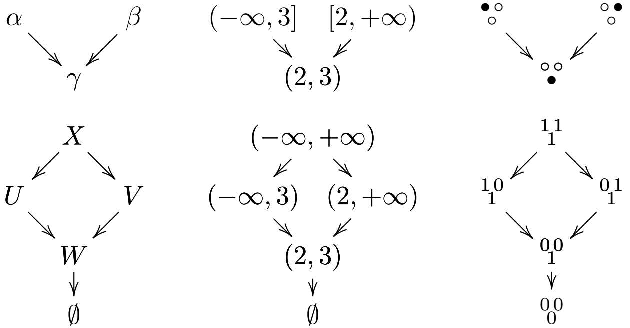
“Acyclic” for us is “acyclic modulo reflexive arrows”...
Consider the set of DAGs on a finite set of vertices A.
The equivalence relation $R \sim S \iff R^* = S^*$
partitions it into equivalent classes that are “diamond-shaped”,
i.e., “everything between a top and a bottom element”:
$[R] = \{\, R' \mid R^{\mathsf{ess}} \subseteq R' \subseteq R^* \,\}$.
To build R^{ess} from R we drop all “non-essential arrows”.
(This is the dual of the saturation $R \mapsto R^*$).

Moral: we can represent finite partial orders canonically
by their minimal DAGs (that only have “essential arrows”).
ZDags are finite, acyclic, and minimal. 8-)

10. Our favorite topological space: $\mathbb{V}$

Here it is:
as a DAG, $\mathbb{V} = (V, \mathsf{BPM}_V) = (\{\alpha, \beta, \gamma\}, \{(\alpha \to \gamma), (\beta \to \gamma)\})$
as a partial order, $\mathbb{V} = (V, \mathsf{BPM}_V^*)$
as a top. space, $\mathbb{V} = (X, \mathcal{O}(X))$ ← note the renaming!
$= (X, \{\{\alpha, \beta, \gamma\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, \{\gamma\}, \{\}\})$
$= (X, \{X, U, V, W, \emptyset\})$ ← names for the open sets
$= (X, \{{}^{1\,1}_{\ 1}, {}^{1\,0}_{\ 1}, {}^{0\,1}_{\ 1}, {}^{0\,0}_{\ 1}, {}^{0\,0}_{\ 0}\})$ ← positional notation!

We can think of it as a quotient topology on $\mathbb{R}$...



I draw X on top because it "covers" the other open sets,
and because ${}^{1\,1}_{\ 1}$ is $\top$ ("Top") in the Heyting algebra
(but $\top$ is also the terminal... the HA must $\mathbb{K}^{\mathrm{op}}$).
Surprise: $(\mathcal{O}(X), \supseteq^{\mathsf{ess}})$ is a ZDag!

11. Our favorite sheaves and presheaves

Let's write $\mathcal{O}(\mathbb{R})$ for $(\mathcal{O}(\mathbb{R}), \subseteq)$ $\quad \leftarrow$ a category $(\nearrow\nwarrow)$
and $\mathcal{O}(\mathbb{R})^{\mathrm{op}}$ for $(\mathcal{O}(\mathbb{R}), \supseteq)$. $\quad \leftarrow$ another $(\swarrow\searrow)$
Then $\mathcal{C}^\infty \in \mathbf{Set}^{\mathcal{O}(\mathbb{R})^{\mathrm{op}}}$ is a sheaf.
Bad news: it is too big to visualize.

We write $\mathbb{V} \equiv$  and $\mathbb{K} = \mathbb{V}' \equiv$ .
Let's define presheaves $C^\infty, E \in \mathbf{Set}^{\mathbb{K}}$.
A presheaf in $\mathbf{Set}^{\mathbb{D}}$ is just a functor from $\mathbb{D}$ to $\mathbf{Set}$.
Sheafness is separatedness plus collatedness.
C^∞ will obey both, and E will fail both.

$$C^\infty \quad = \quad \begin{array}{ccc} & C^\infty(X) & \\ \swarrow & & \searrow \\ C^\infty(U) & & C^\infty(V) \\ \searrow & & \swarrow \\ & C^\infty(W) & \\ & \downarrow & \\ & C^\infty(\emptyset) & \end{array} \quad = \quad \begin{array}{ccc} & \mathcal{C}^\infty(\mathbb{R}) & \\ \swarrow & & \searrow \\ \mathcal{C}^\infty((-\infty,1)) & & \mathcal{C}^\infty((0,+\infty)) \\ \searrow & & \swarrow \\ & \mathcal{C}^\infty((0,1)) & \\ & \downarrow & \\ & \mathcal{C}^\infty(\emptyset) & \end{array}$$

12. Compatibility

Let $U = (-\infty, 3)$ and $V = (2, \infty)$ (temporarily).
Let $f_U \in \mathcal{C}^\infty(U, \mathbb{R})$ and $f_V \in \mathcal{C}^\infty(V, \mathbb{R})$, in:

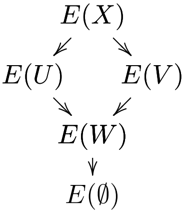
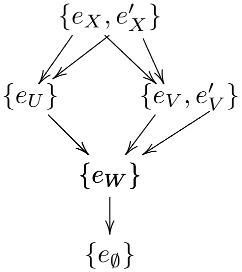
$$\begin{array}{ccc}
 & \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}) & \\
\swarrow & & \searrow \\
\mathcal{C}^\infty((-\infty, 3), \mathbb{R}) & & \mathcal{C}^\infty((2, +\infty), \mathbb{R}) \\
\searrow & & \swarrow \\
 & \mathcal{C}^\infty((2, 3), \mathbb{R}) & \\
 & \downarrow & \\
 & \mathcal{C}^\infty(\emptyset, \mathbb{R}) &
\end{array}$$

We say that two "locally defined functions", f_U and f_V, are **compatible** iff they "coincide wherever they're both defined" (in the example: on $(2, 3)$).
More precisely: f_U and f_V are compatible iff $f_U|_{U \cap V} = f_V|_{U \cap V}$.
Sheafness means that every **compatible family** $\{f_U, \ldots, f_V\}$ has exactly one glueing to an $f_{U \cup \ldots \cup V}$
(collatedness guarantees existence of a glueing, separatedness guarantees that there is at most one).

13. The evil presheaf

Here is the "evil presheaf", $E :$  $\to$ **Set**.

Note that everything here is given explicitly —
restriction functions that are the images of black pawn's moves, e.g., $\rho^X_V : E(X) \to E(V)$, are *drawn*;
restriction functions like ρ^U_U are necessarily $= \mathrm{id}_{E(U)}$, and
restriction functions like ρ^X_W are obtained by composition.
Note (again!) that E is a *functor*.

$E \quad =$  $=$ 

Then $\{e_U, e_V\}$ is a compatible family,
because $e_U|_{U\cap V} := \rho^U_W(e_U) = e^W$ and $e_V|_{U\cap V} := \rho^V_W(e_V) = e^W$,
but $\{e_U, e_V\}$ has two different glueings, e_X and e'_X,
so separatedness doesn't hold in E...
Also, $\{e_U, e'_V\}$ is another compatible family,
and this one has *no glueings*.
So collatedness also doesn't hold in E.

14. Stack operations

The fastest way to formalize all this is by using **stacks**.
(This is not the standard way at all! I learned it from Harold Simmons's "The point-free approach to sheafification".)

This is E as a stack:
$\Sigma E = E(X) \sqcup E(U) \sqcup E(V) \sqcup E(W) \sqcup E(\emptyset)$
We have an operation called "extent", $[e_U] = U$,
going from ΣE to $\Omega = \{X, U, V, W, \emptyset\}$,
and a non-commutative '$\cdot$', heavily overloaded,
that behaves as *restriction* when its left arg is in ΣE
and as *intersection* when its left arg is in Ω:

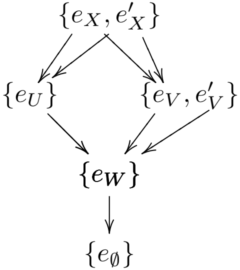
$$\begin{array}{rcl} U \cdot V & := & U \wedge V \\ & = & W \\ U \cdot e_V & := & U \cdot [e_V] \\ & = & U \cdot V \\ & = & W \\ e_U \cdot V & := & e_U|_{([e_U] \cdot V)} \\ & = & e_W \\ e_U \cdot e_V & := & e_U|_{([e_U] \cdot [e_V])} \\ & = & e_W \end{array}$$

15. Stack operations (2)

The '$\cdot$' also accepts sets as arguments, with the usual conventions:
$\{a, b\} \cdot \{c, d\} = \{a \cdot c, a \cdot d, b \cdot c, b \cdot d\}$,
$a \cdot \{b, c\} = \{a \cdot b, a \cdot c\}$,
$\{a, b\} \cdot c = \{a \cdot c, b \cdot c\}$.
(Also: $[\{a, b\}] = \{[a], [b]\}$).

16. Covers and families

Def: a **cover** is a subset of Ω. (Example: $\{U, V\}$)
Def: a **family** is a subset of ΣE "where $[\cdot]$ is injective".
Def: a **compatible family** is a family "where '$\cdot$' commutes".
Example 1: $\{e_V, e'_V\}$ is not a family.
Example 2: $\{e_U, e_V\}$ is a compatible family.
Example 3: $\{e_X, e'_V\}$ is non-compatible family.



Notation for covers: $\mathcal{U}, \mathcal{V}, \ldots$, where $\bigcup \mathcal{V} = V$.
Notation for families: $e_\mathcal{U}$, where $[e_\mathcal{U}] = \mathcal{U}$.
Def: a cover $\mathcal{U}$ is (downward) **saturated** when $\mathcal{U} \cdot \Omega = \mathcal{U}$.
Def: a family $e_\mathcal{U}$ is (downward) **saturated** when $e_\mathcal{U} \cdot \Omega = e_\mathcal{U}$.
Example 4: $\{U, V\} \cdot \Omega = \{U, V, W, \emptyset\}$.
Example 5: $\{e_U, e'_V\} \cdot \Omega = \{e_U, e'_V, e_W, e_\emptyset\}$.
Example 6: $e_X \cdot \Omega = \{e_X, e_U, e_V, e_W, e_\emptyset\}$.
Example 7: $e_X \cdot \{U, V\} \cdot \Omega = \{e_U, e_V, e_W, e_\emptyset\}$.

17. Saturated families

Let's annotate saturated covers with a '$\bullet$'.
So: $\mathcal{U}, \mathcal{U}', \mathcal{U}^\bullet, \mathcal{U}^{\bullet\prime}$ are saturated families,
possibly different, all "covering U".

Let's write the saturation operation, '$\cdot\Omega$', as '$()^\bullet$',
and let's say that $\mathcal{U} \approx \mathcal{V}$ when $(\mathcal{U})^\bullet = (\mathcal{V})^\bullet$,
and write the equivalence classes as $[\mathcal{U}]$.

On finite DAGs each equivalence class has both a top element
and a bottom element:
$[\mathcal{U}] = \{\, \mathcal{U}' \mid (\mathcal{U})^\circ \subseteq \mathcal{U}' \subseteq (\mathcal{U})^\bullet \,\}$.
The operation $(\mathcal{U})^\circ$, that drops all "non-essential open sets"
in a cover, is new...
and it also makes sense for families.

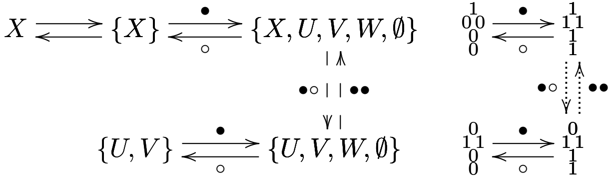
Examples:
$\{U, V, W\}^\bullet = \{U, V, W, \emptyset\}$
$\{U, V, W\}^\circ = \{U, V\}$
$\{e_U, e_V, e_W\}^\bullet = \{e_U, e_V, e_W, e_\emptyset\}$
$\{e_U, e_V, e_W\}^\circ = \{e_U, e_V\}$

18. Adding unions

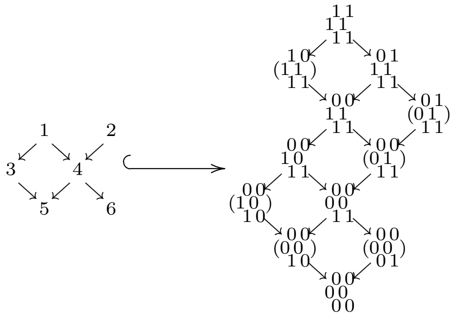
In a sheaf $F : \mathbb{K} \to \mathbf{Set}$ every compatible family $f_{\mathcal{U}}$ can be glued in a unique way to obtain a f_U, and we can obtain $f_{\mathcal{U}}$ back from f_U: $f_{\mathcal{U}} = f_U \cdot \mathcal{U}$.

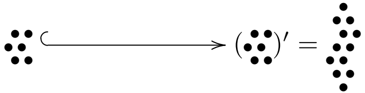
To understand what is going on here we need another notion of saturation...

The '•' saturation adds *smaller opens sets* to a cover;
The '••' saturation also adds *unions* to a cover.



19. Priming





To understand "topological sheaves" we take a DAG (e.g., $\mathbb{V}$) and prime it twice; the operations '••' and '•○' work on $\mathbb{V}''$.

For "generic" sheaves ("sheaves on a site") we take any DAG $\mathbb{D}$ to play the role of $\mathbb{V}'$ and an operation '$*$' on $\mathbb{D}$ that obeys three rules (obeyed by '••', of course), and from there on we treat what were "open sets" as "truth-values" (!!!), and the '$*$' as a modality (!!!!!).

20. What next?

...but that doesn't fit in 20 minutes! 8-(
Look for the complete version of these slides in my home page!

Goodbye! 8-)