Intuitionistic Logic for Children, or:
Planar Heyting Algebras for Children

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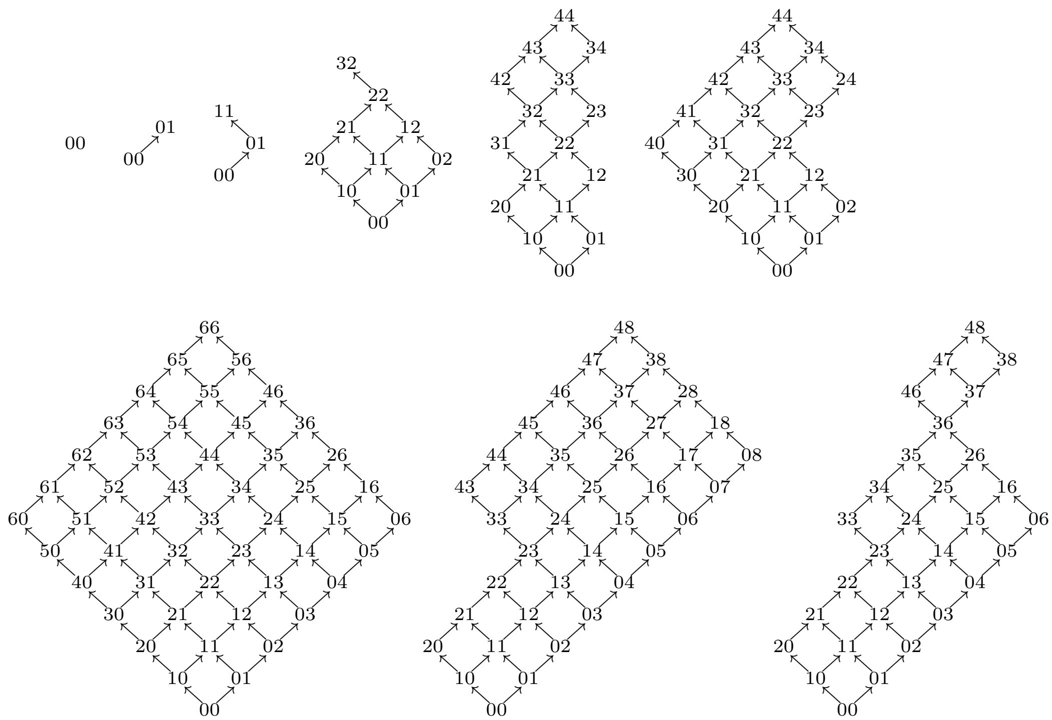
http://angg.twu.net/math-b.html#zhas-for-children
http://angg.twu.net/LATEX/2015planar-has.pdf

This is a work in progress...
It has a funny formatting because it is:
part seminar notes (for humanities people),
part handouts,
part a demo of dednat6,
part a draft for something more serious.

Also, the "seminar notes" format allowed me
to focus on examples and figures instead of
on formal definitions.
For more on archetypal examples, see:
http://angg.twu.net/math-b.html#idarct
http://angg.twu.net/LATEX/idarct-preprint.pdf
Feedback very welcome!

One page intro (to the main theorem)

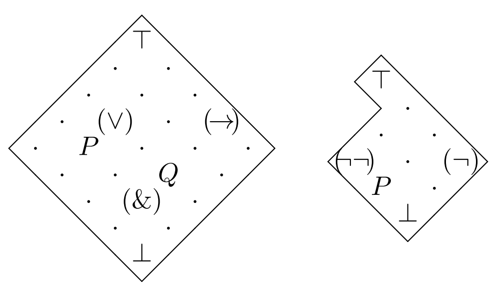
Each one of the posets below is a Heyting Algebra:



The connectives '&', '∨', '→' can be *defined* by:

$$
\begin{array}{rcllll}
ab \,\&\, cd & := & \mathsf{min}(a,c)\,\mathsf{min}(b,d) & & \\
ab \vee cd & := & \mathsf{max}(a,c)\,\mathsf{max}(b,d) & & \\
P \to Q & := & \mathsf{if} & (P\ \mathsf{below}\ Q) & \mathsf{then}\ \top \\
 & & \mathsf{elseif} & (P\ \mathsf{leftof}\ Q) & \mathsf{then}\ \mathsf{ne}(P\&Q) \\
 & & \mathsf{elseif} & (P\ \mathsf{rightof}\ Q) & \mathsf{then}\ \mathsf{nw}(P\&Q) \\
 & & \mathsf{elseif} & (P\ \mathsf{above}\ Q) & \mathsf{then}\ Q \\
 & & \mathsf{end} & &
\end{array}
$$

which are easy to interpret graphically - for example:



$$
\begin{array}{rcl}
(\&) & := & P\&Q \\
(\vee) & := & P \vee Q \\
(\to) & := & P \to Q \\
(\neg) & := & \neg P \\
(\neg\neg) & := & \neg\neg P
\end{array}
$$

Connectives (via brute force)

The best way to see that the definitions

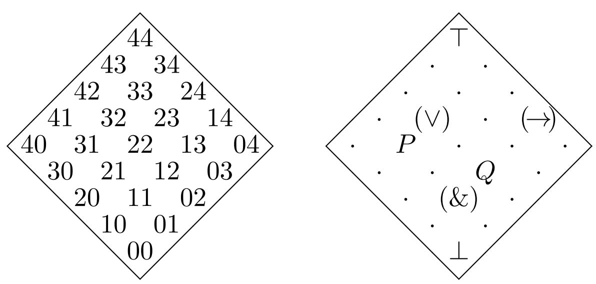
$$\begin{array}{rcllll}
ab \mathbin{\&} cd & := & \mathsf{min}(a,c)\,\mathsf{min}(b,d) \\
ab \vee cd & := & \mathsf{max}(a,c)\,\mathsf{max}(b,d) \\
P \to Q & := & \mathsf{if} & (P \mathrel{\mathsf{below}} Q) & \mathsf{then}\ \top \\
& & \mathsf{elseif} & (P \mathrel{\mathsf{leftof}} Q) & \mathsf{then}\ \mathsf{ne}(P\&Q) \\
& & \mathsf{elseif} & (P \mathrel{\mathsf{rightof}} Q) & \mathsf{then}\ \mathsf{nw}(P\&Q) \\
& & \mathsf{elseif} & (P \mathrel{\mathsf{above}} Q) & \mathsf{then}\ Q \\
& & \mathsf{end}
\end{array}$$

obey the expected properties, which are

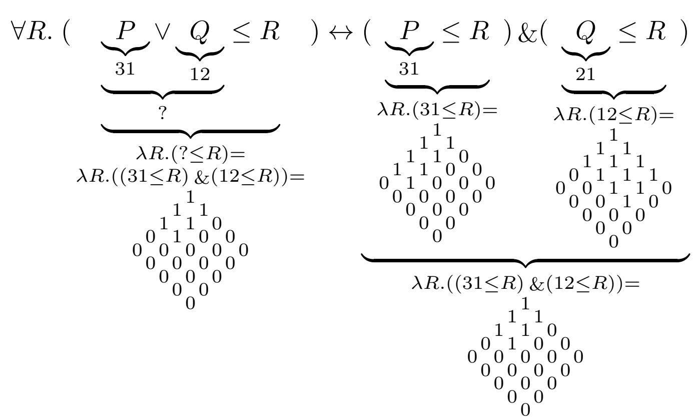
$$\begin{array}{rrcl}
\forall P. & (P \le Q\&R) & \leftrightarrow & (P \le Q)\&(P \le R) \\
\forall R. & (P{\vee}Q \le R) & \leftrightarrow & (P \le R)\&(Q \le R) \\
\forall P. & (P \le Q{\to}R) & \leftrightarrow & (P\&Q \le R)
\end{array}$$

is by brute force.

For example, in this case,



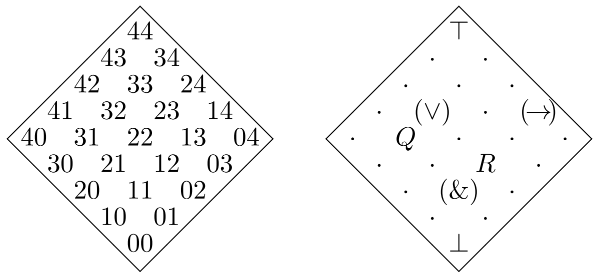
we can do:



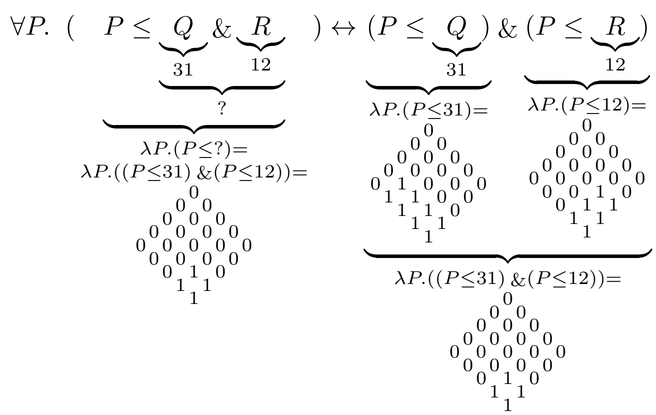
we get $(31 \vee 12) =$ '?' $= 32$.

Connectives (via brute force, 2)

$$\forall P.\quad (P \le Q \& R) \quad \leftrightarrow \quad (P \le Q) \& (P \le R)$$
$$\forall P.\quad (P \le Q \to R) \quad \leftrightarrow \quad (P \& Q \le R)$$

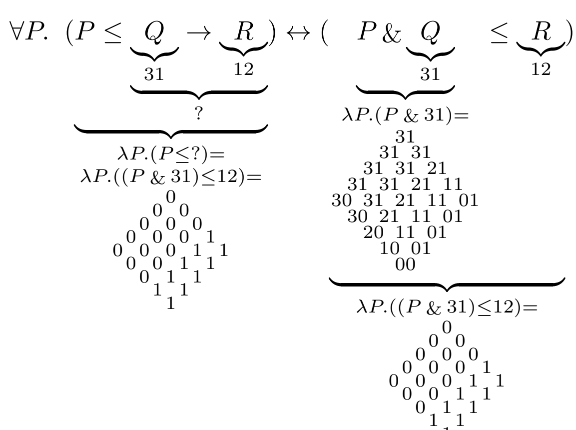


Here's how to calculate $31 \,\&\, 12$:



We get $(31 \,\&\, 12) =$'?'$= 11$.

Once we learn how to calculate '&'s quickly,
we can calculate '→'s - they need $\lambda P.(P \,\&\, Q)$:

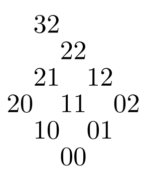


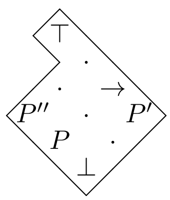
Some non-tautologies
Some propositions that are always true classically,

P	$\neg P$	$\neg\neg P$	$(\neg\neg P)\to P$
0	1	0	1
1	0	1	1

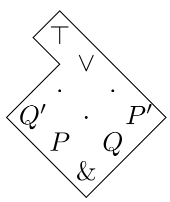
P	Q	$P \,\&\, Q$	$\neg(P \,\&\, Q)$	$\neg P$	$\neg Q$	$\neg P \lor \neg Q$	$\neg(P \,\&\, Q) \to (\neg P \lor \neg Q)$
0	0	0	1	1	1	1	1
0	1	0	1	1	0	1	1
1	0	0	1	0	1	1	1
1	1	1	0	0	0	0	1

are not always true intuitionistically,
and we can use ZHAs to exhibit cases where they are not $\top$:





$$\underbrace{(\underbrace{\neg\underbrace{\neg\underbrace{P}_{10}}_{02}}_{20}) \to \underbrace{P}_{10}}_{12}$$



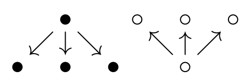
$$\underbrace{\underbrace{\neg(\underbrace{\underbrace{P}_{10} \,\&\, \underbrace{Q}_{01}}_{00})}_{32} \to \underbrace{(\underbrace{\neg\underbrace{P}_{10}}_{02} \lor \underbrace{\neg\underbrace{Q}_{01}}_{20})}_{22}}_{22}$$

I have *some* material that helps in telling the full story -
classical and intuitionistic theorems and tautologies, for children -
and I will try to put it in the last section of these notes
as I typeset it for the seminars.

Basic definitions.
A *ZSet* is a finite nonempty subset of $\mathbb{N}^2$ that touches boths axes.
The *black moves* and the *white moves* on a ZSet A are defined as:

$$\begin{array}{rcl} \mathsf{BM}(A) & := & \{\, ((x,y),(x+dx,y-1)) \in A^2 \mid dx \in \{-1,0,1\} \,\} \\ \mathsf{WM}(A) & := & \{\, ((x,y),(x+dx,y+1)) \in A^2 \mid dx \in \{-1,0,1\} \,\} \end{array}$$

Mnemonic:
a black piece, '$\bullet$', is solid/heavy/wants to sink and move down;
a white piece, '$\circ$', is hollow/light/wants to float and move up.
Figure:



A *ZDAG* is a graph of the form $(A, \mathsf{BM}(A))$ or $(A, \mathsf{WM}(A))$, and
A *ZPoset* is a graph of the form $(A, \mathsf{BM}(A)^*)$ or $(A, \mathsf{WM}(A)^*)$,
where A is a ZSet, and (A, R^*) is transitive-reflexive closure of (A, R).

We say that triple (maxy, L, R) *generates a ZHA* when:
1) $\mathsf{maxy} \in \mathbb{N}$, and L and R are functions from $\{0, 1, ..., \mathsf{maxy}\}$ to $\mathbb{N}$,
2) $L(y) \leq R(y)$ always holds,
3) $L(y+1) = L(1) \pm 1$ and $R(y+1) = R(1) \pm 1$ always hold,
4) $L(0) = R(0)$ and $L(\mathsf{maxy}) = R(\mathsf{maxy})$,
5) $L(y) = 0$ for some y.

The *parity* of $(x, y) \in \mathbb{N}^2$ is the parity of $x + y$.
The *left wall* and the *right wall* of a ZHA are the sets

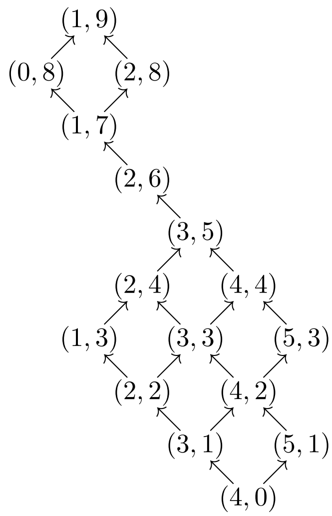
$$\begin{array}{rcl} \mathsf{LW}(\mathsf{maxy}, L, R) & := & \{\, (x,y) \in \mathbb{N}^2 \mid x = L(y) \,\}, \\ \mathsf{RW}(\mathsf{maxy}, L, R) & := & \{\, (x,y) \in \mathbb{N}^2 \mid x = R(y) \,\}. \end{array}$$

The *ZSet generated by* (maxy, L, R), $\mathsf{ZS}(\mathsf{maxy}, L, R)$, is the set of all points between $\mathsf{LW}(\mathsf{maxy}, L, R)$ and $\mathsf{RW}(\mathsf{maxy}, L, R)$ with the same parity as $(L(0), 0)$.
The *ZHA generated by* (maxy, L, R) is this ZPoset:

$$\mathsf{ZHA}(\mathsf{maxy}, L, R) \;:=\; (\mathsf{ZS}(\mathsf{maxy}, L, R), \mathsf{WM}(\mathsf{ZS}(\mathsf{maxy}, L, R))^*)$$

We use the *lr-coordinates* to refer to points of a ZHA.
The point $(L(0), 0)$ is denoted by "00".
The l-coordinate increases when we walk northwest.
The r-coordinate increases when we walk northeast.

ZHAs, visually



$L(9) = 1$	$R(9) = 1$	$L(9) = R(9)$	$\mathsf{maxy} = 9$
$L(8) = 0$	$R(8) = 0$		
$L(7) = 1$	$R(7) = 1$		
$L(6) = 2$	$R(6) = 2$		
$L(5) = 3$	$R(5) = 3$		
$L(4) = 2$	$R(4) = 4$		
$L(3) = 1$	$R(3) = 5$		
$L(2) = 2$	$R(2) = 4$		
$L(1) = 3$	$R(1) = 5$		
$L(0) = 4$	$R(0) = 4$	$L(0) = R(0)$	

We say that triple (maxy, L, R) *generates a ZHA* when:
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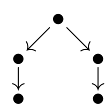
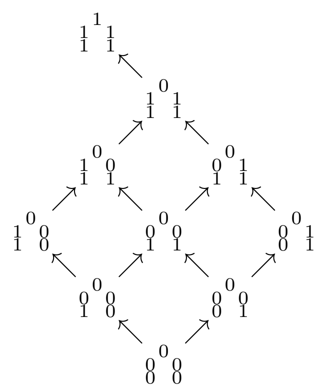
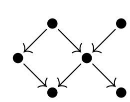
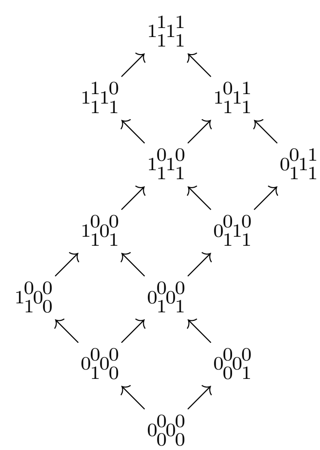
The *ZHA generated by* (maxy, L, R) is this ZPoset:

$$\mathsf{ZHA}(\mathsf{maxy}, L, R) \ := \ (\mathsf{ZS}(\mathsf{maxy}, L, R), \mathsf{WM}(\mathsf{ZS}(\mathsf{maxy}, L, R))^*)$$

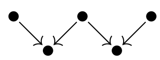
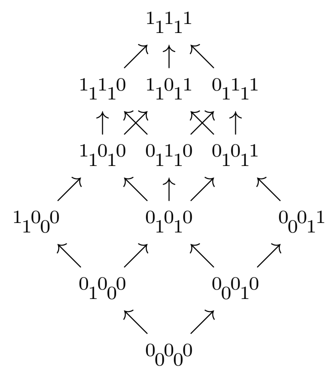
Background story.
Several years ago I was looking for finite, easy-to-draw Heyting Algebras, because I was trying to understand sheaves, and I had no intuition at all about what those "closure operators" were doing...
When I tried to generate Heyting Algebras from order topologies - if $D = (A, R)$ is a DAG, then $D' := (\mathcal{O}(A), \subseteq)$ is a Heyting Algebra - the results had very regular shapes, and were often planar.
For example:

$H =$  $H'(\mathcal{O}(H), \subseteq) =$ 

$G =$  $G'(\mathcal{O}(G), \subseteq) =$ 

Background story, 2: planarity, '↓'

Everytime that I started with a DAG D with three independent points then D' would contain a cube, and would be non-planar. For example:

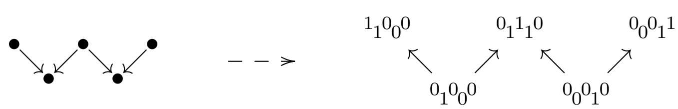
$W =$  $\quad W' = (\mathcal{O}(W), \subseteq) =$ 

Everytime that I started with a "thin" DAG D - "thin" meaning "does not have three independent points" - then D' would be planar.

It turns out that we can always recover D from D'.
For $C \subseteq D$ let $\downarrow C$ be the smallest down-set of D containing C.
For $d \subseteq D$ let $\downarrow d$ be the smallest down-set of D containing $\{d\}$.
The map

$$\begin{array}{rccc} \downarrow\colon & D & \to & D' \\ & d & \mapsto & \downarrow d \end{array}$$

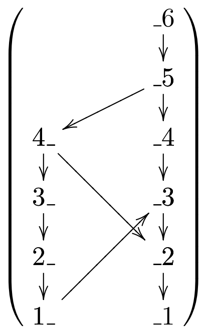
is always a (contra-variant) embedding of D into D', and its image is exactly the set of points of D' with exactly one arrow coming in:



The isomorphism between $\downarrow D \subseteq D'$ and D^{op} is (part of) *Birkhoff's representation theorem for finite distributive lattices* - See Davey & Priestley's "Introduction to Lattices and Order (2nd ed)", pages 116-118, for its properties.

2-Column graphs

This is a *2-column graph*, and our short notation for it:



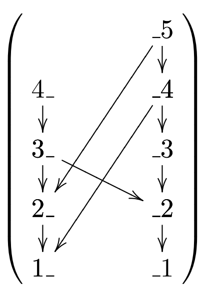
$$\equiv \left(4, 6, \left\{\begin{matrix}4_-\!\to {}_-2,\\ 1_-\!\to {}_-3\end{matrix}\right\}, \left\{4_-\!\leftarrow {}_-5\right\}\right)$$

This is a *2-pile*, and our short notation for it:

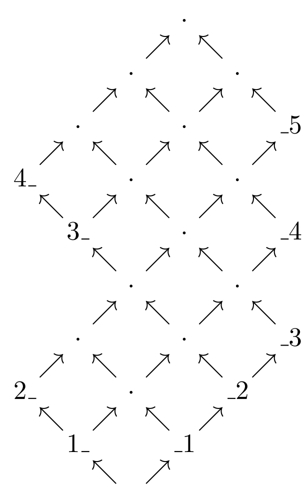
$$\begin{pmatrix} & 0\\ 0 & 1\\ 0 & 1\\ 1 & 1\end{pmatrix} \equiv 13$$

Note that the '1' and '3' tell only the number of '1's
in each column; the total heights are omitted.

$D \equiv$



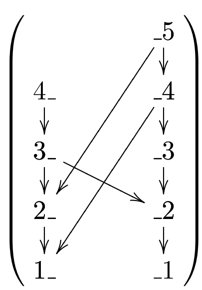
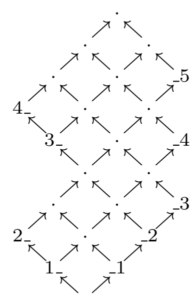
$(\mathcal{O}(D), \subseteq) \equiv$



Missing digits

The *generators* of a ZHA are the points with exactly one arrow coming in.
The *left generators* are the ones of the form '$\circ\!\nwarrow$'.
The *right generators* are the ones of the form '$\nearrow\!\circ$'.

Let C be a 2-column graph, and $C' := (\mathcal{O}(C), \subseteq)$ (a ZHA).
The inclusion $\downarrow\colon C \to C'$ takes the
left column of C to the left generators of C', and the
right column of C to the right generators of C', and the
Example:

$D \equiv$  $(\mathcal{O}(D), \subseteq) \equiv$ 

To obtain the "missing digits" in $1\_, 2\_, \ldots, \_1, \_2, \ldots$ we can do:

$$\downarrow \_5 = \downarrow \begin{pmatrix} & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = 25$$

$$\downarrow 4\_ = \downarrow \begin{pmatrix} & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = 42 \qquad \downarrow \_4 = \downarrow \begin{pmatrix} & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = 14$$

$$\downarrow 3\_ = \downarrow \begin{pmatrix} & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = 32 \qquad \downarrow \_3 = \downarrow \begin{pmatrix} & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = 03$$

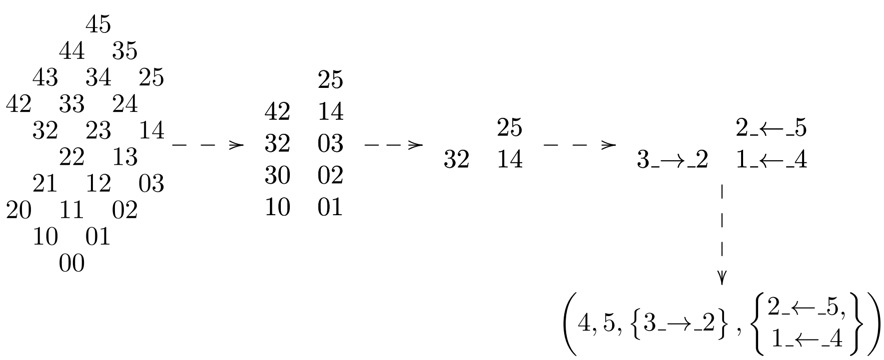
$$\downarrow 2\_ = \downarrow \begin{pmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} = 20 \qquad \downarrow \_2 = \downarrow \begin{pmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = 02$$

$$\downarrow 1\_ = \downarrow \begin{pmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} = 10 \qquad \downarrow \_1 = \downarrow \begin{pmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = 01$$

Once we draw $1\_ \equiv 10$, $2\_ \equiv 10$, $3\_ \equiv 32$, … in the lr-plane,
drawing the rest of the ZHA is automatic.

45
44 35
43 34 25
42 33 24
32 23 14
22 13
21 12 03
20 11 02
10 01
00

From ZHAs to 2-column graphs
Here's how to go in the opposite direction.
Starting from a ZHA H, write its generators in two columns.
The leftmost and righmost digits increase in unit steps always,
but the middle digits correspond to the "missing digits" we discussed before.
Starting from the bottom of each of the two columns,
look at when the "missing"/"middle" digit changes.
Each one of these "generators after change" becomes an arrow
in the 2-column graph C.



Part 2:
J-operators and ZQuotients
(For older children)

J-operators are a basic tool for constructing *sheaves* and for moving back and forth between different logics...
But we will not see the categorical part here.

J-operators
A *J-operator* on a Heyting Algebra H is a function $J : H \to H$, that obeys the three axioms below.
We usually write J as $\cdot^* : H \to H$, and write the axioms as rules.

$$\frac{}{P \leq P^*}\text{J1} \qquad \frac{}{P^* = P^{**}}\text{J2} \qquad \frac{}{(P\&Q)^* = P^*\&Q^*}\text{J3}$$

J1 says that the operation $\cdot^*$ is increasing.
J2 says that the operation $\cdot^*$ is idempotent.
J3 is something mysterious (for now).

A J-operator induces an equivalence relation and equivalence classes:

$$\begin{array}{rcl} P \sim Q & \text{iff} & P^* = Q^* \\ [P]^* & := & \{\, Q \in H \mid P^* = Q^* \,\} \end{array}$$

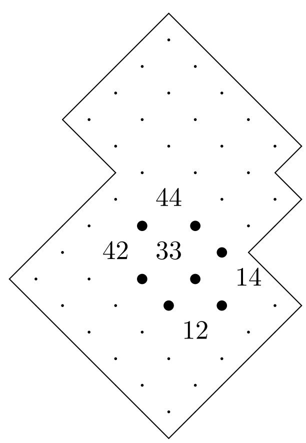
We will use the interval notation,

$$[P, R] := \{\, Q \in H \mid P \leq Q \leq R \,\}$$

to denote all truth-values between P and R (inclusive).

The proofs in the next pages will show that every equivalence class is closed by '&', '∨', and "sandwiching".
For example, if 42, 33, and 14 belong to the same equivalence class, E, then:

$$\begin{array}{c} 44 = 42 \vee 33 \vee 14 \in E \\ 12 = 42 \,\&\, 33 \,\&\, 14 \in E \\ [12, 44] = [42 \,\&\, 33 \,\&\, 14, 42 \vee 33 \vee 14] \subseteq E \end{array}$$



Moreover, if $E = \{Q_1, \ldots, Q_n\}$
then $Q_1 \,\&\, \ldots \,\&\, Q_n \in E$ and $Q_1 \vee \ldots \vee Q_n \in E$,
and $E = [Q_1 \,\&\, \ldots \,\&\, Q_n, Q_1 \vee \ldots \vee Q_n]$.

Derived rules

All the rules below,

Monotonicity: $P \le Q$ implies $P^* \le Q^*$,

Sandwich lemma: all truth values between P and P^* are equivalent,

EC&, EC$\lor$, ECS: equivalence classes are closed by '&', '$\lor$', and sandwiching,

are consequences of just the Heyting Algebra rules plus J1, J2, J3.

$$\dfrac{}{(P\&Q)^* \le Q^*}\,\mathsf{Mop} \quad := \quad \dfrac{\dfrac{}{(P\&Q)^* = P^*\&Q^*}\,\mathsf{J3} \quad \dfrac{}{P^*\&Q^* \le Q^*}}{(P\&Q)^* \le Q^*}$$

$$\dfrac{P \le Q}{P^* \le Q^*}\,\mathsf{Mo} \quad := \quad \dfrac{\dfrac{\dfrac{P \le Q}{P = P\&Q}}{P^* = (P\&Q)^*} \quad \dfrac{}{(P\&Q)^* \le Q^*}\,\mathsf{Mop}}{P^* \le Q^*}$$

$$\dfrac{P \le Q \le P^*}{P^* = Q^*}\,\mathsf{Sand} \quad := \quad \dfrac{\dfrac{P \le Q}{P^* \le Q^*}\,Mo \quad \dfrac{\dfrac{Q \le P^*}{Q^* \le P^{**}}\,\mathsf{Mo} \quad \dfrac{}{P^{**} = P^*}\,\mathsf{J2}}{Q^* \le P^*}}{P^* = Q^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P\&Q)^*}\,\mathsf{EC\&} \quad := \quad \dfrac{\dfrac{P^* = Q^*}{P^* = Q^* = P^*\&Q^*} \quad \dfrac{}{P^*\&Q^* = (P\&Q)^*}\,\mathsf{J3}}{P^* = Q^* = (P\&Q)^*}$$

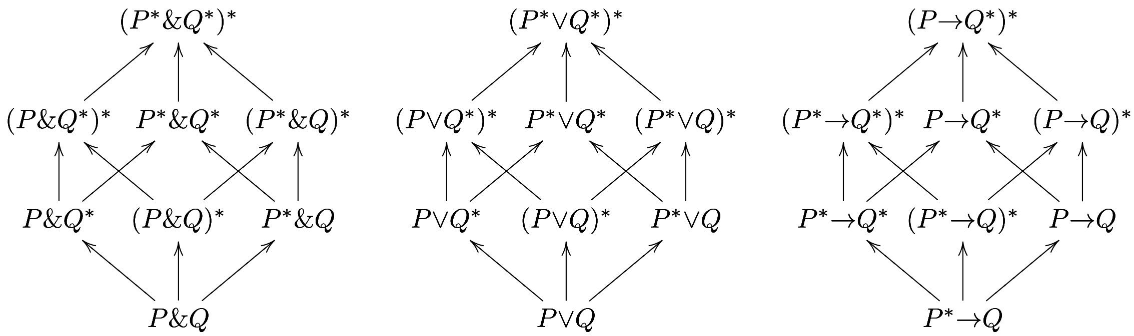
$$\dfrac{P^* = Q^*}{P^* = Q^* = (P \lor Q)^*}\,\mathsf{EC\lor} \quad := \quad \dfrac{P^* = Q^* \quad \dfrac{\dfrac{\dfrac{}{P \le P \lor Q} \quad \dfrac{\dfrac{}{P \le P^*}\,\mathsf{J1} \quad \dfrac{\dfrac{}{Q \le Q^*}\,\mathsf{J1} \quad \dfrac{P^* = Q^*}{Q^* = P^*}}{Q \le P^*}}{P \lor Q \le P^*}}{P \le P \lor Q \le P^*}}{P^* = (P \lor Q)^*}\,\mathsf{Sand}}{P^* = Q^* = (P \lor Q)^*}$$

$$\dfrac{P \le Q \le R \quad P^* = R^*}{P^* = Q^* = R^*}\,\mathsf{ECS} \quad := \quad \dfrac{\dfrac{\dfrac{P \le Q \le R \quad \dfrac{}{R \le R^*}\,\mathsf{J1} \quad \dfrac{P^* = R^*}{R^* = P^*}}{P \le Q \le P^*}}{P^* = Q^*}\,\mathsf{Sand} \quad P^* = R^*}{P^* = Q^* = R^*}$$

(Todo: use these rules to prove the figure in the previous page.)

How J-operators interact with the connectives
For the next result about how J-operators divide a ZHA into equivalence classes we need one of the facts that will be proved below - one arrow of the cubes.

The implications in the cubes below



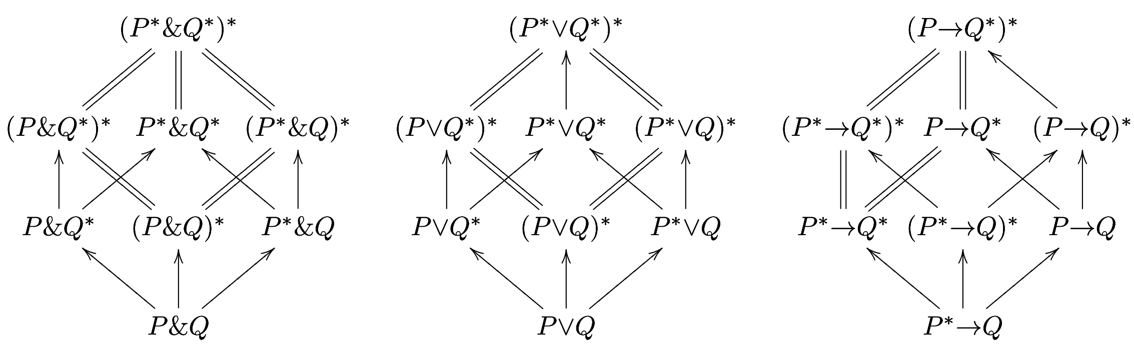
can be proved easily using just Mo plus the derived HA rules that say that '&', '∨', '→' are functorial.

If we add the arrows corresponding to the proofs below (that are done explicitly in the next page),

$$\overline{\overline{(P^*\&Q^*)^* = P^*\&Q^* = (P\&Q)^*}}$$

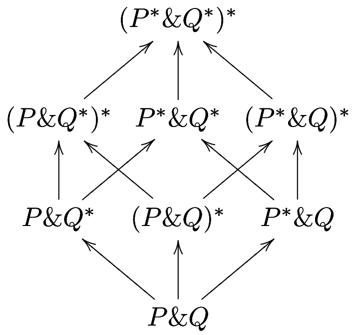
$$\overline{\overline{(P^*{\vee}Q^*)^* \le (P{\vee}Q)^*}} \qquad \overline{\overline{(P{\to}Q^*)^* \le P^*{\to}Q^*}}$$

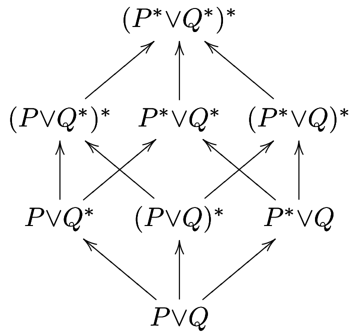
the partial orders on the cubes becomes (equivalent to the one generated by) this:

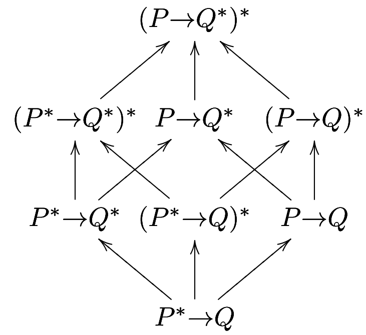


We will call the cubes above, and the rules coming from them, the $\&^*$Cube, $\vee^*$Cube, and $\to^*$Cube,

How J-operators interact with the connectives: proofs





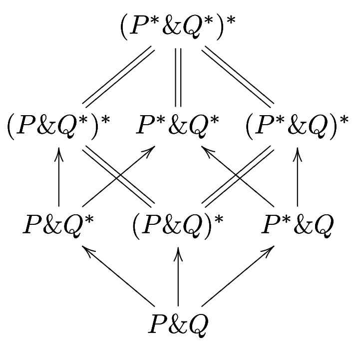


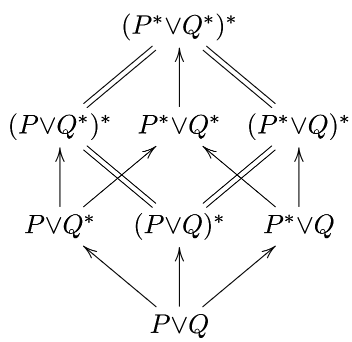
plus:

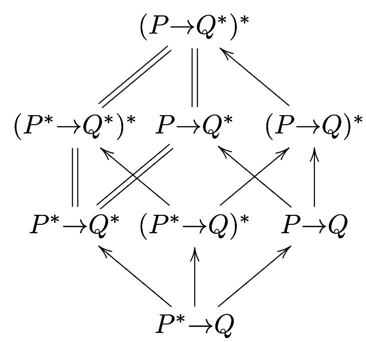
$$\dfrac{\dfrac{\dfrac{}{P^{**}=P^{*}}\ \mathsf{J2}\qquad \dfrac{}{Q^{**}=Q^{*}}\ \mathsf{J2}}{(P^*\&Q^*)^* = P^{**}\&Q^{**} = P^*\&Q^* = (P\&Q)^*}\ \mathsf{J3}}{(P^*\&Q^*)^* = P^*\&Q^* = (P\&Q)^*}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P\le P\vee Q}}{P^*\le (P\vee Q)^*}\ \mathsf{Mo}\qquad \dfrac{\dfrac{}{Q\le P\vee Q}}{Q^*\le (P\vee Q)^*}\ \mathsf{Mo}}{P^*\vee Q^*\le (P\vee Q)^*}}{(P^*\vee Q^*)^*\le (P\vee Q)^{**}}\ \mathsf{Mo}}{(P^*\vee Q^*)^*\le (P\vee Q)^*}\ \mathsf{J2}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P{\to}Q^*\le P{\to}Q^*}}{(P{\to}Q^*)\&P\le Q^*}}{((P{\to}Q^*)\&P)^*\le Q^{**}}\ \mathsf{Mo}}{((P{\to}Q^*)\&P)^*\le Q^*}\ \mathsf{J2}}{(P{\to}Q^*)^*\&P^*\le Q^*}\ \mathsf{J3}}{(P{\to}Q^*)^*\le P^*{\to}Q^*}$$

yields:







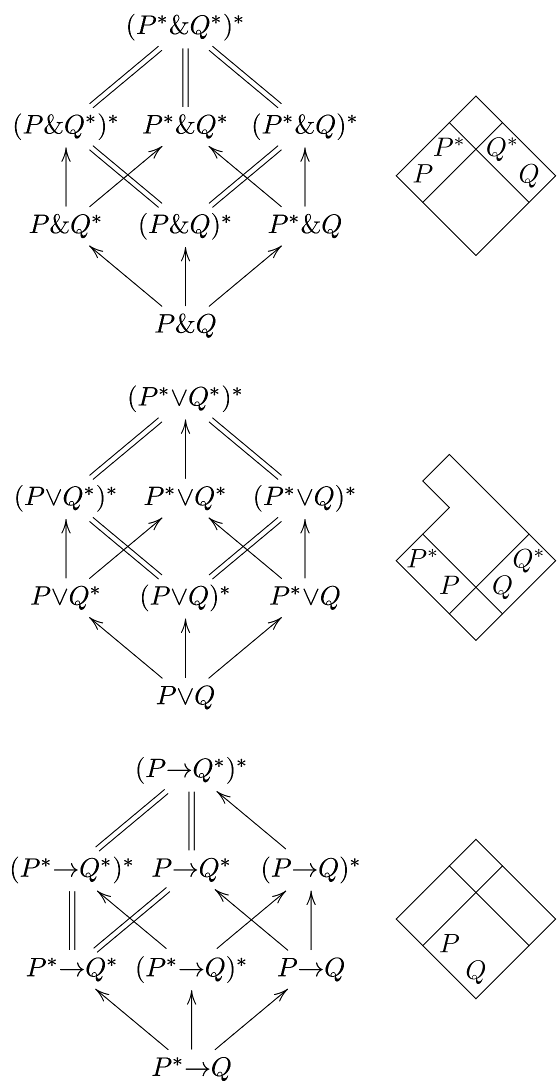
How J-operators interact with the connectives: completeness
Take a 4-uple (H, J, P, Q) made of a Heyting Algebra,
a J-operator on it, and two truth-values $P, Q \in H$.
The arrows in $\&^*\mathsf{Cube}$, $\vee^*\mathsf{Cube}$, $\to^*\mathsf{Cube}$ are *theorems*,
so they are true on all '(H, J, P, Q)'s.
Take an arrow that is not in the cubes – for example, $P^*\vee Q^* \leq (P\vee Q)^*$.
Maybe it is true in all '(H, J, P, Q)'s.
Maybe it is a theorem, that we forgot to prove.
Maybe our cubes are *incomplete*.

They *are* complete, though.

Here is a way to:
1) prove that the arrows in the cubes are the only theorems,
2) exhibit countermodels for all arrows not in the cubes,
3) remember which arrows are and are not in the cubes.
We just need one model for each of the cubes/connectives.

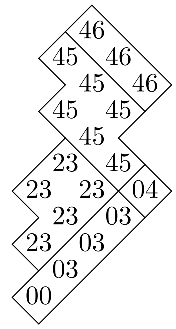
It is in the next page.

How J-operators interact with the connectives: figure



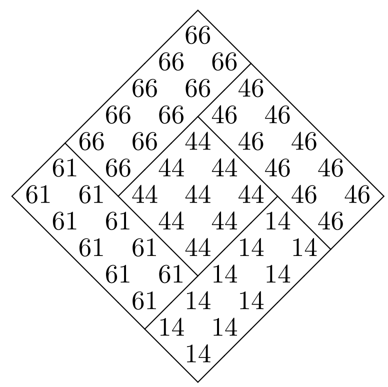
There are Y-cuts or λ-cuts
We saw that the equivalence classes of a J-operator are *intervals* - i.e., lozenges, except maybe for dents coming from irregular contours of ZHAs, like:

```
   46
 45  36
   35  26
 34  25
   24
 23  14         - - ➢
22  13  04
 12  03
11  02
 01
00
```



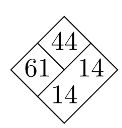
From what we know now this may be a J-operator:

```
            66
          65  56
        64  55  46
      63  54  45  36
    62  53  44  35  26
  61  52  43  34  25  16
60  51  42  33  24  15  06      - - ➢
  50  41  32  23  14  05
    40  31  22  13  04
      30  21  12  03
        20  11  02
          10  01
            00
```



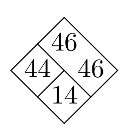
It has some cuts stopping midway
instead of going NW-SE or SW-NE as far as possible...
To show that this can't happen we will show that a J-operator
cannot have four neighboring points, like $\left(\begin{smallmatrix} & 22 & \\ 21 & & 12 \\ & 11 & \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} & 25 & \\ 24 & & 15 \\ & 14 & \end{smallmatrix}\right)$,
in three different equivalence classes.

```
  22
21  12    - - ➢
  11
```



⇐ this a Y-cut

```
  25
24  15    - - ➢
  14
```



⇐ this a λ-cut

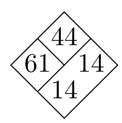
There are Y-cuts or λ-cuts: proofs
We need these two derived rules:

$$\dfrac{Q^* = R^*}{(P \vee Q)^* = (P \vee R)^*}\;\textsf{NoYcuts} \quad := \quad \dfrac{\dfrac{\dfrac{Q^* = R^*}{P \vee Q^* = P \vee R^*}}{(P \vee Q^*)^* = (P \vee R^*)^*}}{(P \vee Q)^* = (P \vee R)^*}\;\vee^*\textsf{Cube}$$

$$\dfrac{Q^* = R^*}{(P \& Q)^* = (P \& R)^*}\;\textsf{No}\lambda\textsf{cuts} \quad := \quad \dfrac{\dfrac{Q^* = R^*}{P^* \& Q^* = P^* \& R^*}}{(P \& Q)^* = (P \& R)^*}\;\textsf{J3}$$

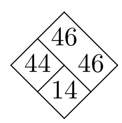
Now let's use them to prove the the Y-cut and the λ-cut
of the example in the previous page are inadmissible
in a J-operator.

$$\begin{matrix} & 22 & \\ 21 & & 12 \\ & 11 & \end{matrix} \quad - - \succ$$



$\Leftarrow$ this a Y-cut

$$\begin{matrix} & 25 & \\ 24 & & 15 \\ & 14 & \end{matrix} \quad - - \succ$$



$\Leftarrow$ this a λ-cut

Look:

$$\dfrac{\dfrac{\dfrac{11^* = 12^*}{(21 \vee 11)^* = (21 \vee 12)^*}\;\textsf{NoYcuts}}{21^* = 22^*}}{61 = 14} \qquad \dfrac{\dfrac{\dfrac{25^* = 15^*}{(24 \& 25)^* = (24 \& 15)^*}\;\textsf{No}\lambda\textsf{cuts}}{24^* = 14^*}}{44 = 14}$$

Examples of J-operators: Fourman and Scott

(i) *The closed quotient.*

$$J_a p = a \vee p.$$

(ii) *The open quotient.*

$$J^a p = a \to p.$$

(iii) *The Boolean quotient.*

$$B_a p = (p \to a) \to a.$$

(iv) *The forcing quotient.*

$$(J_a \,\&\, J^b)p = (a \vee p) \,\&(b \to p).$$

(vi) *A mixed quotient.*

$$(B_a \,\&\, J^a)p = (p \to a) \to p.$$

(i)	$J_a \vee J_b$	=	$J_{a \vee b}$	(ii)	$J^a \vee J^b$	=	$J^{a \,\&\, b}$
(iii)	$J_a \,\&\, J_b$	=	$J_{a \,\&\, b}$	(iv)	$J^a \,\&\, J^b$	=	$J^{a \vee b}$
(v)	$J_a \,\&\, J^a$	=	$\bot$	(vi)	$J_a \vee J^a$	=	$\top$
(vii)	$J_a \vee K$	=	$K \circ J_a$	(viii)	$J^a \vee K$	=	$J^a \circ K$
(ix)	$J_a \vee B_a$	=	B_a	(x)	$J^a \vee B_b$	=	$B_{a \to b}$

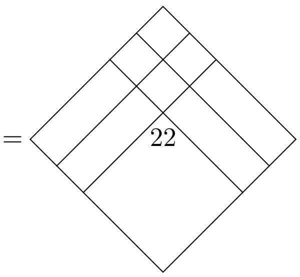
This above is from M.P. Fourman and D.S. Scott's "Sheaves and Logic" (1979), that was published in SLNM0753 ("Applications of Sheaves: Proceedings of the Research Symposium on Applications of Sheaf Theory to Logic, Algebra and Analysis - Durham, july 9-21, 1977"). Relevant pages: 329-331.

How do we visualize the J-operators J_a, J^a, B_a, etc?
And what are the '&' and '$\vee$' in the algebra of J-operators?
How do we visualize these '&' and '$\vee$'?

Examples of J-operators: diagrams

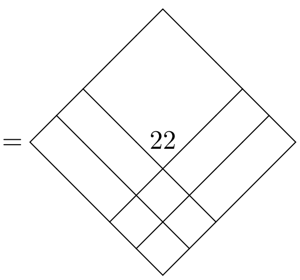
$J_a p := a \vee p$
(closed quotient)

$J_{22} =$



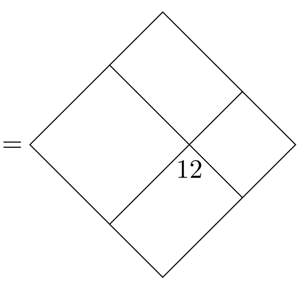
$J^a p := a \to p$
(open quotient)

$J^{22} =$



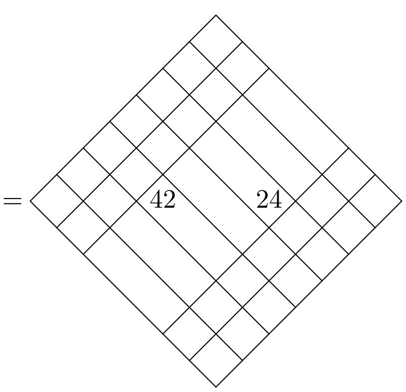
$B_a p := (p \to a) \to a$
(Boolean quotient)

$B_{12} =$



$(J_a \,\&\, J^b)p := (a \vee p) \,\&(b \to p)$
(forcing quotient)

$(J_{42} \,\&\, J^{24})p =$



Partitions into contiguous classes ("piccs")
A good way to understand the algebra of J-operators
is to start by the one-dimensional case.
(ZHAs are two-dimensional things.)

A partition of $\{0,\dots,n\}$ into contiguous classes (a "picc")
is one in which this holds: if $a,b,c \in \{0,\dots,n\}$, $a<b<c$ and $a \sim c$,
then $a \sim b \sim c$.
So, for example, $\{\{0,1\},\{2\},\{3,4,5\}\}$ is a picc,
but $\{\{0,2\},\{1\}\}$ is not.

A partition of $\{0,\dots,n\}$ into contiguous classes induces:
1) an equivalence relation $\cdot \sim_P \cdot$,
2) a function $[\cdot]_P$ that returns the equivalence class of an element,
3) a function

$$\begin{array}{rrcl} \cdot^P: & \{0,\dots,n\} & \to & \{0,\dots,n\} \\ & a & \mapsto & \max\,[a]_P \end{array}$$

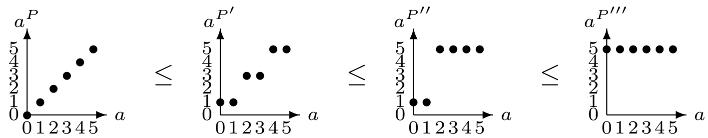
that takes each element to the top element in its class,
4) a set $\mathsf{St}_P := \{\, a \in \{0,\dots,n\} \mid a^P = a \,\}$ of the "stable"
elements of $\{0,...,n\}$, and
5) a graphical representation with a bar between a and $a+1$
when they are in different classes:

$$01|2|345 \quad \equiv \quad \{\{0,1\},\{2\},\{3,4,5\}\},$$

which will be our favourite notation for piccs from now on.

The algebra of piccs

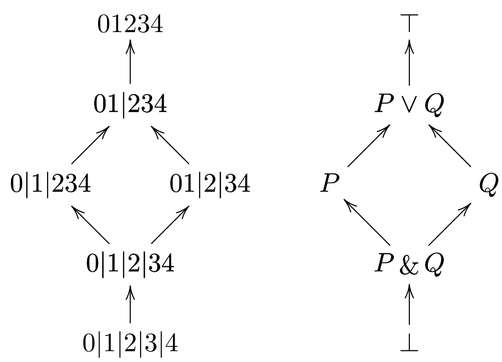
When P and P' are two piccs on $\{0, \dots, n\}$ we say that $P \leq P'$ when $\forall a \in \{0, \dots, n\}. a^P \leq a^{P'}$.
The intuition is that $P \leq P'$ means that the graph of the function $\cdot^P$ is under the graph of $\cdot^{P'}$:



$$\begin{array}{ccccccc} 0|1|2|3|4|5 & \leq & 01|23|45 & \leq & 01|2345 & \leq & 012345 \\ P & \leq & P' & \leq & P'' & \leq & P''' \end{array}$$

This yields a partial order on piccs, whose bottom element is the identity function $0|1|\dots|n$, and the top element is $01\dots n$, that takes all elements to n.

It turns out that the piccs form a (Heyting!) algebra, in which we can define $\top$, $\bot$, &, $\vee$, and even $\to$.

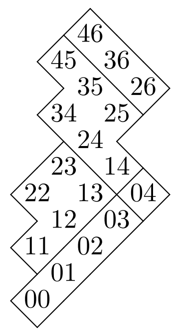


ZQuotients

A *ZQuotient* for a ZHA with top element 46 is:
a picc on $\{0,\ldots,4\}$ (a "partition of the left wall"), plus
a picc on $\{0,\ldots,6\}$ (a "partition of the right wall").

Our favourite short notation for ZQuotients is with "/"s and "\"s, like this, "4321/0 0123\45\6", because we regard the cuts in a ZQuotient as diagonal cuts on the ZHA.

The graphical notation is this (for 4321/0 0123\45\6 on ):



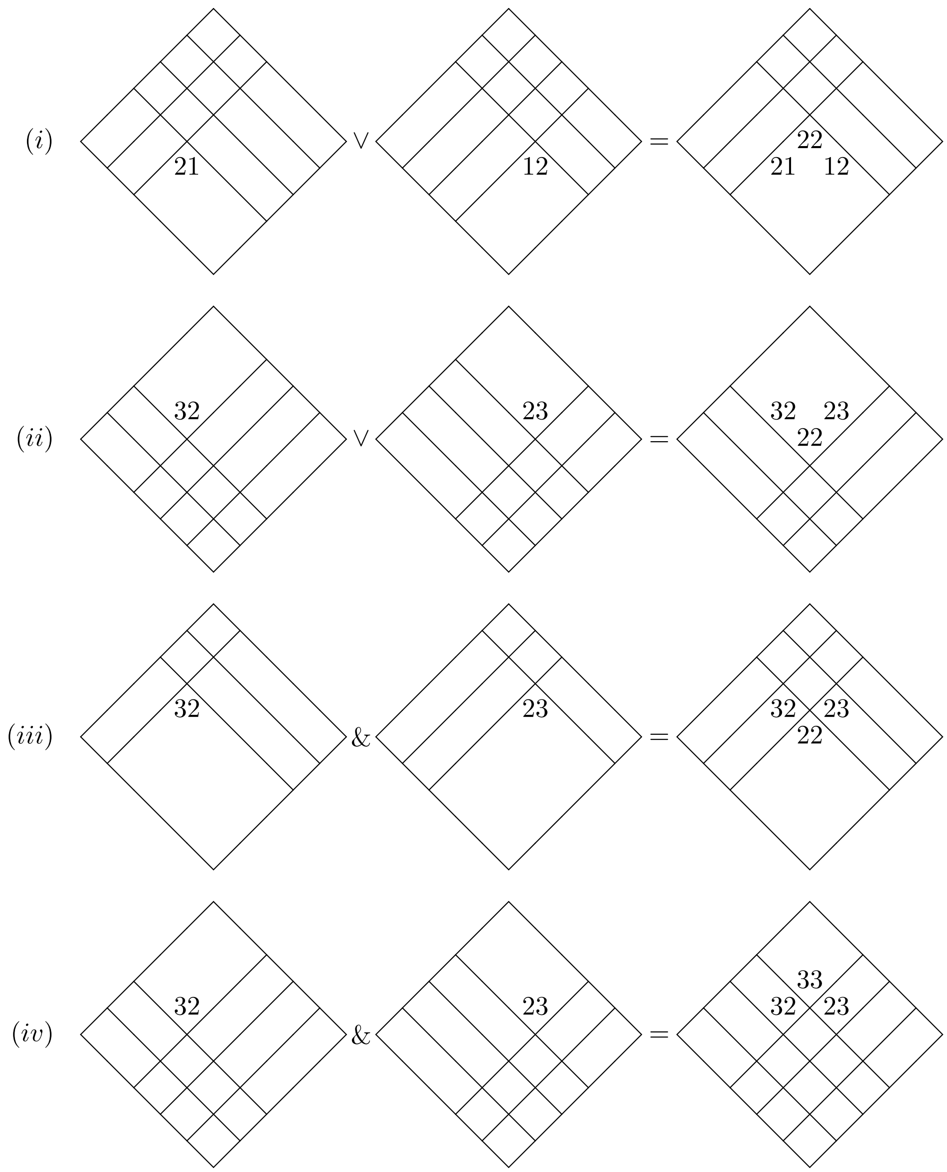
which makes clear how we can adapt the definitions of $\cdot \sim_P \cdot$, $[\cdot]_P$, $\cdot^P$, St_P, which were on (one-dimensional!) piccs, to their two-dimensional counterparts on ZQuotients.
If P is the ZQuotient of the figure above, then:

$$\begin{array}{rcl}
34 \sim_P 25 & \text{is} & \text{true}, \\
23 \sim_P 24 & \text{is} & \text{false}, \\
[12]_P & = & \{11, 12, 13, 22, 23\}, \\
22^P & = & 23, \\
\mathsf{St}_P & = & \{03, 04, 23, 45, 46\}.
\end{array}$$

The algebra of J-operators

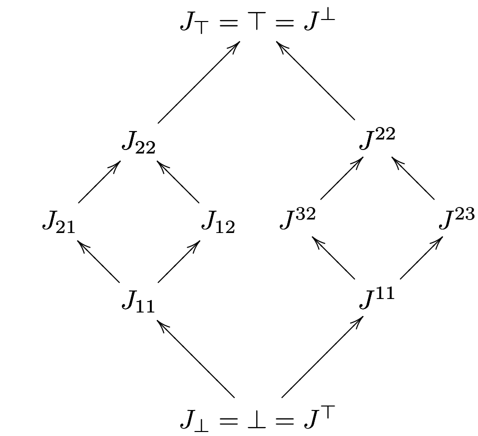
$$\begin{array}{rll} (i) & J_a \vee J_b = J_{a \vee b} & J_{21} \vee J_{12} = J_{21 \vee 12} \\ (ii) & J^a \vee J^b = J^{a \& b} & J^{32} \vee J^{23} = J^{32 \& 23} \\ (iii) & J_a \,\&\, J_b = J_{a \& b} & J_{32} \,\&\, J_{23} = J_{32 \& 23} \\ (iv) & J^a \,\&\, J^b = J^{a \vee b} & J^{32} \,\&\, J^{23} = J^{32 \vee 23} \end{array}$$

(↑ used in the examples below)

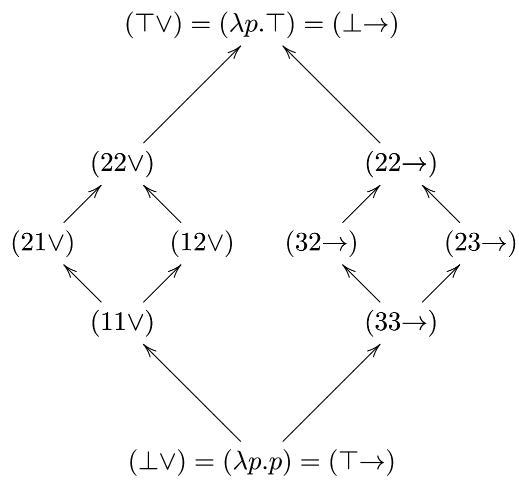


The algebra of J-operators, 2

We can depict the four equations of the previous page as:



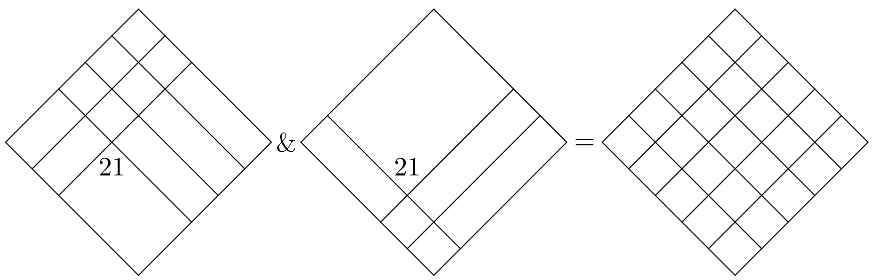
using Fourman and Scott's notation, or as

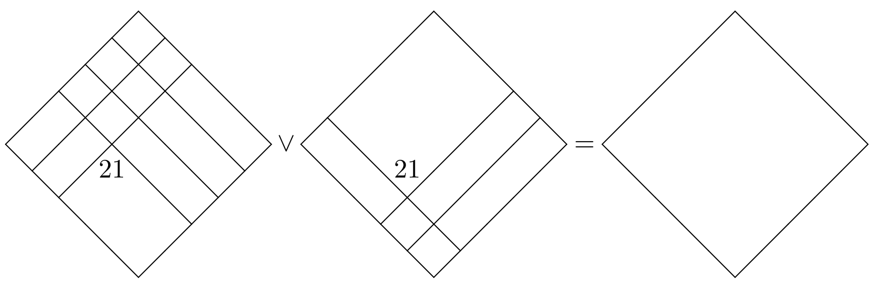


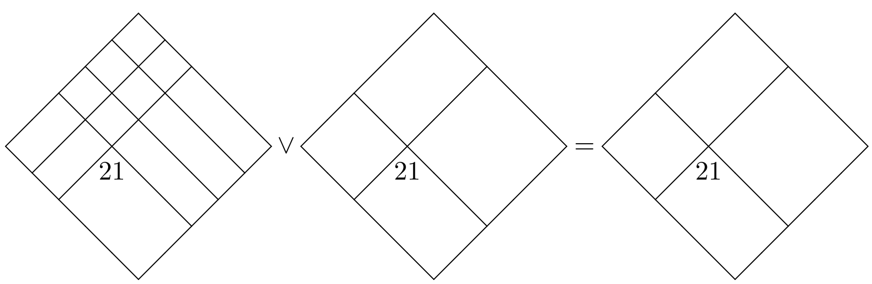
using a notation that I think is obvious.

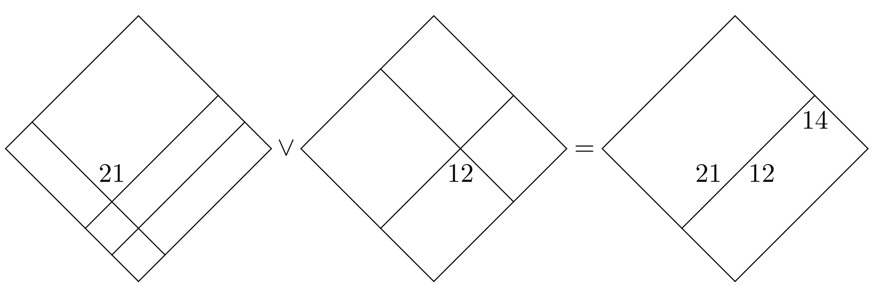
The algebra of J-operators, 3

(v)	$J_a \,\&\, J^a = \bot$	$J_{21} \,\&\, J^{21} = \bot$
(vi)	$J_a \vee J^a = \top$	$J_{21} \vee J^{21} = \top$
(ix)	$J_a \vee B_a = B_a$	$J_{21} \vee B_{21} = B_{21}$
(x)	$J^a \vee B_b = B_{a \to b}$	$J^{21} \vee B_{12} = B_{21 \to 12}$
		($\uparrow$ used in the examples below)

(v)


(vi)


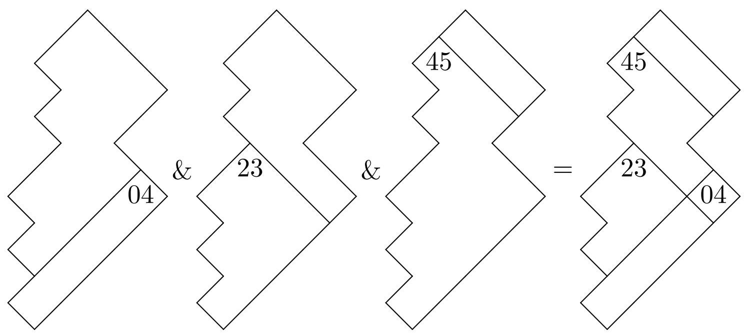
(ix)


(x)


ZQuotients as polynomials

Fourman and Scott, p.331:
If we take a polynomial in $\to$, &, $\vee$, $\bot$, *say* $f(p,a,b,\ldots)$, *it is a decidable question whether for all* $a,b,\ldots$ *it defines a J-operator.*

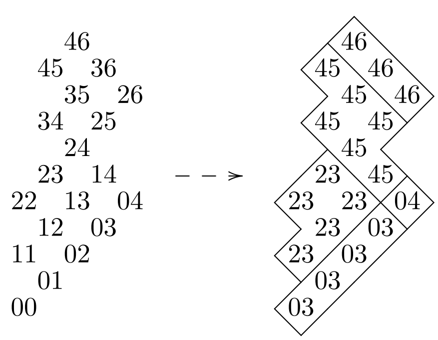
All ZQuotients are polynomials in that sense.
Moreover, they can be built from elementary J-operators using just B_P and &.
Example:



It is easy to check by hand (test it for a few 'P's!) that

$$\begin{array}{rcl} B_{04}\,\&\,B_{23}\,\&\,B_{45} & = & \lambda P.((B_{04}\,\&\,B_{23}\,\&\,B_{45})(P)) \\ & = & \lambda P.((B_{04}(P)\,\&\,B_{23}(P)\,\&\,B_{45}(P)) \\ & = & \lambda P.(((P{\to}04){\to}04)\,\&((P{\to}23){\to}23)\,\&((P{\to}45){\to}45)) \end{array}$$

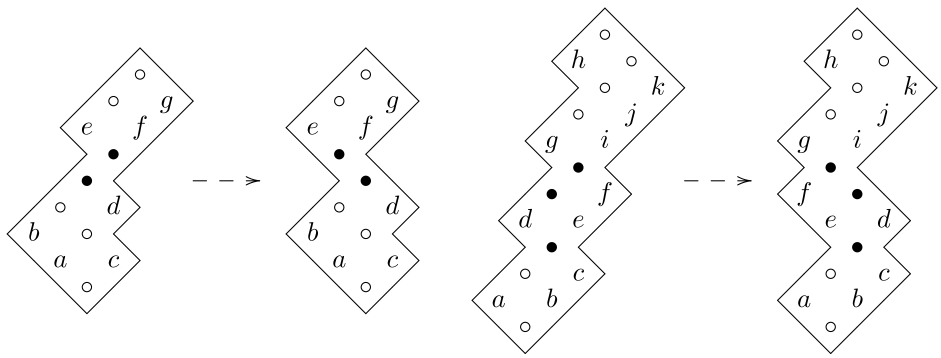
acts as:



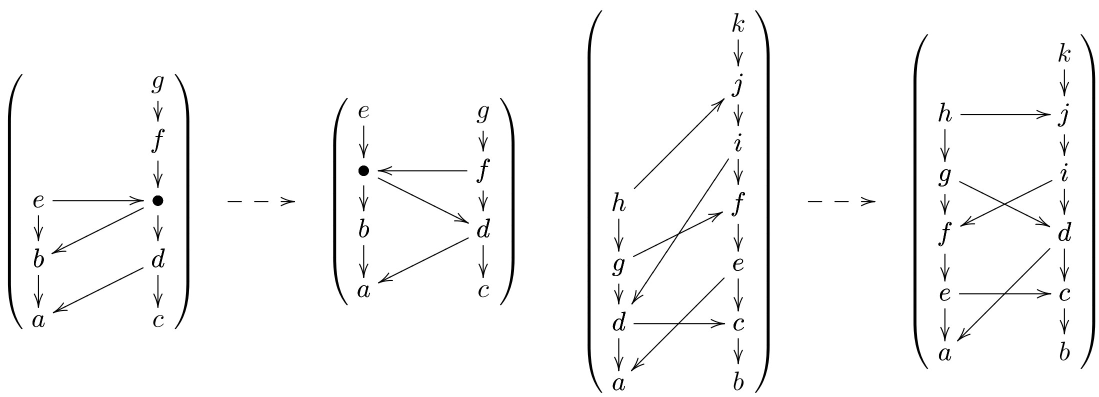
Now we know that on ZHAs
1) J-operators are ZQuotients,
2) ZQuotients are (polynomial!) J-operators.

Bottlenecks and flipping

A *bottleneck* in a ZHA is a point where $L(y) = R(y)$.
We can flip everything in a ZHA between two consecutive bottlenecks and obtain a ZHA that is isomorphic to the previous one.



Their 2-column graphs will be isomorphic, too,
but that may not be evident when we look at them.



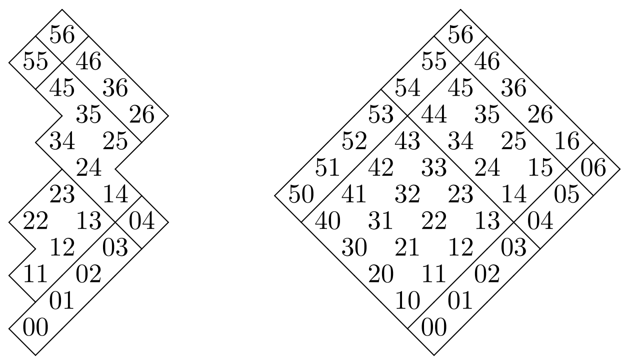
How ZQuotients act on 2-column graphs

Here is one way to understand how a ZQuotient acts on a 2-column graph.
It will take several slides.

Let $C := \left(5, 6, \left\{\begin{smallmatrix} 4\_\to\_5 \\ 3\_\to\_4 \\ 2\_\to\_2 \\ 1\_\to\_1 \end{smallmatrix}\right\}, \{2\_\leftarrow\_5\}\right)$.
Let $C^{\Diamond} := (5, 6, \{\}, \{\})$.
Let H be the ZHA for C.
Let $H^{\Diamond}$ be the ZHA for $C^{\Diamond}$ (a lozenge).
Let $J : H \to H$ be a J-operator on H.
We can describe J by its cuts.
Draw the same cuts on $H^{\Diamond}$.
This induces a J-operator $J^{\Diamond} : H^{\Diamond} \to H^{\Diamond}$ on $H^{\Diamond}$.
For example, if the cuts are

$$5/4321/0\ 0123\backslash 45\backslash 6,$$

then (H, J) and $(H^{\Diamond}, J^{\Diamond})$ are:



The operation '$\cdot^*$' takes each element in H
to the top element in its equivalence class.
Let's create a dual operation, '$\cdot^{\mathsf{co}*}$', that takes
each element in H to the *bottom* element in its equivalence class.
The corresponding operations on $H^{\Diamond}$
will be denoted by '$\cdot^{\Diamond}$' and '$\cdot^{\mathsf{co}\Diamond}$'.
For example:

$$\begin{array}{cc} 12^* = 23 & 12^{\Diamond} = 43 \\ 12^{\mathsf{co}*} = 11 & 12^{\mathsf{co}\Diamond} = 10 \end{array}$$

Now look at the cuts, and at the left and right piccs...

$$\begin{array}{rcl} [1]^L &=& \{1,2,3,4\} \\ 1^L &=& 4 \\ 1^{\mathsf{co}L} &=& 1 \end{array} \qquad \begin{array}{rcl} [2]^R &=& \{0,1,2,3\} \\ 2^R &=& 3 \\ 2^{\mathsf{co}R} &=& 0 \end{array}$$

We have:

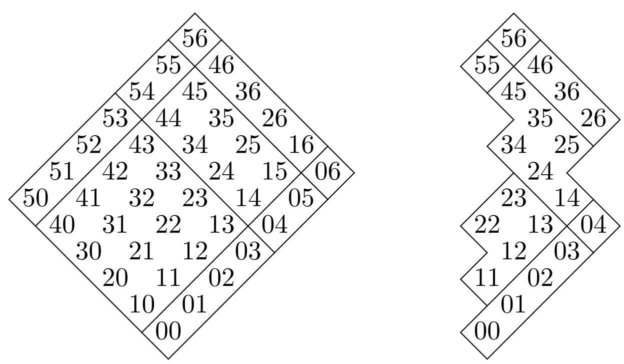
$$\begin{array}{rcl} ab^{\Diamond} &=& a^L b^R \\ ab^{\mathsf{co}\Diamond} &=& a^{\mathsf{co}L} b^{\mathsf{co}R} \end{array} \qquad \begin{array}{rcccl} 12^{\Diamond} &=& 1^L 2^R &=& 43 \\ 12^{\mathsf{co}\Diamond} &=& 1^{\mathsf{co}L} 2^{\mathsf{co}R} &=& 10 \end{array}$$

How ZQuotients act on 2-column graphs, 2

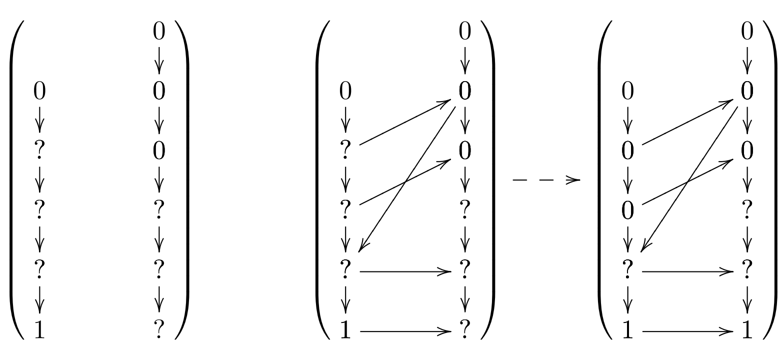
Let $C := \left(5, 6, \left\{ \begin{smallmatrix} 4\_\to\_5 \\ 3\_\to\_4 \\ 2\_\to\_2 \\ 1\_\to\_1 \end{smallmatrix} \right\}, \{2\_\leftarrow\_5\} \right)$.
Let $C^\Diamond := (5, 6, \{\}, \{\})$.
Let H be the ZHA for C.
Let $H^\Diamond$ be the ZHA for $C^\Diamond$ (a lozenge).
Let $J : H \to H$ be a J-operator on H,
and $J^\Diamond : H^\Diamond \to H^\Diamond$ be a J-operator on $H^\Diamond$,
both with these cuts:

$$5/4321/0\ 0123\backslash 45\backslash 6.$$

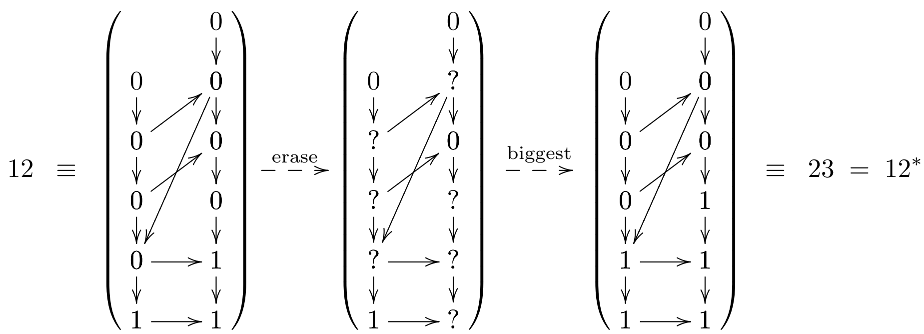
Then $(H^\Diamond, J^\Diamond)$ and (H, J) and are:



The equivalence classes for 12 in $(H^\Diamond, J^\Diamond)$ and (H, J) are
$[12]^\Diamond = [12^{\mathsf{co}\Diamond}, 12^\Diamond] = [10, 43] \subseteq H^\Diamond$ and
$[12]^* = [12^{\mathsf{co}*}, 12^*] = [11, 23] \subseteq H$.
The elements of $[12]^\Diamond$ and $[12]^*$
are simply the open sets of these forms:

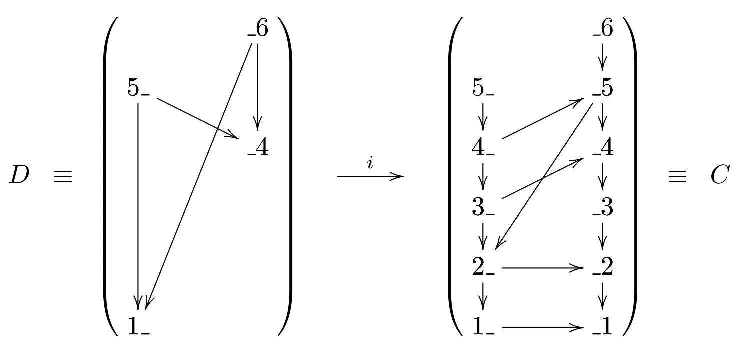


How ZQuotients act on 2-column graphs, 3
The best way to see the action of a J-operator
on a 2-column graph C is this.
An open set on C is a map $C \to \{0, 1\}$.
We erase some of its information, replacing it by ‘?’s,
then we try to reconstruct it.
There are two natural ways.
One, depicted below, that yields ‘$\cdot^*$’, takes the *biggest*
open set with ‘0’ and ‘1’s in the specified places.



The other way, that takes the *smallest* open set with ‘0’ and ‘1’s
in the specified places, yields ‘$\cdot^{\mathsf{co}*}$’.

Here is the *right* way (for adults!!!) to see that.
Choose a subset D of the points of C.
Endow D with the topology inherited from C.
(In our case, D has to inherit the *order*).

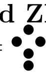


The inclusion map $i : D \to C$ induces a map $i^* : \mathcal{O}(D) \leftarrow \mathcal{O}(C)$,
that can be extended to a functor $i^* : \mathbf{Set}^D \leftarrow \mathbf{Set}^C$
having both adjoints — $i_! \dashv i^* \dashv i_*$.
This $i_! \dashv i^* \dashv i_*$ is an *essential geometric morphism*
that is an *inclusion*.

Part 3:
Seminar handouts
(For younger children -
including some who have
never seen a theorem)

This part is *very* incomplete
at this moment!

Handouts: ZSets and ZDAGs for children

As a subset of $\mathbb{Z}^2$, $K =$  ("kite") is:

$$\left\{\begin{matrix} & (1,3), & \\ (0,2), & & (2,2), \\ & (1,1), & \\ & (1,0) & \end{matrix}\right\}.$$

The *reading order* on K, $\mathsf{read}_K : K \to \mathbb{N}$, is $\begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 4 & \\ & 5 & \end{smallmatrix}$.

The two natural DAGs on K are:

$$(K, \mathsf{BM}(K)) = \begin{array}{ccccc} & & (1,3) & & \\ & \swarrow & & \searrow & \\ (0,2) & & & & (2,2) \\ & \searrow & & \swarrow & \\ & & (1,1) & & \\ & & \downarrow & & \\ & & (1,0) & & \end{array} = (\left\{\begin{matrix} (1,3), \\ (0,2),\ (2,2), \\ (1,1), \\ (1,0) \end{matrix}\right\}, \left\{\begin{matrix} ((1,3),(0,2)),\ ((1,3),(2,2)) \\ ((0,2),(1,1)),\ ((2,2),(1,1)), \\ ((1,1),(1,0)) \end{matrix}\right\})$$

$$(K, \mathsf{WM}(K)) = \begin{array}{ccccc} & & (1,3) & & \\ & \swarrow & & \searrow & \\ (0,2) & & & & (2,2) \\ & \searrow & & \swarrow & \\ & & (1,1) & & \\ & & \downarrow & & \\ & & (1,0) & & \end{array} = (\left\{\begin{matrix} (1,3), \\ (0,2),\ (2,2), \\ (1,1), \\ (1,0) \end{matrix}\right\}, \left\{\begin{matrix} ((0,2),(1,3)),\ ((2,2),(1,3)) \\ ((1,1),(0,2)),\ ((1,1),(2,2)), \\ ((1,0),(1,1)) \end{matrix}\right\})$$

which are isomorphic to:

$$\begin{array}{ccccc} & & 1 & & \\ & \swarrow & & \searrow & \\ 2 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 4 & & \\ & & \downarrow & & \\ & & 5 & & \end{array} = (\left\{\begin{matrix} & 1 & \\ 2, & & 3, \\ & 4, & \\ & 5, & \end{matrix}\right\}, \left\{\begin{matrix} (1,2),\ (1,3) \\ (2,4),\ (3,4), \\ (4,5) \end{matrix}\right\})$$

$$\begin{array}{ccccc} & & 1 & & \\ & \swarrow & & \searrow & \\ 2 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 4 & & \\ & & \downarrow & & \\ & & 5 & & \end{array} = (\left\{\begin{matrix} & 1 & \\ 2, & & 3, \\ & 4, & \\ & 5, & \end{matrix}\right\}, \left\{\begin{matrix} (2,1),\ (3,1) \\ (4,2),\ (4,3), \\ (5,4) \end{matrix}\right\})$$

Handouts: Notation for characteristic functions.

By default $\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \\ & 1 & \end{smallmatrix}$ would be the function $\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \\ & 1 & \end{smallmatrix} : \begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix} \to \{0,1\}$,

but when we say $\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \\ & 1 & \end{smallmatrix} \subseteq \begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}$ we mean:

$$\left\{\begin{matrix} & \cdot & \\ (0,2), & & \cdot \\ & \cdot & \\ & (1,0) & \end{matrix}\right\} \subseteq \left\{\begin{matrix} (1,3), \\ (0,2),\ (2,2), \\ (1,1), \\ (1,0) \end{matrix}\right\}, \text{ or } \left\{\begin{matrix} & \cdot & \\ 2, & & \cdot \\ & \cdot & \\ & 5 & \end{matrix}\right\} \subseteq \left\{\begin{matrix} & 1 & \\ 2, & & 3, \\ & 4, & \\ & 5 & \end{matrix}\right\}.$$

Handouts: Order topologies.

Example: $(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}, \mathcal{O}(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}))$ is $\left(\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}, \left\{\begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 0 & \\ & 0 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 0 & \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 1 & \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 1 & \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 0 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}, \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}\right\}\right)$.

Note that $\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \\ & 1 & \end{smallmatrix}$ is *not open* - because when we draw it like this,

$$\begin{array}{ccccc} & & 0 & & \\ & \swarrow & & \searrow & \\ 1 & & & & 0 \\ & \searrow & & \swarrow & \\ & & 0 & & \\ & & \downarrow & & \\ & & 1 & & \end{array}$$

there is an arrow '$1 \to 0$' in it.

Order topologies can be defined formally interpreting each arrow as a condition. For example, on this DAG,

$$\begin{array}{ccccc} & & 1 & & \\ & \swarrow & & \searrow & \\ 2 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 4 & & \\ & & \downarrow & & \\ & & 5 & & \end{array}$$

the set of open sets is:

$$\left\{ A \subseteq \{1,2,3,4,5\} \;\middle|\; \begin{pmatrix} 1{\in}A \to 2{\in}A \ \ \& \ \ 1{\in}A \to 3{\in}A \ \ \& \\ 2{\in}A \to 4{\in}A \ \ \& \ \ 3{\in}A \to 4{\in}A \ \ \& \\ 4{\in}A \to 5{\in}A \end{pmatrix} \right\}$$