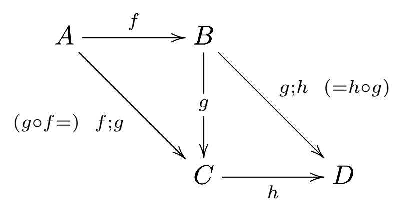
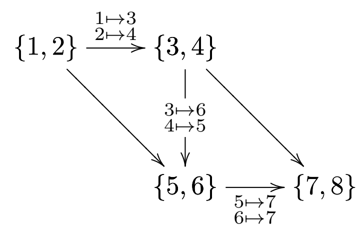
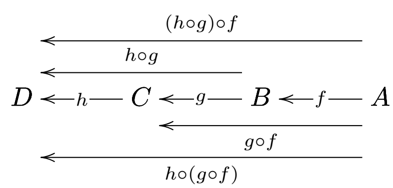
Composition
(is associative)

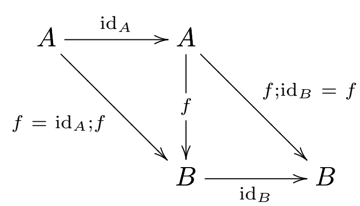


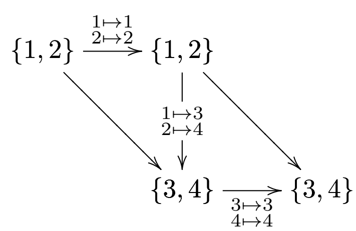


$$\begin{aligned} ((h \circ g) \circ f)(a) &= (h \circ g)(f(a)) \\ &= h(g(f(a))) \\ &= h((g \circ f)(a)) \\ &= (h \circ (g \circ f))(a) \\ ((h \circ g) \circ f)(a) &= (h \circ (g \circ f))(a) \\ (h \circ g) \circ f &= h \circ (g \circ f) \end{aligned}$$

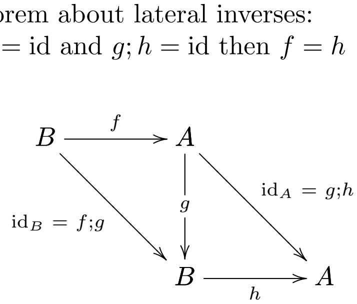


Identities:
If $f : A \to B$ then $\mathrm{id}_A; f = f = f; \mathrm{id}_B$





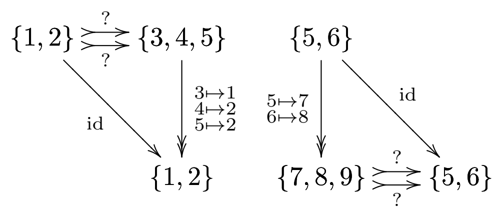
A theorem about lateral inverses:
If $f; g = \mathrm{id}$ and $g; h = \mathrm{id}$ then $f = h$



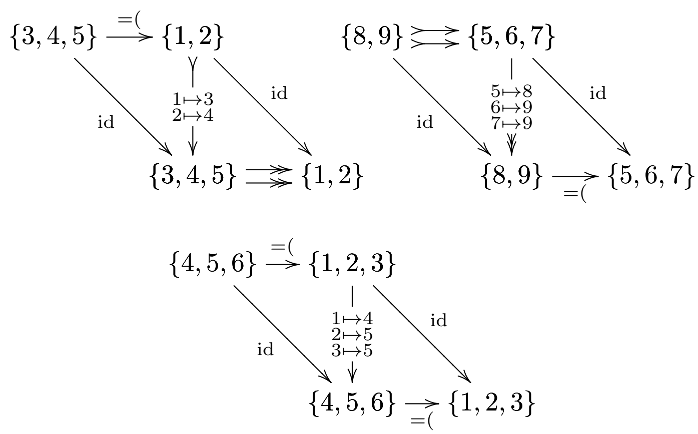
$(f; g); h = \mathrm{id}_B; h = h$
$f; (g; h) = f; \mathrm{id}_A = f$

$$\begin{aligned} f &= f; \mathrm{id}_A \\ &= f; (g; h) \\ &= (f; g); h \\ &= \mathrm{id}_B; h \\ &= h \end{aligned}$$

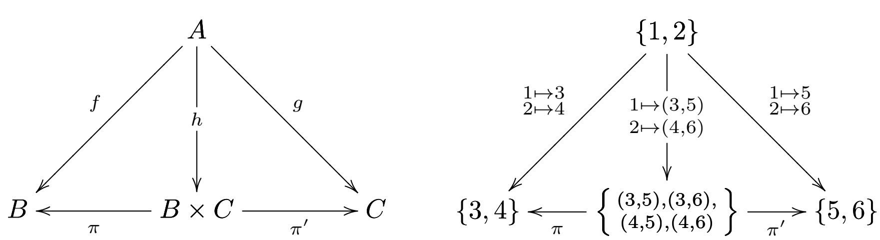
Multiple inverses



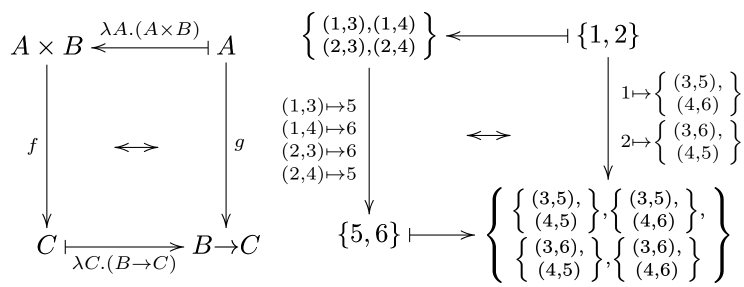
No inverses



Products
Property: $\forall f, g. \exists ! h. (f = h; \pi \ \& \ g = h; \pi')$
Solution: $h = \lambda a : A.(f(a), g(a))$
Bijection: $(f, g) \leftrightarrow h$
$(\rightarrow)$: $\lambda(f, g).(\lambda a : A.(f(a), g(a)))$
$(\leftarrow)$: $\lambda h.((h; \pi), (h; \pi'))$



Exponentials
Bijection: $f \leftrightarrow g$
$(\to)$ ("currying"): $g := \operatorname{cur} f := \lambda a : A.\lambda b : B.f(a, b)$
$(\leftarrow)$ ("uncurrying"): $f := \operatorname{uncur} g := \lambda(a, b) : A \times B.((g(a))(b))$



Properties: $\operatorname{cur}\operatorname{uncur} f = f, \quad \operatorname{uncur}\operatorname{cur} g = g$
where: $f \times f' := \langle \pi; f, \pi'; f' \rangle, \quad \operatorname{uncur} g := (g \times \mathrm{id}); \mathrm{ev}$
solving type equations:

$$\cfrac{\cfrac{\cfrac{\pi \quad f}{\pi; f} \qquad \cfrac{\pi' \quad f'}{\pi'; f'}}{\langle \pi; f, \pi'; f' \rangle}}{f \times f'}\ \mathrm{ren} \qquad \cfrac{\cfrac{\cfrac{\pi : A\times ? \to A}{\pi; f : A\times ? \to B} \qquad \cfrac{\pi' :? \times A' \to A' \quad f' : A' \to B'}{\pi'; f' :? \times A' \to B'}}{\langle \pi; f, \pi'; f' \rangle : A \times A' \to B \times B'}}{f \times f' : A \times A' \to B \times B'}\ \mathrm{ren}$$

$$\cfrac{\cfrac{\cfrac{g \quad \mathrm{id}}{g \times \mathrm{id}} \qquad \mathrm{ev}}{(g \times \mathrm{id}); \mathrm{ev}}}{\operatorname{uncur} g}\ \mathrm{ren} \qquad \cfrac{\cfrac{\cfrac{g : A \to (B\to C) \quad \mathrm{id} :? \to ?}{g \times \mathrm{id} : A\times ? \to (B\to C)\times ?} \qquad \mathrm{ev} : (B\to C) \times B \to C}{(g \times \mathrm{id}); \mathrm{ev} : A\times ? \to C}}{\operatorname{uncur} g : A\times ? \to C}$$