

Notes on notation: CWM

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<http://angg.twu.net/LATEX/2017cwm.pdf><http://angg.twu.net/math-b.html#notes-on-notation>From <http://angg.twu.net/math-b.html#idarct>:

Different people have different measures for “mental space”; someone with a good algebraic memory may feel that an expression like $\text{Frob}^b : \Sigma_f(P \wedge f^*Q) \xrightarrow{\cong} \Sigma_f P \wedge Q$ is easy to remember, while I always think diagrammatically, and so what I do is that I remember this diagram [...] and I reconstruct the formula from it.

These are very informal notes showing my favourite ways to draw the “missing diagrams” in MacLane’s *Categories for the Working Mathematician*, and my favourite choices of letters for them. Work in progress changing often, contributions and chats very welcome, etc. My plan is to do something similar for parts of the *Sketches of an Elephant* next.

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II. Constructions on Categories

p.45: 6. Comma Categories (in my notation)

The most general case is with functors $\mathbf{A} \xrightarrow{F} \mathbf{B} \xleftarrow{G} \mathbf{C}$.
 The comma category $(F \downarrow G)$ is

$$\begin{array}{ccc}
 A \longrightarrow FA \xrightarrow{h} GC \longleftarrow C & (A, h, C) \\
 \alpha \downarrow & F\alpha \downarrow & F\gamma \downarrow & \gamma \downarrow \\
 A' \longrightarrow FA' \xrightarrow{h'} GC' \longleftarrow C' & (A', h', C') \\
 \downarrow (\alpha, \gamma) \\
 \mathbf{A} \xrightarrow{F} \mathbf{B} \longleftarrow \mathbf{B} \xleftarrow{G} \mathbf{C} & (F \downarrow G)
 \end{array}$$

To obtain the other 8 cases I replace the functors F and G by $\text{Id}_{\mathbf{B}}$ or S_B , where $S_B : 1 \rightarrow \mathbf{B}$ is the functor that “selects” the object B — it takes the only object $\bullet \in 1$ to B . For example, $(S_B, \text{Id}_{\mathbf{B}})$ is:

$$\begin{array}{ccc}
 \bullet \longrightarrow B \xrightarrow{h} B' \longleftarrow B' & (\bullet, h, B') \\
 \parallel & \parallel & \beta \downarrow & \beta \downarrow \\
 \bullet \longrightarrow B \xrightarrow{h'} B'' \longleftarrow B'' & (\bullet, h', B'') \\
 \downarrow (\text{id}_{\bullet}, \beta) \\
 \mathbf{A} \xrightarrow{S_B} \mathbf{B} \longleftarrow \mathbf{B} \xleftarrow{\text{Id}_{\mathbf{B}}} \mathbf{B} & (S_B \downarrow \text{Id}_{\mathbf{B}})
 \end{array}$$

Shorthands:

1) Use ‘ $_$ ’ in the pairs and triples in the positions where the information there is trivial — $(\text{id}_{\bullet}, \beta) : (\bullet, h, B') \rightarrow (\bullet, h', B'')$ becomes $(_, \beta) : (_, h, B') \rightarrow (_, h', B'')$.

2) Use B instead of S_B .

3) Use \mathbf{B} instead of $\text{Id}_{\mathbf{B}}$.

The correspondence with the names in CWM is:

The comma category $(F \downarrow G)$

The category $(B \downarrow \mathbf{B})$ of objects under B

The category $(\mathbf{B} \downarrow B)$ of objects over B

The category $(B \downarrow G)$ of objects G -under B

The category $(F \downarrow B)$ of objects F -over B

The nine cases:

$$\begin{array}{ccc}
 (F \downarrow G) & (F \downarrow \mathbf{B}) & (F \downarrow B) \\
 (\mathbf{B} \downarrow G) & (\mathbf{B} \downarrow \mathbf{B}) & (\mathbf{B} \downarrow B) \\
 (B \downarrow G) & (B \downarrow \mathbf{B}) & (B \downarrow B')
 \end{array}
 \Rightarrow
 \begin{array}{ccc}
 (F \downarrow G) & (F \downarrow \text{Id}_{\mathbf{B}}) & (F \downarrow S_B) \\
 (\text{Id}_{\mathbf{B}} \downarrow G) & (\text{Id}_{\mathbf{B}} \downarrow \text{Id}_{\mathbf{B}}) & (\text{Id}_{\mathbf{B}} \downarrow B) \\
 (S_B \downarrow G) & (S_B \downarrow \text{Id}_{\mathbf{B}}) & (S_B \downarrow S_{B'})
 \end{array}$$

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II. Constructions on Categories

p.45: 6. Comma Categories

MacLane uses a notation with lots of names and shorthands.

Fix $C, b \in C$. The category $(b \downarrow C)$ of *objects under b* has objects like $\langle f, c \rangle$, where $c \in C$ and $f : b \rightarrow c$.

Fix $C, a \in C$. The category $(C \downarrow a)$ of *objects over a* has objects like $\langle c, f \rangle$, where $c \in C$ and $f : c \rightarrow a$.

Fix $C, D, b \in C$. $S : D \rightarrow C$. The category $(b \downarrow S)$ of *objects S -under b* has objects like $\langle f, d \rangle$, where $d \in D$ and $f : b \rightarrow Sd$.

Fix $C, E, a \in C$. $T : E \rightarrow C$. The category $(T \downarrow a)$ of *objects T -over a* has objects like $\langle f, d \rangle$, where $d \in D$ and $f : b \rightarrow Sd$.

Fix C, D, E , and S, T with $E \xrightarrow{T} C \xleftarrow{S} D$. The *comma category* (T, S) has objects like $\langle e, d, f \rangle$, where $d \in D, e \in E$ and $f : Te \rightarrow Sd$.

An object $b \in C$ may be regarded as a functor $b : 1 \rightarrow C$ with image b .

A category C may be regarded as the identity functor $C \rightarrow C$.

We have:

$(b \downarrow C)$ has objects like $\langle *, c, f \rangle$, where $c \in C$ and $f : b \rightarrow c$.

$(C \downarrow a)$ has objects like $\langle c, *, f \rangle$, where $c \in C$ and $f : c \rightarrow a$.

$(b \downarrow S)$ has objects like $\langle *, d, f \rangle$, where $d \in D$ and $f : b \rightarrow Sd$.

$(T \downarrow a)$ has objects like

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III. Universals and Limits

p.55: Definition: universal arrow from c to S Fix $D, C, S : D \rightarrow C, r \in D, c \in C$.

Then we have functors

 $D(r, -) : D \rightarrow \mathbf{Set}$ and $C(c, S-) : D \rightarrow \mathbf{Set}$.

Every $u : c \rightarrow Sr$ induces a NT

$$\begin{array}{lcl} (S \circ u) & : & D(r, -) \rightarrow C(c, S-), \\ (S \circ u)d & : & D(r, d) \rightarrow C(c, Sd) \\ & & f' \mapsto Sf' \circ u. \end{array}$$
We say that a pair $\langle r, u \rangle$ is a *universal arrow from c to S* when $(S \circ u)$ (i.e., $\lambda d. \lambda f'. Sf' \circ u$) is a natural isomorphism, i.e., when every $(S \circ u)d$ (i.e., $\lambda f'. Sf' \circ u$) is a bijection.

In MacLane's and in my notation:

$$\begin{array}{ccc} \begin{array}{ccc} & c & \\ & \downarrow u & \\ r & \xrightarrow{S} & Sr \\ f' \downarrow & & \downarrow Sf' \\ d & \xrightarrow{S} & Sd \\ & & \downarrow Sf' \circ u \\ & & f = \end{array} & \begin{array}{ccc} & A & \\ & \downarrow u & \\ B & \xrightarrow{\quad} & RB \\ \beta \downarrow & & \downarrow R\beta \\ B' & \xrightarrow{\quad} & RB' \\ & & \downarrow g = \\ & & u; R\beta \end{array} \\ D \xrightarrow{S} C & & B \xrightarrow{R} A \\ D(r, -) \xrightarrow[\cong]{(S \circ u)} C(c, S-) & & (B, -) \longrightarrow (A, R-) \\ D(r, d) \xrightarrow[\cong]{(S \circ u)d} C(c, Sd) & & (B, B') \longrightarrow (A, RB') \end{array}$$

As comma categories (universal arrows are initial in comma categories):

$$\begin{array}{ccc} * \mapsto c \xrightarrow{u} Sr \leftarrow r & \langle r, u \rangle & \bullet \mapsto A \xrightarrow{u} RB \leftarrow B & (_, u, B) \\ \parallel & \parallel & \parallel & \parallel \\ * \mapsto c \xrightarrow{f} Sd \leftarrow d & \langle d, f \rangle & \bullet \mapsto A \xrightarrow{g} RB' \leftarrow B' & (_, g, B') \\ 1 \xrightarrow{c} C \xleftarrow{S} D & (c \downarrow S) & 1 \xrightarrow{S_A} \mathbf{A} \xleftarrow{R} \mathbf{B} & (S_A \downarrow R) \end{array}$$

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III. Universals and Limits

p.58: Definition: universal arrow from S to c Fix $D, C, S : D \rightarrow C, r \in C, c \in C$.

Then we have functors

 $D(-, r) : D^{\text{op}} \rightarrow \mathbf{Set}$ and $C(S-, c) : D^{\text{op}} \rightarrow \mathbf{Set}$.

Every $v : Sr \rightarrow c$ induces a NT

$$\begin{array}{lll} (v \circ S-) & : & D(-, r) \xrightarrow{\sim} C(S-, c), \\ (v \circ S-)d & : & D(d, r) \xrightarrow{\sim} C(Sd, c) \\ & & f' \mapsto v \circ Sf'. \end{array}$$
We say that a pair $\langle r, v \rangle$ is a *universal arrow from S to c*

when $(v \circ S-)$ (i.e., $\lambda d. \lambda f'. v \circ Sf'$) is a natural isomorphism, i.e., when every $(v \circ S-)d$ (i.e., $\lambda f'. v \circ Sf'$) is a bijection.

$$\begin{array}{ccc} C & \xleftarrow{S} & D \\ \\ \begin{array}{ccc} \mathbf{S}d & \xleftarrow{S} & d \\ \downarrow Sf' & & \downarrow f' \\ \mathbf{S}r & \xleftarrow{S} & r \\ \downarrow v & & \\ c & & \end{array} & & \end{array}$$

$$C(S-, c) \xleftarrow[\cong]{(S-ou)} D(-, r)$$

$$C(Sd, c) \xleftarrow[\cong]{(S-ou)d} D(d, r)$$

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III. Universals and Limits

p.59: 2. The Yoneda Lemma

The lemma behind Yoneda, in my notation

$$\begin{array}{ccc}
& & A \\
& & \downarrow \eta \\
B & \longrightarrow & RB \\
& & \downarrow \uparrow U \\
& & D \\
(B, -) & \xrightarrow{S} & (A, R-)
\end{array}$$

There is a bijection between morphisms $\eta : A \rightarrow RB$

and natural transformations $S : (B, -) \rightarrow (A, R-)$.

D is $SCf := \eta; Rf$, i.e., $S := \lambda C.\lambda f.(\eta; Rf)$ and $D := \lambda \eta.\lambda C.\lambda f.(\eta; Rf)$.

U is $\epsilon := SBid_B$, i.e., $U := \lambda S.SBid_B$.

We want to check that $U(D\eta) = \eta$ and $D(US) = S$.

Using just (untyped) λ -calculus we can prove $U(D\eta) = \eta$ easily,

but the proof of $D(US) = S$ stops halfway...

$$\begin{aligned}
U(D\eta) &= (\lambda S.SB(id_B))((\lambda \eta.\lambda C.\lambda f.(\eta; Rf))(\eta)) \\
&= (\lambda S.SB(id_B))(\lambda C.\lambda f.(\eta; Rf)) \\
&= (\lambda C.\lambda f.(\eta; Rf))B(id_B) \\
&= (\lambda f.(\eta; Rf))(id_B) \\
&= \eta; R(id_B) \\
&= \eta; id_{RB} \\
&= \eta
\end{aligned}$$

$$\begin{aligned}
D(US) &= (\lambda \eta.\lambda C.\lambda f.(\eta; Rf))(SB(id_B)) \\
&= \lambda C.\lambda f.((SB(id_B)); Rf)
\end{aligned}$$

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III. Universals and Limits

p.59: 2. The Yoneda Lemma

The lemma behind Yoneda, in my notation (2)We need the naturality (a.k.a. the “condition on squares”) of S :

$$\begin{array}{ccccc}
& A & & C & (B, C) \xrightarrow{SC} (A, RC) & f \dashv \rightarrow SCf \\
& \downarrow \eta & & \downarrow g & \downarrow \lambda g.(f;g) \quad \downarrow \lambda h.(h;Rg) & \downarrow \quad \downarrow \\
B \dashv \rightarrow RB & & & D & (B, D) \xrightarrow{SD} (A, RD) & \downarrow \quad (SCf); Rg \\
\downarrow f & & & & & \downarrow f; g \dashv \rightarrow SD(f;g) \\
C \dashv \rightarrow RC & & & & & \\
\downarrow g & & & & (B, -) \xrightarrow{S} (A, R-) & \\
D \dashv \rightarrow RD & & & & &
\end{array}$$

which yields $(SCf); Rg = SD(f;g)$. Substituting $\left[\begin{array}{l} C := B \\ D := C \\ f := \text{id}_B \\ g := f \end{array} \right]$ in that we get $(SB(\text{id}_B)); Rf = SC(\text{id}_B; f)$, and so:

$$\begin{aligned}
D(US) &= (\lambda\eta.\lambda C.\lambda f.(\eta; Rf))(SB(\text{id}_B)) \\
&= \lambda C.\lambda f.((SB(\text{id}_B)); Rf) \\
&= \lambda C.\lambda f.SC(\text{id}_B; f) \\
&= \lambda C.\lambda f.SCf \\
&= S.
\end{aligned}$$

The last step can be explained as:

$$\begin{aligned}
D(US)Cf &= (\lambda C.\lambda f.SCf)Cf \\
&= (\lambda f.SCf)f \\
&= SCf
\end{aligned}$$

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III. Universals and Limits

p.59: 2. The Yoneda Lemma

p.61: Lemma (Yoneda).

Yoneda in my notation:

$$\begin{array}{ccccc}
& A & & 1 & & 1 \\
& \downarrow \alpha & & \downarrow r & & \downarrow f \\
C \dashv \rightarrow & RC & & C \dashv \rightarrow & RC & & C \dashv \rightarrow & (B, C) \\
\downarrow \uparrow & & \rightarrow & \downarrow \uparrow y & & \rightarrow & Y \downarrow \uparrow & \\
(C, -) \xrightarrow{T} & (A, R-) & & (C, -) \rightarrow & (1, R-) & & (C, -) \rightarrow & (1, (B, -)) \\
& & & \searrow T' & \uparrow & & \searrow f^* & \uparrow \\
& & & & R & & & (B, -)
\end{array}$$

Left part:

Fix categories \mathbf{A} and \mathbf{C} , a functor $R : \mathbf{C} \rightarrow \mathbf{A}$, and objects $A \in \mathbf{A}$, $C \in \mathbf{C}$.We have functors $(C, -) : \mathbf{C} \rightarrow \mathbf{Set}$ and $(A, R-) : \mathbf{C} \rightarrow \mathbf{Set}$.Each map $\alpha : A \rightarrow RC$ induces a NT $T : (C, -) \rightarrow (A, R-)$ and vice-versa.The formulas are $T := \lambda D : \mathbf{C}. \lambda f : (C, D). (a; Rf)$ and $\alpha = T_C(\text{id}_C)$,and the ' $\downarrow \uparrow$ ' is a bijection.

Middle part:

We take the left part and substitute $\mathbf{A} := \mathbf{Set}$ and $A := 1$.The functor R becomes a functor from \mathbf{C} to \mathbf{Set} .There is a natural iso (' $\downarrow \uparrow$ ', unnamed) between the functors $(1, R-)$ and R .We have a bijection between arrows $r : 1 \rightarrow RC$ (or elements of RC)and natural transformations $T' : (C, -) \rightarrow R$.The *Yoneda map* ' y ' in ' $\downarrow \uparrow y$ ' is a bijection $y : \text{Nat}((C, -), R) \cong RC$.

Right part:

Choose an object $B \in \mathbf{C}$. Take the middle part and substitute $R := (B, -)$.We get a bijection $Y \downarrow \uparrow$ between maps $f : B \rightarrow C$ and NTs $f^* : (C, -) \rightarrow (B, -)$. The *Yoneda Functor* $Y : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{C}}$ behaves as:

$$\begin{array}{ccc}
B & B^{\text{op}} \mapsto & (B, -) \\
f \downarrow & f^{\text{op}} \uparrow & \uparrow Yf \\
C & C^{\text{op}} \mapsto & (C, -)
\end{array}$$

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III. Universals and Limits

p.59: 2. The Yoneda Lemma

p.61: Lemma (Yoneda).

$$\begin{array}{ccccc}
 \begin{array}{c} c \\ \downarrow u \\ r \dashrightarrow Sr \end{array} & & \begin{array}{c} * \\ \downarrow u \\ r \dashrightarrow Kr \end{array} & & \begin{array}{c} * \\ \downarrow f \\ r \dashrightarrow D(s, r) \end{array} \\
 \downarrow \uparrow & \longrightarrow & \downarrow \uparrow y & \longrightarrow & Y \downarrow \uparrow \\
 D(r, -) \xrightarrow{\overline{T}} C(c, S-) & & D(r, -) \longrightarrow \mathbf{Set}(*, K-) & & D(r, -) \longrightarrow \mathbf{Set}(*, D(s, -)) \\
 & & \searrow T' & & \searrow D(f, -) \\
 & & K & & D(s, -)
 \end{array}$$

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 III. Universals and Limits
 p.59: 2. The Yoneda Lemma
 Proposition 1

$$\begin{array}{ccc}
 D & \xrightarrow{S} & C \\
 \\
 \begin{array}{ccc}
 & & c \\
 & & \downarrow u \\
 r & \xrightarrow{S} & \mathbf{S}r \\
 f' \downarrow & & \downarrow Sf' \\
 d & \xrightarrow{S} & \mathbf{S}d \\
 k \downarrow & & \downarrow Sk \\
 d' & \xrightarrow{S} & \mathbf{S}d'
 \end{array}
 \end{array}
 \quad \begin{array}{c}
 \downarrow Sf' \circ u \\
 \\
 \\
 \\
 \end{array}$$

$$D(r, -) \xrightarrow[\cong]{(S \circ u)} C(c, S-) \quad D(r, -) \xrightarrow[\cong]{\varphi} C(c, S-)$$

$$D(r, d) \xrightarrow[\cong]{(S \circ u)d} C(c, Sd)$$

$$D(r, r) \xrightarrow[\cong]{(S \circ u)r} C(c, Sr) \quad D(r, r) \xrightarrow[\cong]{\varphi_r} C(c, Sr)$$

$$\begin{array}{ccc}
 1_r \xrightarrow{(S \circ u)r} & S1_r \circ u & 1_r \xrightarrow{\varphi_r} \varphi_r(1_r) \\
 & = 1_{Sr} \circ u & \\
 & = u &
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 r & & \\
 \rho \downarrow & & \\
 \mathbf{r} & \xrightarrow{D(r, -)} & D(r, r) \\
 f' \downarrow & \mapsto & \downarrow D(r, f') = \lambda_{\rho, f' \circ \rho} \\
 \mathbf{d} & \xrightarrow{} & D(r, d) \\
 k \downarrow & \mapsto & \downarrow D(r, k) = \lambda_{f', k \circ f'} \\
 \mathbf{d}' & \xrightarrow{} & D(r, d')
 \end{array}
 &
 \begin{array}{ccc}
 r & \xrightarrow{C(s, S-)} & C(c, Sr) \\
 f' \downarrow & \mapsto & \downarrow C(c, Sf') = \lambda_{u, Sf' \circ u} \\
 d & \xrightarrow{} & C(c, Sd) \\
 k \downarrow & \mapsto & \downarrow C(c, Sk) = \lambda_{g, Sk \circ g} \\
 d' & \xrightarrow{} & C(c, Sd')
 \end{array}
 &
 \begin{array}{ccc}
 D(r, r) & \xrightarrow{\varphi_r} & C(c, Sr) \\
 D(r, f') \downarrow & & \downarrow C(c, Sf') \\
 D(r, d) & \xrightarrow{\varphi_r} & C(c, Sd) \\
 \\
 1_r & \xrightarrow{\varphi_r} & \varphi_r(1_r) \\
 \lambda_{\rho, f' \circ \rho} \downarrow & & \downarrow C(c, Sf') \\
 f' & \xrightarrow{\varphi_r} & C(c, Sd)
 \end{array}
 \end{array}$$

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III. Universals and Limits

p.59: 2. The Yoneda Lemma

p.60: Proposition 2

$$\begin{array}{ccc}
 D & \xrightarrow{K} & \mathbf{Set} \\
 \\
 \begin{array}{ccc}
 & & * \\
 & & \downarrow u \\
 r & \longrightarrow & Kr \\
 f' \downarrow & & \downarrow Kf' \\
 d & \longrightarrow & Kd
 \end{array}
 \end{array}$$

$$D(r, d) \xrightarrow[\cong]{(K \circ u)d} \mathbf{Set}(*, Kd) \xrightarrow[\cong]{} Kd$$

$$D(r, -) \xrightarrow[\cong]{(K \circ u)} \mathbf{Set}(*, K-) \xrightarrow[\cong]{} K$$

Yoneda: GF

$$\begin{array}{lcl}
 f & : & A \rightarrow B \\
 & & \text{Nat}(yB, F) \mapsto \text{Nat}(yA, F) \\
 & & c \mapsto c \circ (f \circ -) : yA \rightarrow F \\
 & & c \circ (f \circ -)_C : yAC \rightarrow FC \\
 & & : \mathbf{C}(C, A) \rightarrow FC \\
 & & g \mapsto \mathbf{Cat}_c(f \circ g)
 \end{array}$$

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IV. Adjoints

p.79: Adjunctions - the naturality of φ Fix $X, A, F : X \rightarrow A, G : A \rightarrow X, \langle F, G, \varphi \rangle : X \dashv A$.

Remember that we have functors

 $A(F-, -) : X^{\text{op}} \times A \rightarrow \mathbf{Set}$ and $X(-, G-) : X^{\text{op}} \times A \rightarrow \mathbf{Set}$,and $\varphi : A(F-, -) \rightarrow X(-, G-)$ is a natural transformation (and a natural iso)...Let $\langle h, k \rangle : \langle x, a \rangle \rightarrow \langle x', a' \rangle$ be a morphism in $X^{\text{op}} \times A$.The naturality of φ is easier to see in this diagram:

$$\begin{array}{ccc}
\langle F, G, \varphi \rangle : X \dashv A & & \\
A \xleftarrow{F} X & & \\
& & \\
\begin{array}{ccc}
Fx' \xleftarrow{F} x' & & \\
\downarrow Fh & \xrightarrow{F} & \downarrow h \\
Fx \xleftarrow{F} x & & \\
\downarrow f & \xrightarrow{\varphi} & \downarrow \varphi f \\
a \xrightarrow{G} Ga & & \\
\downarrow k & \xrightarrow{G} & \downarrow Gk \\
a' \xrightarrow{G} Ga' & &
\end{array} & \begin{array}{ccc}
A(F-, -)\langle x, a \rangle \xrightarrow{\varphi_{\langle a, x \rangle}} X(-, G-)\langle x, a \rangle & & \\
\downarrow A(F-, -)\langle k, h \rangle & & \downarrow X(-, G-)\langle k, h \rangle \\
A(F-, -)\langle x', a' \rangle \xrightarrow{\varphi_{\langle a', x' \rangle}} X(-, G-)\langle x', a' \rangle & & \\
& & \\
A(Fx, a) \xrightarrow{\varphi_{x, a}} X(x, Ga) & & \\
\downarrow A(Fh, k) & & \downarrow X(h, Gk) \\
A(Fx', a') \xrightarrow{\varphi_{x', a'}} X(x', Ga') & & \\
& & \\
f \dashv \longrightarrow \varphi f & & \\
\downarrow & & \downarrow \\
k \circ f \circ Fh \dashv \longrightarrow \varphi(k \circ f \circ Fh) & &
\end{array}
\end{array}$$

$$A(F-, -) \xrightleftharpoons[\varphi_{-, a}^{-1}]{\varphi_{-, a}^{-1}} X(-, G-)$$

$$A(Fx, a) \xrightleftharpoons[\varphi_{x, a}]{\varphi_{x, a}^{-1}} X(x, Ga)$$

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IV. Adjoints

p.82: Adjunctions - interdefinabilities

(In MacLane's notation; unrevised)

 $F \dashv G, \quad \langle F, G, \varphi \rangle : X \rightarrow A,$ $A \xleftarrow{F} X, \quad A(F-, -) \xleftarrow{\varphi^{-1}} X(-, G-).$

$$\begin{array}{ccc}
 Fx' & \longleftarrow & x' \\
 Fh \downarrow & & \downarrow h \\
 Fx & \longleftarrow & x \\
 \varphi^{-1}g \downarrow & \longleftarrow \varphi^{-1} & \downarrow g \\
 f \downarrow & \longleftarrow \varphi & \downarrow \varphi f \\
 a & \longrightarrow & Ga \\
 k \downarrow & & \downarrow Gk \\
 a' & \longrightarrow & Ga'
 \end{array}$$

$$\begin{array}{ccc}
 Fx \longleftarrow x & \xrightarrow{\varphi^{-1}g = \epsilon_a \circ Fg} & Fx' \longleftarrow x' \\
 \text{id}_{Fx} \downarrow & \longleftarrow \varphi & \downarrow Fh = \varphi^{-1}(\eta_x \circ h) \\
 Fx \longrightarrow GFx & & Fx \longrightarrow GFx \\
 \eta_x = \downarrow & & \downarrow \eta_x \\
 & & x \\
 & & \downarrow h \\
 & & Ga \\
 & & \downarrow Gk \\
 & & Ga'
 \end{array}$$

Theorem 2. $\langle L, R, \natural \rangle : \mathbf{A} \rightarrow \mathbf{B}$ is completely determined by:

- (i) L, R, η , with each η_A universal
- (ii) G, F_0 and universal arrows η_A
- (iii) F, G, ϵ with each ϵ_a universal
- (iv) F, G_0 and universal arrows ϵ_a
- (v)

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IV. Adjoints

p.82: Adjunctions - interdefinabilities

(In my notation)

 $L \dashv R, \langle L, R, \sharp \rangle : \mathbf{A} \rightarrow \mathbf{B},$

$$\mathbf{B} \begin{array}{c} \xleftarrow{L} \\ \xrightarrow{R} \end{array} \mathbf{A}, \quad \mathbf{B}(L-, -) \begin{array}{c} \xleftarrow{b} \\ \xrightarrow{\sharp} \end{array} \mathbf{A}(-, R-).$$

$$\begin{array}{ccc} LA' & \longleftarrow & A' \\ L\alpha \downarrow & & \downarrow \alpha \\ LA & \longleftarrow & A \\ g^b \downarrow & \begin{array}{c} \xleftarrow{b} \\ \xrightarrow{\sharp} \end{array} & \downarrow g \\ B & \longrightarrow & RB \\ \beta \downarrow & & \downarrow R\beta \\ B' & \longrightarrow & RB' \end{array}$$

$$\begin{array}{ccc} LA & \longleftarrow & A \\ \text{id}_{LA} \downarrow & \xrightarrow{\quad} & \downarrow \eta_A \\ LA & \longrightarrow & RLA \\ \epsilon_B = & Lg; \epsilon_B & \\ LRB & \longleftarrow & RB \\ \epsilon_B \downarrow & \nearrow & \downarrow g \\ B & & \end{array} \quad \begin{array}{ccc} LA' & \longleftarrow & A' \\ \downarrow & & \downarrow \alpha \\ A & & \\ \nearrow & & \downarrow \eta_A \\ LA & \longrightarrow & RLA \end{array}$$

$$\begin{array}{ccc} LRB & \longleftarrow & RB \\ (\text{id}_{RB})^\beta \downarrow & \xleftarrow{\quad} & \downarrow \text{id}_{RB} \\ B & \longrightarrow & RB \\ \epsilon_B = & & \\ A & & \\ \nearrow & \downarrow \eta_A & \\ LA & \longrightarrow & RLA \\ f \downarrow & \downarrow Rf & \downarrow \\ B & \longrightarrow & RB \end{array} \quad \begin{array}{ccc} LRB & \longrightarrow & RB \\ \epsilon_B \downarrow & \nearrow & \downarrow \\ B & & \\ \beta \downarrow & & \downarrow R\beta \\ B' & \longrightarrow & RB' \end{array}$$

Theorem 2. $\langle L, R, \sharp \rangle : \mathbf{A} \rightarrow \mathbf{B}$ is completely determined by:

- (i) L, R, η , with each η_A universal
- (ii) G, F_0 and universal arrows η_A
- (iii) F, G, ϵ with each ϵ_a universal
- (iv) F, G_0 and universal arrows ϵ_a
- (v)

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VI. Monads and Algebras

p.137: Monads

Fix $X, T : X \rightarrow X, \mu : T^2 \rightarrow T, \eta : I_X \rightarrow T$.

Then we can make a diagram:

$$\begin{array}{ccc}
 I & \xrightarrow{\eta} & T \xleftarrow{\mu} T^2 & & x & \xrightarrow{\eta x} & Tx \xleftarrow{\mu x} T^2 x \\
 \\
 T & \xrightarrow{T\eta} & T^2 \xleftarrow{T\mu} T^3 & & Tx & \xrightarrow{T(\eta x)} & T^2 x \xleftarrow{T(\mu x)} T^3 x \\
 \eta T \downarrow & \searrow \text{id} & \downarrow \mu & & \eta(Tx) \downarrow & \searrow \text{id} & \downarrow \mu x \\
 T^2 & \xrightarrow{\mu} & T \xleftarrow{\mu} T^2 & & T^2 x & \xrightarrow{\mu x} & Tx \xleftarrow{\mu x} T^2 x \\
 & & \downarrow \mu T & & & & \downarrow \mu(Tx)
 \end{array}$$

A monad $T = \langle T, \eta, \mu \rangle$ in a category X is a triple as above that obeys $\mu \circ \eta T = I_X = \mu \circ T\eta$ and $\mu \circ T\mu = \mu \circ \mu T$.

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2. Algebras for a monad

p.140: T -algebrasFix X and a monad $T = \langle T, \eta, \mu \rangle$ in X .A T -algebra is a pair $\langle x, h \rangle$ with $x \in X$ and $h : Tx \rightarrow x$ that obeys $\text{id}_x = h \circ \eta_x$, $h \circ \mu_x = h \circ Th$:

$$\begin{array}{ccc}
 x & \xrightarrow{\eta_x} & Tx & \xleftarrow{\mu_x} & T^2x \\
 & \searrow \text{id} & \downarrow h & & \downarrow Th \\
 & & x & \xleftarrow{h} & Tx
 \end{array}
 \quad \text{or} \quad
 \begin{array}{ccc}
 x & \xrightarrow{\eta_x} & Tx & \xleftarrow{\mu_x} & T^2x \\
 & \searrow \text{id} & \swarrow h & & \swarrow Th \\
 & & x & \xleftarrow{h} & Tx
 \end{array}$$

A morphism $f : \langle x, h \rangle \rightarrow \langle x', h' \rangle$ (in the category X^T of T -algebras) is a morphism $f : x \rightarrow x'$ obeying $f \circ h = h' \circ Tf$.

$$\begin{array}{ccc}
 x & \xleftarrow{h} & Tx \\
 f \downarrow & & \downarrow Tf \\
 x' & \xleftarrow{h'} & Tx'
 \end{array}$$

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First examples

Let M be a monoid.We will call its identity e and its elements a, b, c , etc.Multiplication in M will be written as ab .Let Q, R, S be (arbitrary) sets.Then $T = (\times M) : \mathbf{Set} \rightarrow \mathbf{Set}$ and $\langle \times M, \eta, \mu \rangle$ is a monad on \mathbf{Set} , where:

$$\eta S : S \rightarrow S \times M \quad \text{and} \quad \mu S : (S \times M) \times M \rightarrow S \times M$$

$$s \mapsto \langle s, e \rangle \quad \text{and} \quad \langle \langle s, a \rangle, b \rangle \mapsto \langle s, ab \rangle.$$

In λ -notation they are $\eta = \lambda S. \lambda s. \langle s, e \rangle$ and $\mu = \lambda S. \lambda \langle \langle s, a \rangle, b \rangle. \langle s, ab \rangle$.Note that the conditions on η and μ , that we gave abstractly as:

$$x \xrightarrow{\eta x} Tx \xleftarrow{\mu x} T^2x$$

$$Tx \xrightarrow{T(\eta x)} T^2x \xleftarrow{T(\mu x)} T^3x$$

$$\begin{array}{ccccc} & & \eta(Tx) & & \\ & & \downarrow & \text{id} & \downarrow \\ Tx & & & & T^2x \\ & & \downarrow & \downarrow \mu x & \downarrow \mu(Tx) \\ T^2x & \xrightarrow{\mu x} & Tx & \xleftarrow{\mu x} & T^2x \end{array}$$

become:

$$q \mapsto \langle q, e \rangle \quad \langle q, ab \rangle \longleftarrow \langle \langle q, a \rangle, b \rangle$$

$$\langle q, a \rangle \mapsto \langle \langle q, e \rangle, a \rangle \quad \langle \langle q, ab \rangle, c \rangle \longleftarrow \langle \langle \langle q, a \rangle, b \rangle, c \rangle$$

$$\begin{array}{ccccc} & & \downarrow & & \downarrow \\ \langle q, a \rangle & & \downarrow & \text{id} & \downarrow \\ & & \langle q, ae \rangle & & \langle q, (ab)c \rangle \\ & & \downarrow & \downarrow \mu x & \downarrow \mu(Tx) \\ \langle \langle q, a \rangle, e \rangle & \mapsto & \langle q, ea \rangle & & \langle q, a(bc) \rangle \longleftarrow \langle \langle q, a \rangle, bc \rangle \end{array}$$

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First examples (2)

Fix a monoid M and sets Q, R .

An *action of M on a set Q* is a map $h : Q \times M \rightarrow Q$
 $\langle q, a \rangle \mapsto qa$

obeying $q(ab) = (qa)b$ and $qe = q$.

An *action of M on a set R* is a map $h' : R \times M \rightarrow R$
 $\langle r, a \rangle \mapsto ra$

obeying $r(ab) = (ra)b$ and $re = r$.

Note that we don't write h or h' .

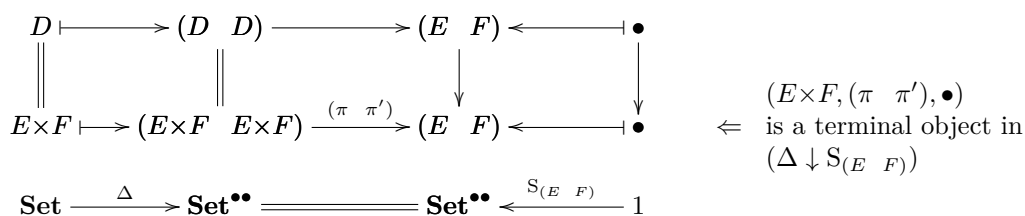
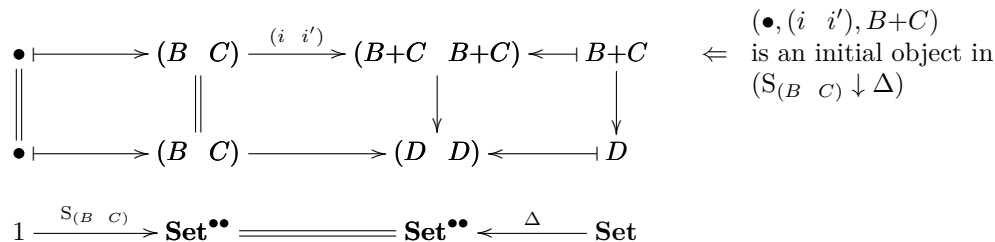
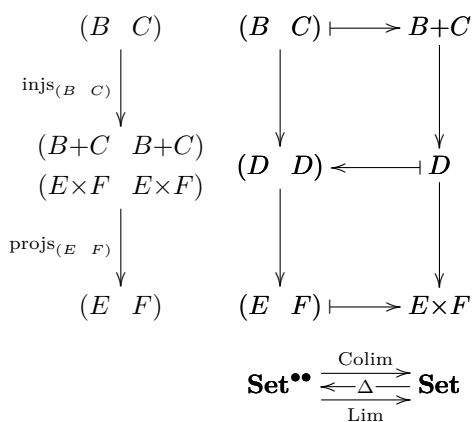
Kan extensions in my notation

Archetypal example: the functor $\Delta : \mathbf{Set} \rightarrow \mathbf{Set}^{\bullet\bullet}$

has both adjoints: $\text{Colim} \dashv \Delta \dashv \text{Lim}$.

I will refer to the unit η of $\text{Colim} \dashv \Delta$ as *injs*

and to the counit ϵ of $\Delta \dashv \text{Lim}$ as *projs*.



Kan extensions in my notation

Archetypal example: the functor $f^* : \mathbf{Set}^4 \rightarrow \mathbf{Set}^6$

has both adjoints: $\text{Colim} \dashv f^* \dashv \text{Lim}$.

I will refer to the unit η of $\text{Colim} \dashv f^*$ as *injs*

and to the counit ϵ of $f^* \dashv \text{Lim}$ as *projs*.

$$\begin{array}{ccc}
 \begin{pmatrix} & C_2 \\ C_3 \rightarrow & \downarrow \\ & C_4 \\ \downarrow & \\ C_5 & \end{pmatrix} & \dashv \rightarrow & \begin{pmatrix} 0 \rightarrow & C_2 \\ \downarrow & \downarrow \\ C_3 \rightarrow & C_4 \\ \downarrow & \downarrow \\ C_5 \rightarrow & C_{\text{po}} \end{pmatrix} & & \begin{pmatrix} 0 \rightarrow & D_2 \\ \downarrow & \downarrow \\ D_3 \rightarrow & D_4 \\ \downarrow & \downarrow \\ D_5 \rightarrow & D_{\text{po}} \end{pmatrix} \\
 \downarrow & & \downarrow & & \downarrow \text{injs}_{(B \ C)} \\
 \begin{pmatrix} & D_2 \\ D_3 \rightarrow & \downarrow \\ & D_4 \\ \downarrow & \\ D_5 & \end{pmatrix} & \dashv \leftarrow & \begin{pmatrix} D_1 \rightarrow & D_2 \\ \downarrow & \downarrow \\ D_3 \rightarrow & D_4 \\ \downarrow & \downarrow \\ D_5 \rightarrow & D_6 \end{pmatrix} & & \begin{pmatrix} D_1 \rightarrow & D_2 \\ \downarrow & \downarrow \\ D_3 \rightarrow & D_4 \\ \downarrow & \downarrow \\ D_5 \rightarrow & D_6 \end{pmatrix} \\
 \downarrow & & \downarrow & & \downarrow \text{projs}_{(E \ F)} \\
 \begin{pmatrix} & E_2 \\ E_3 \rightarrow & \downarrow \\ & E_4 \\ \downarrow & \\ E_5 & \end{pmatrix} & \dashv \rightarrow & \begin{pmatrix} E_{\text{pb}} \rightarrow & E_2 \\ \downarrow & \downarrow \\ E_3 \rightarrow & E_4 \\ \downarrow & \downarrow \\ E_5 \rightarrow & 1 \end{pmatrix} & & \begin{pmatrix} D_{\text{pb}} \rightarrow & D_2 \\ \downarrow & \downarrow \\ D_3 \rightarrow & D_4 \\ \downarrow & \downarrow \\ D_5 \rightarrow & 1 \end{pmatrix} \\
 \mathbf{Set}^4 & \begin{array}{c} \xrightarrow{\text{Colim}} \\ \xleftarrow{f^*} \\ \xrightarrow{\text{Lim}} \end{array} & \mathbf{Set}^6
 \end{array}$$

Kan extensions in my notation

$$\begin{array}{ccc}
 SK & \xleftarrow{A^K} & S \\
 \sigma K \downarrow & & \downarrow \sigma \\
 RK & \xleftarrow{A^K} & R = \text{Ran}_K T \\
 \epsilon \downarrow & & \\
 T & & \\
 \\
 A^M & \xleftarrow{A^K} & A^C \\
 \\
 M & \xrightarrow{K} & C
 \end{array}$$