Planar Heyting Algebras for Children
Eduardo Ochs - EBL2017 - Pirenópolis, may 2017
http://angg.twu.net/math-b.html#ebl-2017
http://angg.twu.net/math-b.html#zhas-for-children-2
http://angg.twu.net/LATEX/2017planar-has.pdf (paper)
http://angg.twu.net/LATEX/2017ebl-slides.pdf (slides)
http://angg.twu.net/LATEX/2017ebl-handouts.pdf **(these handouts)**
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LR-coordinates and ZHAs:

$$\langle l,r\rangle = (0,0) + l\,\underbrace{\overrightarrow{(-1,1)}}_{\nwarrow} + r\,\underbrace{\overrightarrow{(1,1)}}_{\nearrow}$$

Shorthand: $lr = \langle l,r\rangle$.
$\mathbb{N}^2 \subset \mathbb{Z}^2$ is a quarter-plane.
$\mathbb{LR} \subset \mathbb{Z}^2$ is a "quarter-plane turned 45° to the left":
$\mathbb{LR} = \{\langle l,r\rangle \mid l,r \in \mathbb{N}\}$.
An LRSet is a finite, non-empty subset of $\mathbb{LR}$ containing $(0,0)$.
ZHAs are LRSets obeying extra conditions...

A "height-left-right-wall" (h,L,R) generates this ZHA:
$\{(x,y) \in \mathbb{LR} \mid y{\le}h, L(y){\le}x{\le}R(y)\}$

Example:

$$\left(9, \left\{\begin{array}{c}(5,-1),\\(4,0),\\(3,-1),\\(2,-2),\\(1,-1),\\(0,0)\end{array}\right\}, \left\{\begin{array}{c}(5,-1),\\(4,0),\\(3,1),\\(2,2),\\(1,1),\\(0,0)\end{array}\right\}\right) \rightsquigarrow$$

$$\rightsquigarrow \left\{\begin{array}{ccccc} & (-1,5) & & & \\ & & (0,4) & & \\ & (-1,3) & & (1,3) & \\ (-2,2) & & (0,2) & & (2,2) \\ & (-1,1) & & (1,1) & \\ & & (0,0) & & \end{array}\right\} \rightsquigarrow \begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array}$$

A ZHA is made of all points in $\mathbb{LR}$ between a left and a right wall.
Theorem: every ZHA is a Heyting Algebra. Details: paper, secs.1–9.

Logic in a ZHA

Abbreviations: b, l, r, a mean below, leftof, rightof, above.

In a ZHA Ω we have:

$$\begin{array}{rcl} \top &:=& \sup(\Omega) \\ \bot &:=& 00 \\ ab \land cd &:=& \min(a,c)\min(b,d) \\ ab \lor cd &:=& \max(a,c)\max(b,d) \\ cd \to ef &:=& \end{array}$$

$$\left(\begin{array}{llll} \mathsf{if} & cd \;\mathsf{b}\; ef & \mathsf{then} & \top \\ \mathsf{elseif} & cd \;\mathsf{l}\; ef & \mathsf{then} & \mathsf{ne}(ef) \\ \mathsf{elseif} & cd \;\mathsf{r}\; ef & \mathsf{then} & \mathsf{nw}(ef) \\ \mathsf{elseif} & cd \;\mathsf{a}\; ef & \mathsf{then} & ef \\ \mathsf{end} & & & \end{array}\right)$$

where:

$$\begin{array}{rcl} cd \;\mathsf{b}\; ef &:=& c \le e \land d \le f \\ cd \;\mathsf{l}\; ef &:=& c \ge e \land d \le f \\ cd \;\mathsf{r}\; ef &:=& c \le e \land d \ge f \\ cd \;\mathsf{r}\; ef &:=& c \le e \land d \le f \end{array}$$

and ne means "go northeast as most as possible", and nw means "go northwest as most as possible"

Partial order:

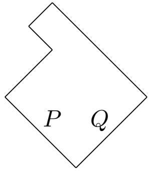
$$ab \le cd \quad := \quad a \le c \land b \le d$$

Theorem:
$P \le (Q \to R)$ iff $(P \land Q) \le R$
(with the weird '→' above).

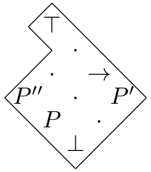
Two non-tautologies

In this ZHA, with this valuation,

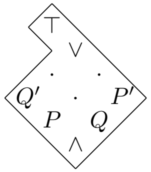
$$H = \begin{array}{ccccc} & 32 & & & \\ & & 22 & & \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array} \qquad v =$$



we have:



$$\underbrace{(\underbrace{\neg\underbrace{\neg\underbrace{P}_{10}}_{02}}_{20}) \to \underbrace{P}_{10}}_{12}$$



$$\underbrace{\underbrace{\neg(\underbrace{\underbrace{P}_{10} \land \underbrace{Q}_{01}}_{00})}_{32} \to \underbrace{(\underbrace{\neg\underbrace{P}_{10}}_{02} \lor \underbrace{\neg\underbrace{Q}_{01}}_{20})}_{22}}_{22}$$

...these two classical tautologies are not $=\top$ (=32) in v,
so they are not true in all Heyting Algebras,
and they can't be theorems of intuitionistic logic...

Note that $\neg P = \neg 10 = 10 \to 00 = \mathsf{ne}(00) = 02$
because 10 below 00 is false
and 10 leftof 00 is true.

Calculating '→' by brute force

If $H = 5 \times 5$, $Q = 31$ and $R = 12$, then
we can calculate $Q \to R$ $(= 14)$ by:

$$((\underbrace{P \land \underbrace{Q}_{31}}_{(\lambda P.P\land 31)=}) \le \underbrace{R}_{12}) \quad = \quad (P \le (\underbrace{\underbrace{Q}_{31} \to \underbrace{R}_{12}}_{??????}))$$

$(\lambda P.P\land 31)=$

$$\begin{array}{ccccccccc} & & & & 31 & & & & \\ & & & 31 & & 31 & & & \\ & & 31 & & 31 & & 21 & & \\ & 31 & & 31 & & 21 & & 11 & \\ 30 & & 31 & & 21 & & 11 & & 01 \\ & 30 & & 21 & & 11 & & 01 & \\ & & 20 & & 11 & & 01 & & \\ & & & 10 & & 01 & & & \\ & & & & 00 & & & & \end{array}$$

$(\lambda P.P\land 31\le 12)=$

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & & 1 & \\ 0 & & 0 & & 0 & & 1 & & 1 \\ & 0 & & 0 & & 1 & & 1 & \\ & & 0 & & 1 & & 1 & & \\ & & & 1 & & 1 & & & \\ & & & & 1 & & & & \end{array}$$

??????
must be 14,
look below:

copy the left side:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & & 1 & \\ 0 & & 0 & & 0 & & 1 & & 1 \\ & 0 & & 0 & & 1 & & 1 & \\ & & 0 & & 1 & & 1 & & \\ & & & 1 & & 1 & & & \\ & & & & 1 & & & & \end{array}$$

Theorem: the formula for '→' in terms of b, l, r, a yields the same results as the brute force method.