Abstract

A *closure operator*, or a *J-operator*, on a Heyting Algebra H is an operator $J : H \to H$ obeying $P \leq P^* = P^{**}$ and $(P \wedge Q)^* = P^* \wedge Q^*$, where P^* is a shorthand for $J(P)$. Each J-operator induces an equivalence relation, with $P \sim_J Q$ iff $P^* = Q^*$; if we write $[P]^J$ for the equivalence class of P it is easy to see that P^* is always the maximal element of $[P]^J$.

In this paper we use finite, planar HAs — "ZHAs", in the terminology of the preceding paper in this series — to understand visually how J-operators and other related operations work.

Our first result is that the the boundaries between J-equivalence classes on a ZHA H are diagonal lines without any "cuts stopping midway", and, conversely, any "slashing" of a ZHA by diagonal cuts without any cuts stopping midway induces a J-operator; there is a correspondence between J-operators on ZHAs and slashings — and this correspondence yields a nice way to visualize the algebra of J-operators on a ZHA.

Our second result is that there is a correspondence between slashings and "sets of question marks", in the following sense. In the previous paper we showed that every ZHA H "is" the set of open sets, in the order topology, of a two-column graph (P, A), and we showed a way to interpret truth-values $Q, R \in H$ as open sets $Q', R' \subseteq P$. Any way to express P as a disjoint union $P_! \cup P_?$ yields an equivalence relation on H, in which $Q \sim R$ iff $Q' \backslash P_? = R' \backslash P_?$; i.e., Q and R are equivalent iff they are equal when we erase the information on $P_?$.

Our third result is about ways to reconstruct the missing information on the points of $P_?$ after the erasal. There are two natural ways to do this reconstruction; one is by taking the biggest possible choice, and another is by taking the smallest possible one. It turns out that these two ways are respectively the right adjoint and the left adjoint to erasing,

$$\text{(smallest choice)} \dashv \text{(erasing)} \dashv \text{(biggest choice)}$$

and that the J-operator J is essentially the unit of the second adjunction. Also, When we draw the categorical diagrams for these adjunctions we get some of the basic diagrams for understanding sheaves on a topos, but in a "miniature case" — the full case will be discussed on the next paper in this series.

Contents

Planar Heyting Algebras for Children 2: Closure Operators

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November 19, 2017

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In this paper we use finite, planar HAs — "ZHAs", in the terminology of the preceding paper in this series — to understand visually how J-operators and other related operations work.

Our first result is that on a ZHA H the boundaries between J-equivalence classes are diagonal lines without any "cuts stopping midway", and, conversely, any "slashing" of a H by diagonal cuts without any cuts stopping midway induces a J-operator. Slashings are easy to visualize, so this yields a way to visualize J-operators (on ZHAs).

This first result will be explained in three parts. In sections 1 to 6 we define slashings, slash-operators, slash-regions, slash-partitions, cuts and cuts stopping midway; in sections 7 to 8 we define J-operators, J-regions, J-partitions, partitions into "intervals", and we show that very J-partition on a ZHA H is a partition of H into intervals without cuts stopping midway; in sections 9 to 12 we show a way to visualize how J-operators interact with the connectives, and prove a lemma pending from sec.8. As a bonus, in sections 13 to 16 we show how to visualize the algebra of slash-operators, and we prove that all slash-operators are polynomial.

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1 Piccs and slashings

A picc ("partition into contiguous classes") of an interval $I = \{0, \dots, n\}$ is a partition P of I that obeys this condition ("picc-ness"):

$$\forall a, b, c \in \{0, \dots, n\}.\ (a < b < c\ \ \&\ \ a \sim_P c) \to (a \sim_P b \sim_P c).$$

So $P = \{\{0\}, \{1, 2, 3\}, \{4, 5\}\}$ is a picc of $\{0, \dots, 5\}$, and $P' = \{\{0\}, \{1, 2, 4, 5\}, \{3\}\}$ is a partition of $\{0, \dots, 5\}$ that is not a picc.

A short notation for piccs is this:

$$0|123|45 \equiv \{\{0\}, \{1, 2, 3\}, \{4, 5\}\}$$

we list all digits in the "interval" in order, and we put bars to indicate where we change from one equivalence class to another.

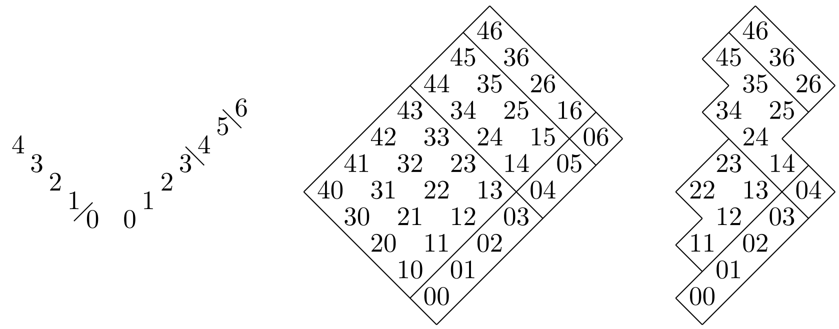
Let's define a notation for "intervals" in $\mathbb{LR}$,

$$[ab, ef] := [\langle a, b\rangle, \langle e, f\rangle] := \{\, \langle c, d\rangle \in \mathbb{LR} \mid a \leq c \leq e\ \ \&\ \ b \leq d \leq f \,\},$$

Note that it can be adapted to define "intervals" in a ZHAs H:

$$\begin{array}{rcl} [ab, ef] \cap H & := & \{\, \langle c, d\rangle \in \mathbb{LR} \mid a \leq c \leq e\ \ \&\ \ b \leq d \leq f \,\} \cap H \\ & = & \{\, \langle c, d\rangle \in H \mid a \leq c \leq e\ \ \&\ \ b \leq d \leq f \,\}. \end{array}$$

A *slashing* S on a ZHA H with top element ab is a pair of piccs, $S = (L, R)$, where L is a picc on $\{0, \dots, a\}$ and R is a picc on $\{0, \dots, b\}$; for example, $S = (4321/0,\ 0123\backslash45\backslash6)$ is a slashing on $[00, 46]$. We write the bars in L as '/'s and the bars in R as '\' as a reminder that they are to be interpreted as northeast and northwest "cuts" respectively; $S = (4321/0,\ 0123\backslash45\backslash6)$ is interpreted as the diagram at the left below, and it "slashes" $[00, 46]$ and the ZHA at the right below as:



A slashing $S = (L, R)$ on a ZHA H with top element ab induces an equivalence relation '$\sim_S$' on H that works like this: $\langle c, d\rangle \sim_S \langle e, f\rangle$ iff $c \sim_L e$ and $d \sim_R f$. We write

$$\begin{array}{rcl}
[c]_L & := & \{\, e \in \{0, \ldots, a\} \mid c \sim_L a \,\} \\
[d]_R & := & \{\, f \in \{0, \ldots, b\} \mid d \sim_L f \,\} \\
[cd]_S & := & \{\, ef \in H \mid cd \sim_S ef \,\}
\end{array}$$

for the equivalence classes, and note that

$$\begin{array}{lrcl}
\text{if} & [c]_L & = & \{c', \ldots, c''\} \\
\text{and} & [d]_L & = & \{d', \ldots, d''\} \\
\text{then} & [cd]_S & = & [c'd', c''d''] \cap H;
\end{array}$$

for example, in the ZHA at the right at the example above we have:

$$\begin{array}{rcl}
[1]_L & = & \{1, 2, 3, 4\}, \\
[2]_R & = & \{0, 1, 2, 3\}, \\
[12]_S & = & [10, 43] \cap H \;=\; \{11, 12, 13, 22, 23\}.
\end{array}$$

We say that a slashing S on a ZHA H partitions H into *slash-regions*; later (sec.7) we will see that a J-operator J also partitions H, and we will refer to its equivalence classes as *J-regions*.

Slash-regions are intervals, but note that neither 10 or 43 belong to the slash-region $[12]_S = [10, 43] \cap H$ above.

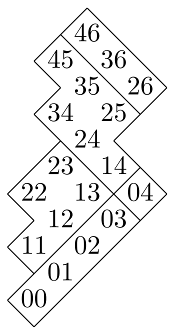
A *slash-partition* is a partition on a ZHA induced by a slashing, and a *slash-equivalence* is an equivalence relation on a ZHA induced by a slashing. Formally, a slash-partition on H is a set of subsets of H, and a slash-equivalence is subset of $H \times H$, but it is so easy to convert between partitions and equivalence relations that we will often use both terms interchangeably. Our visual representation for slash-partitions and slash-equivalences on a ZHA H will be the same: H slashed by diagonal cuts.

2 From slash-partitions back to slashings

We saw how to go from a slashing $S = (L, R)$ on H to an equivalence relation $\sim_S$ on H; let's see now how to recover L and R from $\sim_S$.

Let LW_H be the left wall of H, and RW_H the right wall of H. For example,

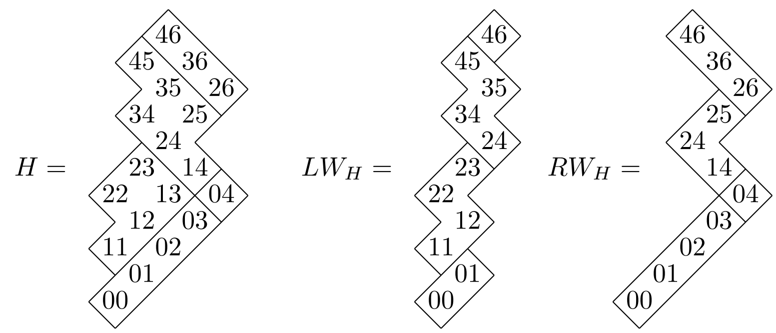
$H =$



$$LW_H = \{00, 01, 11, 12, 22, 23, 24, 34, 35, 45, 46\}$$
$$RW_H = \{00, 01, 02, 03, 04, 14, 24, 25, 26, 36, 46\}$$

To recover the picc L — which is a picc on $\{0, 1, 2, 3, 4\}$ — we need to find where we change from an L-equivalence class to another when we go from one digit to the next; and to recover the picc R — which is a picc on $\{0, 1, 2, 3, 4, 5, 6\}$ — we need to find where we change from an R-equivalence class to another when we go from one digit to the next.

We can recover L and R by walking LW_H (or RW_H) from bottom to top in a series of white pawns moves, and checking when we change from one S-equivalence class to another. Northwest moves give information about L, and northeast moves give information about R. Look at the example below, in which we walk on RW_H:



$$
\begin{array}{lll}
 & & {}_{25}\nearrow^{26} : 25\not\sim_S 26 \Rightarrow 5\not\sim_R 6 \Rightarrow 5\backslash 6 \\
 & & {}_{24}\nearrow^{25} : 24\sim_S 25 \Rightarrow 4\sim_R 5 \Rightarrow 45 \\
{}^{46}\nwarrow_{36} : 36\sim_S 46 \Rightarrow 3\sim_L 4 \Rightarrow 43 & & {}_{03}\nearrow^{04} : 03\not\sim_S 04 \Rightarrow 3\not\sim_R 4 \Rightarrow 3\backslash 4 \\
{}^{36}\nwarrow_{26} : 26\sim_S 36 \Rightarrow 2\sim_L 3 \Rightarrow 32 & & {}_{02}\nearrow^{03} : 02\sim_S 03 \Rightarrow 2\sim_R 3 \Rightarrow 23 \\
{}^{24}\nwarrow_{14} : 14\sim_S 24 \Rightarrow 1\sim_L 2 \Rightarrow 21 & & {}_{01}\nearrow^{02} : 01\sim_S 02 \Rightarrow 1\sim_R 2 \Rightarrow 12 \\
{}^{14}\nwarrow_{04} : 04\not\sim_S 14 \Rightarrow 0\not\sim_L 1 \Rightarrow 1/0 & & {}_{00}\nearrow^{01} : 00\sim_S 01 \Rightarrow 0\sim_R 1 \Rightarrow 01
\end{array}
$$

$$(L,R) \ = \ (4321/0, 0123\backslash 45\backslash 6)$$

3 Slash-regions have maximal elements

...here is how our argument will work, in a particular case:

$$
\begin{array}{rcl}
[1]_L &=& \{1,2,3,4\}, \\
{[2]_R} &=& \{0,1,2,3\}, \\
I &=& [10,43], \\
{[12]_S} &=& I\cap H \ = \ \{11,12,13,22,23\}.
\end{array}
$$

$$
\underbrace{(\underbrace{(\underbrace{(\underbrace{11\vee 12}_{=12\in I})\vee 13}_{=13\in I})\vee 22}_{=23\in I})\vee 23}_{=23\in I}
\qquad
\underbrace{(\underbrace{(\underbrace{(\underbrace{11\vee 12}_{=12\in H})\vee 13}_{=13\in H})\vee 22}_{=23\in H})\vee 23}_{=23\in H}
$$

$$\bigvee [12]_S = \bigvee\{11,12,13,22,23\} = 11\vee 12\vee 13\vee 22\vee 23 \in I\cap H$$

$$11 \le \bigvee [12]_S,\ 12 \le \bigvee [12]_S,\ \ldots,\ 23 \le \bigvee [12]_S$$

We have $[12]_S = I \cap H$, and $\bigvee [12]_S$ belongs to $I \cap H$ and is greter-or-equal than all elements of $I \cap H$, so $\bigvee [12]_S$ is the maximal element of $[12]_S$.

Here is how we can do that in the general case. Let $S = (L, R)$ be a slashing on a ZHA H. Let P be a point of H. The equivalence class $[P]_S$ is a finite set $\{P_1, \ldots, P_n\}$, and we know that $[P]_S = H \cap I$ for some interval I. Look at the elements P_1, $P_1 \vee P_2$, $(P_1 \vee P_2) \vee P_3$, ..., $((P_1 \vee P_2) \vee \ldots) \vee P_n$ We can see that all of them belong to both H and I, so we conclude that $\bigvee [P]_S = ((P_1 \vee P_2) \vee \ldots) \vee P_n$ belongs to $H \cap I$, and it is easy to see that it is greater-or-equal that all elements in $H \cap I$, so it is the maximal element of $H \cap I$.

A similar argument shows that $\bigwedge [P]_S = ((P_1 \wedge P_2) \wedge \ldots) \wedge P_n$ is the smallest element of $[P]_S$.

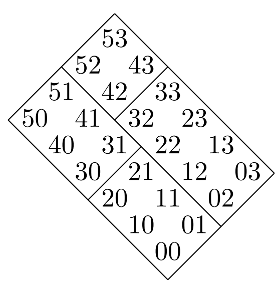
The same argument shows that if C is any non-empty set of the form $I \cap H$, where I is an interval, then $\bigvee C \in C$, $\bigwedge C \in C$, $[\bigwedge C, \bigvee C] \cap H = C$.

Remember that an *interval* in a ZHA H is any set of the form $[P, Q] \cap H$. Let's introduce a new definition: a *closed interval* in a ZHA H is a non-empty set $C \subset H$, with $\bigvee C \in C$, $\bigwedge C \in C$, $[\bigwedge C, \bigvee C] \cap H = C$; informally, a closed interval in a ZHA has a lowest and highest element, and it "is" everything between them.

4 Cuts stopping midway

We saw in the last section that every slash-region is a closed interval. A *partition into closed intervals* of a ZHA H is, as its name says, a partition of H whose equivalence classes are all closed intervals in H.

Some partitions into closed intervals of a ZHA are not slashings — for example, take the partition P with these equivalence classes:



Here is an easy way to prove formally that the partition above does not come from a slashing $S = (L, R)$. We will adapt the idea from sec.2, where we recovered L and R from northwest and northeast steps.

$$\begin{array}{ccccc}
\underbrace{21 \sim_P 31}_{\text{false}} & \leftrightarrow & \underbrace{2 \sim_L 3}_{=(} & \leftrightarrow & \underbrace{22 \sim_P 32}_{\text{true}} \\
\underbrace{31 \sim_P 41}_{\text{true}} & \leftrightarrow & \underbrace{3 \sim_L 4}_{=(} & \leftrightarrow & \underbrace{32 \sim_P 42}_{\text{false}}
\end{array}$$

The problem is that the figure above has “cuts stopping midway”... if its cuts all crossed the ZHA all the way through, we would have this for L and northeast cuts,

$$\begin{array}{ccccccccc}
0 \sim_L 1 & \leftrightarrow & 00 \sim_P 10 & \leftrightarrow & 01 \sim_P 11 & \leftrightarrow & 02 \sim_P 12 & \leftrightarrow & 03 \sim_P 13 \\
1 \sim_L 2 & \leftrightarrow & 10 \sim_P 20 & \leftrightarrow & 11 \sim_P 21 & \leftrightarrow & 12 \sim_P 22 & \leftrightarrow & 13 \sim_P 23 \\
2 \sim_L 3 & \leftrightarrow & 20 \sim_P 30 & \leftrightarrow & 21 \sim_P 31 & \leftrightarrow & 22 \sim_P 32 & \leftrightarrow & 23 \sim_P 33 \\
3 \sim_L 4 & \leftrightarrow & 30 \sim_P 40 & \leftrightarrow & 31 \sim_P 41 & \leftrightarrow & 32 \sim_P 42 & \leftrightarrow & 33 \sim_P 43 \\
4 \sim_L 5 & \leftrightarrow & 40 \sim_P 50 & \leftrightarrow & 41 \sim_P 51 & \leftrightarrow & 42 \sim_P 52 & \leftrightarrow & 43 \sim_P 53 \\
5 \sim_L 6 & \leftrightarrow & 50 \sim_P 60 & \leftrightarrow & 51 \sim_P 61 & \leftrightarrow & 52 \sim_P 62 & \leftrightarrow & 53 \sim_P 63
\end{array}$$

and something similar for R and northwest cuts.

Formally, a partition P on H has an “L-cut between c and c^+ stopping midway” if $cd \sim_P c^+d \not\leftrightarrow cd \sim_P c^+d$ for some d, and it has an “R-cut between d and d^+ stopping midway” if $cd \sim_P cd^+ \not\leftrightarrow c^+d \sim_P c^+d^+$ for some c; here we are writing x^+ for $x+1$.

Theorem: a partition of H into closed intervals is a slash-partition if and only if it doesn’t have any cuts stopping midway. Proof: use the ideas above to recover L and R from $\sim_P$, and then check that $S = (L, R)$ induces an equivalence relation $\sim_S$ that coincides with $\sim_P$.

5 Slash-operators

We can define operations that take each each $P \in H$ to the maximal and to the minimal element of its S-equivalent class, now that we know that these maximal and minimal elements exist:

$$\begin{array}{rcll}
P^S & := & \bigvee [P]_S & \quad \text{(maximal element)}, \\
P^{\mathrm{co}S} & := & \bigwedge [P]_S & \quad \text{(minimal element)}.
\end{array}$$

Note that $[P]_S = [P^{\mathrm{co}S}, P^S] \cap H$.

We will use the operation $\cdot^S$ a lot and $\cdot^{\mathrm{co}S}$ very little. The ‘co’ in ‘co$S$’ means that $\cdot^{\mathrm{co}S}$ is dual to $\cdot^S$, in a sense that will be made precise later.

A *slash-operator* on a ZHA H is a function $\cdot^S : H \to H$ induced by a slashing $S = (L, R)$ on H. It is easy to see that $P \le P^S$ (“ $\cdot^S$ is non-decreasing”) and that $P^S = (P^S)^S$ (“ $\cdot^S$ is idempotent”).

Any idempotent function $\cdot^F : H \to H$ induces an equivalence relation on H: $P \sim_F Q$ iff $P^F = Q^F$. We can use that to test if a given $\cdot^F : H \to H$ is a slash-operator: $\cdot^F$ is a slash-operator iff it obeys all this:

1) $\cdot^F$ is idempotent,
2) $\cdot^F$ is non-decreasing,
3) $\sim_F$ partitions H into closed intervals,
4) $\sim_F$ doesn’t have cuts stopping midway.

6 Slash-operators: a property

Slash-operators obey a certain property that will be very important later. Let's state that property in five equivalent ways:

1) If $cd \sim_S c'd'$ and $ef \sim_S e'f'$ then $cd \land ef \sim_S c'd' \land e'f'$.
2) If $P \sim_S P'$ and $Q \sim_S Q'$ then $P \land Q \sim_S P' \land Q'$.
3) If $P \sim_S P'$ and $Q \sim_S Q'$ then $(P \land Q)^S = (P' \land Q')^S$.
4) If $P \sim_S P'$ and $Q \sim_S Q'$ then

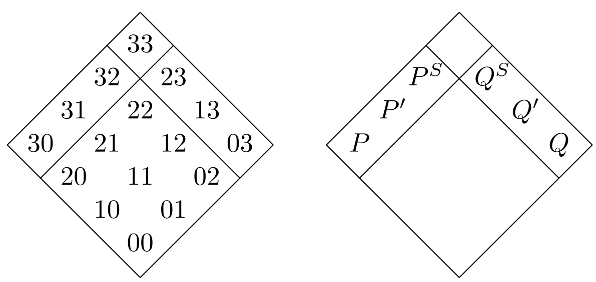
$$\begin{array}{rcll} (P \land Q)^S & = & (P^S \land Q^S)^S & \text{(a)} \\ & = & ((P')^S \land (Q')^S)^S & \text{(b)} \\ & = & (P' \land Q')^S & \text{(c)} \end{array}$$

5) $(P \land Q)^S = (P^S \land Q^S)^S$.

Here's a proof of $1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5$.

$1 \leftrightarrow 2$: we just changed notation,
$2 \leftrightarrow 3$: because $A \sim_S B$ iff $A^S = B^S$,
$3 \to 5$: make the substitution $\begin{bmatrix} P' := P^S \\ Q' := Q^S \end{bmatrix}$ in 3,
$5 \to 4$: 4a is just a copy of 5, and 4c is a copy of 5 with $\begin{bmatrix} P := P' \\ Q := Q' \end{bmatrix}$. For 4b, note that $P \sim_P P'$ implies $P^S = (P')^S$ and $Q \sim_P Q'$ implies $Q^S = (Q')^S$,
$4 \to 3$: 4 is an equality between more expressions than 3,

...and here is a way to visualize what is going on:



$$\underbrace{(\underbrace{\underbrace{P}_{30} \land \underbrace{Q}_{03}}_{00})^S}_{22} = \underbrace{(\underbrace{\underbrace{\underbrace{P}_{30}{}^S}_{32} \land \underbrace{\underbrace{Q}_{03}{}^S}_{23}}_{22})^S}_{22} = \underbrace{(\underbrace{\underbrace{\underbrace{P'}_{31}{}^S}_{32} \land \underbrace{\underbrace{Q'}_{13}{}^S}_{23}}_{22})^S}_{22} = \underbrace{(\underbrace{\underbrace{P'}_{31} \land \underbrace{Q'}_{13}}_{11})^S}_{22}$$

Note that all subexpressions belong to three S-regions: a region with P, P', $P^S = P'^S$, another with Q, Q', $Q^S = Q'^S$, and one with all the '$\land$'s. If we had cuts stopping midway then some of the '$\land$'s could be in different regions.

I think that the clearest way to show (1) is by putting its proof in tree form:

$$\dfrac{\dfrac{\dfrac{cd \sim_S c'd'}{c \sim_L c'} \quad \dfrac{ef \sim_S e'f'}{e \sim_L e'}}{\min(c,e) \sim_L \min(c',e')} \quad \dfrac{\dfrac{cd \sim_S c'd'}{d \sim_R d'} \quad \dfrac{ef \sim_S e'f'}{f \sim_R f'}}{\min(d,f) \sim_L \min(d',f')}}{\dfrac{\min(c,e)\min(d,f) \sim_S \min(c',e')\min(d',f')}{cd \wedge ef \sim_S c'd' \wedge e'f'}}$$

7 J-operators and J-regions

A *J-operator* on a Heyting Algebra $H = (\Omega, \leq, \top, \bot, \wedge, \vee, \to, \leftrightarrow, \neg)$ is a function $J : \Omega \to \Omega$ that obeys the axioms J1, J2, J3 below; we usually write J as $\cdot^* : \Omega \to \Omega$, and write the axioms as rules.

$$\dfrac{}{P \leq P^*}\ \mathsf{J1} \qquad \dfrac{}{P^* = P^{**}}\ \mathsf{J2} \qquad \dfrac{}{(P\&Q)^* = P^*\&Q^*}\ \mathsf{J3}$$

J1 says that the operation $\cdot^*$ is non-decreasing.
J2 says that the operation $\cdot^*$ is idempotent.
J3 is a bit mysterious but will have interesting consequences.

Note that when H is a ZHA then any slash-operator on H is a J-operator on it; see secs.5 and 6.

A J-operator induces an equivalence relation and equivalence classes on Ω, like slashings do:

$$\begin{array}{rcl} P \sim_J Q & \text{iff} & P^* = Q^* \\ [P]^J & := & \{\, Q \in \Omega \mid P^* = Q^* \,\} \end{array}$$

The axioms J1, J2, J3 have many consequences. The first ones are listed in Figure 1 as derived rules, whose names mean:
Mop (monotonicity for products): a lemma used to prove Mo,
Mo (monotonicity): $P \leq Q$ implies $P^* \leq Q^*$,
Sand (sandwiching): all truth values between P and P^* are equivalent,
EC&: equivalence classes are closed by '&',
EC∨: equivalence classes are closed by '∨',
ECS: equivalence classes are closed by sandwiching,

Take a J-equivalence class, $[P]^J$, and list its elements: $[P]^J = \{P_1, \ldots, P_n\}$. Let $P_\wedge := ((P_1 \wedge P_2) \wedge \ldots) \wedge P_n$ and Let $P_\vee := ((P_1 \vee P_2) \vee \ldots) \vee P_n$. It turns out that $[P]^J = [P_\wedge, P_\vee] \cap \Omega$; let's prove that by doing '⊆' first, then '⊇'.
Using EC& and EC∨ several times we see that

$$\begin{array}{rr} P_1 \wedge P_2 \sim_J P & P_1 \vee P_2 \sim_J P \\ (P_1 \wedge P_2) \wedge P_3 \sim_J P & (P_1 \vee P_2) \vee P_3 \sim_J P \\ \vdots & \vdots \\ ((P_1 \wedge P_2) \wedge \ldots) \wedge P_n \sim_J P & ((P_1 \vee P_2) \vee \ldots) \vee P_n \sim_J P \end{array}$$

$$\dfrac{}{(P\&Q)^* \leq Q^*}\,\mathsf{Mop} \quad := \quad \dfrac{\dfrac{}{(P\&Q)^* = P^*\&Q^*}\,\mathsf{J3} \quad \dfrac{}{P^*\&Q^* \leq Q^*}}{(P\&Q)^* \leq Q^*}$$

$$\dfrac{P \leq Q}{P^* \leq Q^*}\,\mathsf{Mo} \quad := \quad \dfrac{\dfrac{\dfrac{P \leq Q}{P = P\&Q}}{P^* = (P\&Q)^*} \quad \dfrac{}{(P\&Q)^* \leq Q^*}\,\mathsf{Mop}}{P^* \leq Q^*}$$

$$\dfrac{P \leq Q \leq P^*}{P^* = Q^*}\,\mathsf{Sand} \quad := \quad \dfrac{\dfrac{P \leq Q}{P^* \leq Q^*}\,Mo \quad \dfrac{\dfrac{Q \leq P^*}{Q^* \leq P^{**}}\,\mathsf{Mo} \quad \dfrac{}{P^{**} = P^*}\,\mathsf{J2}}{Q^* \leq P^*}}{P^* = Q^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P\&Q)^*}\,\mathsf{EC\&} \quad := \quad \dfrac{\dfrac{P^* = Q^*}{P^* = Q^* = P^*\&Q^*} \quad \dfrac{}{P^*\&Q^* = (P\&Q)^*}\,\mathsf{J3}}{P^* = Q^* = (P\&Q)^*}$$

$$\dfrac{P^* = Q^*}{P^* = Q^* = (P \vee Q)^*}\,\mathsf{EC\vee} \quad := \quad \dfrac{P^* = Q^* \quad \dfrac{\dfrac{\dfrac{}{P \leq P \vee Q} \quad \dfrac{\dfrac{}{P \leq P^*}\,\mathsf{J1} \quad \dfrac{\dfrac{}{Q \leq Q^*}\,\mathsf{J1} \quad \dfrac{P^* = Q^*}{Q^* = P^*}}{Q \leq P^*}}{P \vee Q \leq P^*}}{P \leq P \vee Q \leq P^*}}{P^* = (P \vee Q)^*}\,\mathsf{Sand}}{P^* = Q^* = (P \vee Q)^*}$$

$$\dfrac{P \leq Q \leq R \quad P^* = R^*}{P^* = Q^* = R^*}\,\mathsf{ECS} \quad := \quad \dfrac{\dfrac{\dfrac{P \leq Q \leq R \quad \dfrac{}{R \leq R^*}\,\mathsf{J1} \quad \dfrac{P^* = R^*}{R^* = P^*}}{P \leq Q \leq P^*}}{P^* = Q^*}\,\mathsf{Sand} \quad P^* = R^*}{P^* = Q^* = R^*}$$

Figure 1: J-operators: basic derived rules

so $P_\wedge \sim_J P_\vee \sim_J P$, and by the sandwich lemma $([P_\wedge, P_\vee] \cap \Omega) \subseteq [P]^J$.

For any $P_i \in [P]^J$ we have $P_\wedge \le P_i \le P_\vee$, which means that:

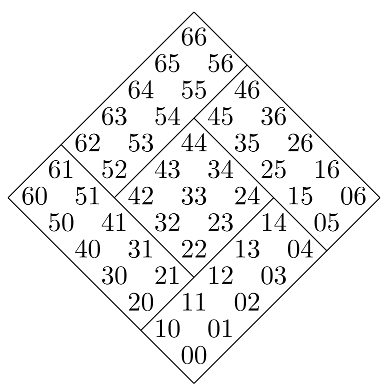
$$\begin{array}{rcl} [P]^J &=& \{P_1, \ldots, P_n\} \\ &\subseteq& \{\, Q \in \Omega \mid P_\wedge \le Q \le P_\vee \,\} \\ &=& [P_\wedge, P_\vee] \cap \Omega, \end{array}$$

so $[P]^J \subseteq [P_\wedge, P_\vee] \cap \Omega$.

As the operation '$\cdot^*$' is increasing and idempotent, each equivalence class $[P]^J$ has exactly one maximal element, which is P^*; but $P_\vee$ is also the maximal element of $[P]^J$, so $P_\vee = P^*$, and we can interpret the operation '$\cdot*$' as "take each P to the top element in its equivalence class", which is similar to how we defined an(other) operation '$\cdot^*$' on slashings in the previous section.

The operation "take each P to the bottom element in its equivalence class" will be useful in a few occasions; we will call it '$\cdot^{\mathrm{co}*}$' to indicate that it is dual to '$\cdot^*$' in some sense. Note that $P^{\mathrm{co}*} = P_\wedge$.

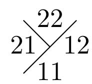
Look at the figure below, that shows a partition of a ZHA $A = [00, 66]$ into five regions, each region being an interval; this partition does not come from a slashing, as it has cuts that stop midway. Define an operation '$\cdot^*$' on A, that works by taking each truth-value P in it to the top element of its region; for example, $30^* = 61$.



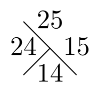
It is easy to see that '$\cdot^*$' obeys J1 and J2; however, it does *not* obey J3 — we will prove that in sec.8. As we will see, *the partitons of a ZHA into intervals that obey* J1, J2, J3 *ae exactly the slashings;* or, in other words, *every J-operator comes from a slashing*.

8 The are no Y-cuts and no λ-cuts

We want to see that if a partition of a ZHA H into intervals has "Y-cuts" or "λ-cuts" like these parts of the last diagram in the last section,



$\Leftarrow$ this is a Y-cut



$\Leftarrow$ this is a λ-cut

then it operation J that takes each element to the top of its equivalence class cannot obey J1, J2 and J3 at the same time. We will prove that by deriving rules that say that if $11 \sim_J 12$ then $21 \sim_J 22$, and that if $15 \sim_J 25$ then $14 \sim_J 24$; actually, our rules will say that if $11^* = 12^*$ then $(11 \vee 21)^* = (12 \vee 21)^*$, and that if $15^* = 25^*$ then $(15 \wedge 24)^* = (25 \wedge 24)^*$. The rules are:

$$\dfrac{P^* = Q^*}{(P \vee R)^* = (Q \vee R)^*}\ \textsf{NoYcuts} \quad := \quad \dfrac{\dfrac{\dfrac{P^* = Q^*}{P \vee R^* = Q \vee R^*}}{(P \vee R^*)^* = (Q \vee R^*)^*}}{\overline{(P \vee R)^* = (Q \vee R)^*}}\ \vee^*\textsf{Cube}$$

$$\dfrac{P^* = Q^*}{(P \& R)^* = (Q \& R)^*}\ \textsf{No}\lambda\textsf{cuts} \quad := \quad \dfrac{\dfrac{P^* = Q^*}{P^* \& R^* = Q^* \& R^*}}{(P \& R)^* = (Q \& R)^*}\ \textsf{J3}$$

The top derivation mentions a rule called '$\vee^*$Cube', which will be defined and proved in sec.10.

9 How J-operators interact with connectives: the obvious cubes

It is easy to prove each one of the arrows below ($A \longrightarrow B$ means $A \leq B$):

$$\begin{tikzcd}[column sep=tiny]
& & (P^*\&Q^*)^* & & \\
(P^*\&Q)^* \arrow[urr] & & P^*\&Q^* \arrow[u] & & (P\&Q^*)^* \arrow[ull] \\
P^*\&Q \arrow[u] \arrow[urr] & & (P\&Q)^* \arrow[ull] \arrow[urr] & & P\&Q^* \arrow[ull] \arrow[u] \\
& & P\&Q \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & (P^*{\vee}Q^*)^* & & \\
(P^*{\vee}Q)^* \arrow[urr] & & P^*{\vee}Q^* \arrow[u] & & (P{\vee}Q^*)^* \arrow[ull] \\
P^*{\vee}Q \arrow[u] \arrow[urr] & & (P{\vee}Q)^* \arrow[ull] \arrow[urr] & & P{\vee}Q^* \arrow[ull] \arrow[u] \\
& & P{\vee}Q \arrow[ull] \arrow[u] \arrow[urr] & &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=tiny]
& & (P^*{\to}Q^*)^* \arrow[drr] & & \\
(P^*{\to}Q)^* \arrow[urr] \arrow[drr] & & P^*{\to}Q^* \arrow[u] \arrow[drr] & & (P{\to}Q^*)^* \\
P^*{\to}Q \arrow[u] \arrow[urr] \arrow[drr] & & (P{\to}Q)^* \arrow[urr] & & P{\to}Q^* \arrow[u] \\
& & P{\to}Q \arrow[u] \arrow[urr] & &
\end{tikzcd}$$

The cubes above will be called the "obvious and-cube", the "obvious or-cube", and the "obvious implication-cube", and they show partial orders between expressions of the form $(P^? \odot Q^?)^?$, where the '$\odot$' stands for one of the connectives '$\land$', '$\lor$' or '$\to$', and each '?' marks a place where we can put either a '*' or nothing; let's be more precise.

The "cube of $\land$-expressions", $\mathsf{ECube}_\land$, is the set of eight expressions of the form $(P^? \land Q^?)^?$; $\mathsf{ECube}_\lor$ is the set of eight expressions of the form $(P^? \lor Q^?)^?$, and $\mathsf{ECube}_\to$ the set of eight expressions of the form $(P^? \to Q^?)^?$.

The "obvious $\land$-cube", $\mathsf{OCube}_\land$, is the directed graph shown above, with 12 arrows between elements of $\mathsf{ECube}_\land$. Its transitive closure, $\mathsf{OCube}^*_\land$, is a partial order on $\mathsf{ECube}_\land$. We define $\mathsf{OCube}_\lor$, $\mathsf{OCube}^*_\lor$, $\mathsf{OCube}_\to$, and $\mathsf{OCube}^*_\to$ similarly.

If we establish that the three '?'s in $(P^? \odot Q^?)^?$ are "worth" 1, 2 and 4 respectively, we get a way to number the elements in $\mathsf{ECube}_\land$ from 0 to 7. We define $(\land)_0, \ldots, (\land)_7$ as:

$$\begin{array}{rcl@{\qquad}rcl}
(\land)_0 &=& P \land Q, & (\land)_4 &=& (P \land Q)^*, \\
(\land)_1 &=& P^* \land Q, & (\land)_{1+4} &=& (P^* \land Q)^*, \\
(\land)_2 &=& P \land Q^*, & (\land)_{2+4} &=& (P \land Q^*)^*, \\
(\land)_{1+2} &=& P^* \land Q^*, & (\land)_{1+2+4} &=& (P^* \land Q^*)^*,
\end{array}$$

and we do the same for $(\lor)_0, \ldots, (\lor)_7, (\to)_0, \ldots, (\to)_7$. We always draw the '$(\odot)_i$'s in this position:

$$\begin{array}{ccc}
& (\odot)_7 & \\
(\odot)_5 & (\odot)_3 & (\odot)_6 \\
(\odot)_1 & (\odot)_4 & (\odot)_2 \\
& (\odot)_0 &
\end{array}
\qquad\qquad
\begin{array}{ccc}
& 7 & \\
5 & 3 & 6 \\
1 & 4 & 2 \\
& 0 &
\end{array}$$

With this numbering we can reinterpret the cubes as subsets of $\{0, \ldots, 7\}^2$; $\{0, \ldots, 7\}^2$ is a ZSet, and so we can use the positional notation and interpret each cube as a grid of '0's and '1's. For example,

$$\begin{tikzcd}[sep=small]
& 7 & \\
5 \arrow[ur] & 3 \arrow[u] & 6 \arrow[ul] \\
1 \arrow[u] \arrow[ur] & 4 \arrow[ul] \arrow[ur] & 2 \arrow[u] \arrow[ul] \\
& 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
=
\left\{\begin{array}{c}
(0,1), (2,3), (4,5), (6,7), \\
(0,2), (1,3), (4,6), (5,7), \\
(0,4), (1,5), (2,6), (3,7)
\end{array}\right\}
=
\begin{array}{c}
0\,0\,0\,1\,0\,1\,1\,0 \\
0\,0\,1\,0\,1\,0\,0\,0 \\
0\,1\,0\,0\,1\,0\,0\,0 \\
1\,0\,0\,0\,0\,0\,0\,0 \\
0\,1\,1\,0\,0\,0\,0\,0 \\
1\,0\,0\,0\,0\,0\,0\,0 \\
1\,0\,0\,0\,0\,0\,0\,0 \\
0\,0\,0\,0\,0\,0\,0\,0
\end{array}$$

The transitive-reflexive closure of a cube yields a different grid:

$$\left(\begin{tikzcd}[sep=small]
& 7 & \\
5 \arrow[ur] & 3 \arrow[u] & 6 \arrow[ul] \\
1 \arrow[u] \arrow[ur] & 4 \arrow[ul] \arrow[ur] & 2 \arrow[u] \arrow[ul] \\
& 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}\right)^*
=
\begin{array}{c}
1\,1\,1\,1\,1\,1\,1\,1 \\
1\,0\,1\,0\,1\,0\,1\,0 \\
1\,0\,0\,0\,1\,1\,0\,0 \\
1\,1\,0\,0\,1\,0\,0\,0 \\
1\,1\,1\,1\,0\,0\,0\,0 \\
1\,0\,1\,0\,0\,0\,0\,0 \\
1\,1\,0\,0\,0\,0\,0\,0 \\
1\,0\,0\,0\,0\,0\,0\,0
\end{array}$$

Note that the grids for $\mathsf{OCube}_\land$ and $\mathsf{OCube}_\lor$ are equal, but the grid for $\mathsf{OCube}_\to$ is different. Also, note that $\mathsf{OCube}_\land$, $\mathsf{OCube}^*_\land$, etc, are directed graphs; sometimes we will need to regard them as pairs, and we will use a lowercase notation for their sets of arrows: $\mathsf{OCube}_\land = (\mathsf{ECube}_\land, \mathsf{ocube}_\land)$, $\mathsf{OCube}^*_\to = (\mathsf{ECube}_\to, \mathsf{ocube}^*_\to)$, etc.

10 How J-operators interact with connectives: the full cubes

We can prove these new derived rules,

$$\dfrac{}{\overline{(P^*\&Q^*)^* = P^*\&Q^* = (P\&Q)^*}}\&^*\mathsf{C}_0 \quad := \quad \dfrac{\dfrac{\dfrac{}{P^{**}=P^*}\mathsf{J2}\quad \dfrac{}{Q^{**}=Q^*}\mathsf{J2}}{\overline{(P^*\&Q^*)^* = P^{**}\&Q^{**} = P^*\&Q^* = (P\&Q)^*}}\mathsf{J3}}{(P^*\&Q^*)^* = P^*\&Q^* = (P\&Q)^*}$$

$$\dfrac{}{\overline{(P^*\lor Q^*)^* \le (P\lor Q)^*}}\lor^*\mathsf{C}_0 \quad := \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P\le P\lor Q}}{P^*\le (P\lor Q)^*}\mathsf{Mo}\quad \dfrac{\dfrac{}{Q\le P\lor Q}}{Q^*\le (P\lor Q)^*}\mathsf{Mo}}{P^*\lor Q^*\le (P\lor Q)^*}}{(P^*\lor Q^*)^*\le (P\lor Q)^{**}}\mathsf{Mo}}{(P^*\lor Q^*)^*\le (P\lor Q)^*}\mathsf{J2}$$

$$\dfrac{}{\overline{(P\to Q^*)^* \le P^*\to Q^*}}\to^*\mathsf{C}_0 \quad := \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P\to Q^*\le P\to Q^*}}{(P\to Q^*)\&P\le Q^*}}{((P\to Q^*)\&P)^*\le Q^{**}}\mathsf{Mo}}{((P\to Q^*)\&P)^*\le Q^*}\mathsf{J2}}{(P\to Q^*)^*\&P^*\le Q^*}\mathsf{J3}}{(P\to Q^*)^*\le P^*\to Q^*}$$

and interpret them as extra arrows on the cubes. The "full $\land$-cube", $\mathsf{FCube}_\land$, is $\mathsf{OCube}_\land$ plus these arrows:

$$(P^*\land Q^*)^* \leftrightarrows P^*\land Q^* \leftrightarrows (P\land Q)^*$$

The "full $\lor$-cube", $\mathsf{FCube}_\lor$, is $\mathsf{OCube}_\lor$ plus this,

$$(P^*\lor Q^*)^* \longrightarrow (P\lor Q)^*$$

and the "full $\to$-cube", $\mathsf{FCube}_\to$, is $\mathsf{OCube}_\to$ plus this,

$$(P\to Q^*)^* \longrightarrow (P^*\to Q^*)$$

We are interested in the transitive-reflexive closures of these full cubes. $\mathsf{FCube}^*_\land$ yields a *non-strict* partial order on $\mathsf{ECube}_\land$ that identifies five of its elements, and $\mathsf{FCube}^*_\lor$ and $\mathsf{FCube}^*_\to$ yield non-strict partial orders that identify four elements each. My favorite way to represent these non-strict partial orders is by the diagrams at the right below, that show very clearly which elements are identified:

$$\left(\begin{tikzcd}
& (\wedge)_7 \arrow[d, shift left] & \\
(\wedge)_5 \arrow[ur] & (\wedge)_3 \arrow[u, shift left] \arrow[d, shift left] & (\wedge)_6 \arrow[ul] \\
(\wedge)_1 \arrow[u] \arrow[ur] & (\wedge)_4 \arrow[ul] \arrow[u, shift left] \arrow[ur] & (\wedge)_2 \arrow[ul] \arrow[u] \\
& (\wedge)_0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}\right)^* = \left(\begin{tikzcd}
& (\wedge)_7 \arrow[dl, equal] \arrow[d, equal] \arrow[dr, equal] & \\
(\wedge)_5 & (\wedge)_3 & (\wedge)_6 \\
(\wedge)_1 \arrow[u] \arrow[ur] & (\wedge)_4 \arrow[ul, equal] \arrow[ur, equal] & (\wedge)_2 \arrow[ul] \arrow[u] \\
& (\wedge)_0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}\right)^*$$

$$\left(\begin{tikzcd}
& (\vee)_7 \arrow[dd, bend left] & \\
(\vee)_5 \arrow[ur] & (\vee)_3 \arrow[u] & (\vee)_6 \arrow[ul] \\
(\vee)_1 \arrow[u] \arrow[ur] & (\vee)_4 \arrow[ul] \arrow[ur] & (\vee)_2 \arrow[ul] \arrow[u] \\
& (\vee)_0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}\right)^* = \left(\begin{tikzcd}
& (\vee)_7 \arrow[dl, equal] \arrow[dr, equal] & \\
(\vee)_5 & (\vee)_3 \arrow[u] & (\vee)_6 \\
(\vee)_1 \arrow[u] \arrow[ur] & (\vee)_4 \arrow[ul, equal] \arrow[ur, equal] & (\vee)_2 \arrow[ul] \arrow[u] \\
& (\vee)_0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}\right)^*$$

$$\left(\begin{tikzcd}
& (\to)_7 \arrow[dr] & \\
(\to)_5 \arrow[ur] \arrow[dr] & (\to)_3 \arrow[u] \arrow[dr] & (\to)_6 \arrow[l] \\
(\to)_1 \arrow[u] \arrow[ur] \arrow[dr] & (\to)_4 \arrow[ur] & (\to)_2 \arrow[u] \\
& (\to)_0 \arrow[u] \arrow[ur] &
\end{tikzcd}\right)^* = \left(\begin{tikzcd}
& (\to)_7 \arrow[d, equal] \arrow[dr, equal] & \\
(\to)_5 \arrow[ur] \arrow[dr] & (\to)_3 \arrow[dr, equal] & (\to)_6 \\
(\to)_1 \arrow[u] \arrow[ur] \arrow[dr] & (\to)_4 \arrow[ur] & (\to)_2 \arrow[u, equal] \\
& (\to)_0 \arrow[u] \arrow[ur] &
\end{tikzcd}\right)^*$$

When the arrow $(\wedge)_i \longrightarrow (\wedge)_j$ belongs to $\mathsf{FCube}^*_\wedge$ we say that $(\wedge)_i \leq (\wedge)_j$ is true "by the full and-cube". We write this as a derived rule as

$$\frac{}{(\wedge)_i \leq (\wedge)_j}\ \&^*\mathsf{Cube}_{ij} \qquad \text{or just as:} \qquad \frac{}{(\wedge)_i \leq (\wedge)_j}\ \&^*\mathsf{Cube}\ ,$$

and when the arrows $(\wedge)_i \rightleftarrows (\wedge)_j$ belongs to $\mathsf{FCube}^*_\wedge$ we say that $(\wedge)_i = (\wedge)_j$ is true "by the full and-cube", and we write that as:

$$\frac{}{(\wedge)_i = (\wedge)_j}\ \&^*\mathsf{Cube}_{ij} \qquad \text{or just as:} \qquad \frac{}{(\wedge)_i = (\wedge)_j}\ \&^*\mathsf{Cube}\ ,$$

and we do the same for '$\vee$' and '$\to$'.

The double-bar rule in sec.8 is a contraction of:

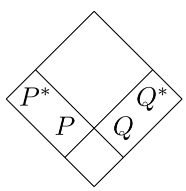
$$\dfrac{(P \vee Q^*)^* = (P \vee R^*)^* \quad \dfrac{}{(P \vee R^*)^* = (P \vee R)^*}\ \vee^*\mathsf{Cube}_{64}}{(P \vee Q)^* = (P \vee R)^*}$$

11 How J-operators interact with connectives: ZHA*-valuations

Let's write $\mathsf{Exprs}(\mathsf{V})$ for the set of well-formed expressions built from a set of variables V, constants $\top$ and $\bot$, and operations $\land$, $\lor$, $\to$, $\leftrightarrow$, $\neg$, $\cdot^*$; each one of the sets $\mathsf{ECube}_\land$, $\mathsf{ECube}_\lor$ and $\mathsf{ECube}_\to$ of the last sections is an 8-element subset of $\mathsf{Exprs}(\{P, Q\})$.

If $\mathsf{E} \subseteq \mathsf{Exprs}(\mathsf{V})$, a *ZHA*-valuation for* E, or an E*-valuation*, is a triple (H, J, v), where H is a ZHA, J is a J-operator on H, and $v : \mathsf{V} \to H$ is a function that assigns a truth-value in H to each variable in V. There is a natural way to extend v to a function $v' : \mathsf{Exprs}(\mathsf{V}) \to H$, and we can restrict v' to a function $v'' : \mathsf{E} \to H$.

We can draw all components of an $\mathsf{ECube}_\lor$-valuation (H, J, v) together by writing 'P' and 'Q' on the positions $v(P)$ and $v(Q)$ on (H, J), as we did in sec.6. We will often also write 'P^*' and 'Q^*' on the positions $v'(P^*)$ and $v'(Q^*)$ for clarity. For example:



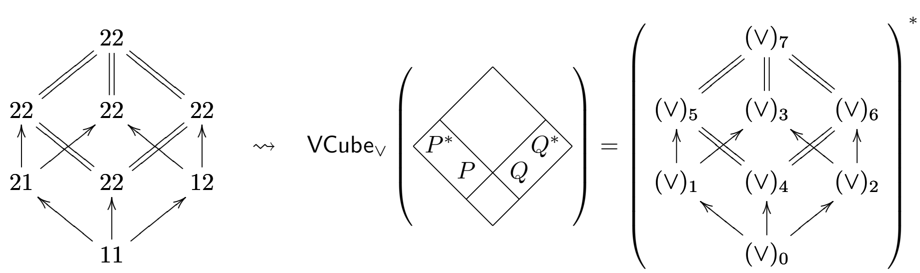
$$\begin{array}{rcl} v(P) &=& 10 \\ v(Q) &=& 01 \\ v'(P^*) &=& 20 \\ v'(Q^*) &=& 02 \end{array} \qquad \begin{array}{rcccl} v'(P \lor Q) &=& 11 &=& v''((\lor)_0) \\ v'(P^* \lor Q) &=& 21 &=& v''((\lor)_1) \\ v'(P \lor Q^*) &=& 12 &=& v''((\lor)_2) \\ v'(P^* \lor Q^*) &=& 22 &=& v''((\lor)_3) \\ v'((P \lor Q)^*) &=& 22 &=& v''((\lor)_4) \\ v'((P^* \lor Q)^*) &=& 22 &=& v''((\lor)_5) \\ v'((P \lor Q^*)^*) &=& 22 &=& v''((\lor)_6) \\ v'((P^* \lor Q^*)^*) &=& 22 &=& v''((\lor)_7) \end{array}$$

Each $\mathsf{ECube}_\lor$-valuation (H, J, v) induces a non-strict partial order on $\mathsf{ECube}_\lor$, in which $(\lor)_i \leq (\lor)_j$ iff $v''((\lor)_i) \leq v''((\lor)_j)$. We will write that partial order as

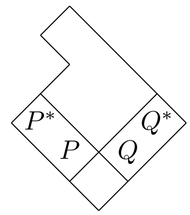
$$\begin{array}{rcll} \mathsf{VCube}_\lor(H, J, v) &=& (\mathsf{ECube}_\lor, \mathsf{vcube}_\lor(H, J, v)) & \qquad \text{or:} \\ \mathsf{VCube}_\lor(v) &=& (\mathsf{ECube}_\lor, \mathsf{vcube}_\lor(v)) & \end{array}$$

We will often omit the 'H' and the 'J' and write just $\mathsf{VCube}_\lor(v)$.

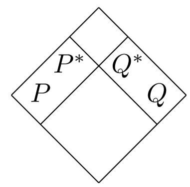
It is easy to calculate by hand the partial orders $\mathsf{VCube}_\lor(v)$, $\mathsf{VCube}_\land(v)$ or $\mathsf{VCube}_\to(v)$ associated to a given valuation (H, J, v): we write in the position corresponding to each '$(\odot)_i$' of the cube the value of the corresponding $v''((\odot)_i)$, then we draw the arrows — some of them will be '='s —, then transfer the arrows to the cube with '$(\odot)_i$'s. For example:



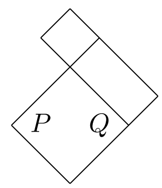
A very important fact. For any i and j,

$(\vee)_i \leq (\vee)_j$ is a theorem iff it is true in 

,

$(\wedge)_i \leq (\wedge)_j$ is a theorem iff it is true in 

,

$(\to)_i \leq (\to)_j$ is a theorem iff it is true in 

.

We will call the valuations at the right above $(H_\vee, J_\vee, v_\vee)$, $(H_\wedge, J_\wedge, v_\wedge)$, $(H_\to, J_\to, v_\to)$. In the language of partial orders, the very important fact can be stated as:

$$\begin{array}{rcl} \mathsf{FCube}^*_\vee &=& \mathsf{VCube}_\vee(v_\vee), \\ \mathsf{FCube}^*_\wedge &=& \mathsf{VCube}_\wedge(v_\wedge), \\ \mathsf{FCube}^*_\to &=& \mathsf{VCube}_\to(v_\to), \end{array}$$

Suppose that (H_1, J_1, v_1), (H_2, J_2, v_2), $\ldots$ are valuations on — say — $\mathsf{ECube}_\to$. This always holds

$$\mathsf{FCube}^*_\to \subseteq \mathsf{VCube}_\to(v_i),$$

because all ZHA*-theorems are true in all valuations. We say that:

$$\begin{array}{rcl} v_i \text{ is } \textit{good} & \text{when} & \mathsf{FCube}^*_\to = \mathsf{VCube}_\to(v_i), \\ v_i \text{ and } v_j \text{ are } \textit{equivalent} & \text{when} & \mathsf{VCube}_\to(v_i) = \mathsf{VCube}_\to(v_j), \\ v_i \text{ is } \textit{better than } v_j & \text{when} & \mathsf{VCube}_\to(v_i) \subseteq \mathsf{VCube}_\to(v_j). \end{array}$$

Also, a *non-theorem* is an arrow $(\to)_i \leq (\to)_j$ that is not in $\mathsf{FCube}^*_\to$; a *countermodel* for a non-theorem $(\to)_i \leq (\to)_j$ is a valuation that "falsifies" $(\to)_i \leq (\to)_j$, i.e., a valuation in which $(\to)_i \leq (\to)_j$ is not true; a valuation is "good" when it is a countermodel for all non-theorems at once; and a valuation v_1 is strictly better than v_2 when v_1 falsifies all non-theorems that v_2 falsifies, plus some.

I found the $\mathsf{ECube}_\to$-good valuation $v_\to$ by a process of trial and error in which I started with a valuation v_1 and tried to make it better step by step, generating valuations v_2, v_3, ... in the process, until I reached a good valuation. Each step consisted of choosing a non-theorem that was not falsified in the current valuation, finding a countermodel for it — often using tableau methods, as in sec.18 of [PH1] — and trying to integrate that countermodel into the current valuation to produce the next one.

In sec.18 of [PH1] we saw that ZHAs do not distinguish as many sentences as arbitrary Heyting Algebras; we saw a sentence $S_P \vee S_Q \vee S_R$ that had a countermodel in a HA, but that ZHAs "think" that its value is always $\top$... for a set $\mathsf{E} \subseteq \mathsf{Exprs}(\mathsf{V})$ of expressions on V we say that E is *ZHA*-good* when there is a single ZHA*-valuation (H, J, v) that falsifies all non-theorems of the form $E_i \leq E_j$ where $E_i, E_j \in \mathsf{E}$. By a slight abuse of language, we will say that the valuation (H, J, v) "distinguishes all elements of E", or "distinguishes E", instead of the more precise "is a countermodel for all non-theorems of the form $E_i \leq E_j$ at once". We have:

$$\begin{array}{rl} \{S_P \vee S_Q \vee S_R, \top\} & \text{is not ZHA}^*\text{-good} \\ \mathsf{ECube}_\vee & \text{is ZHA}^*\text{-good} \\ \mathsf{ECube}_\wedge & \text{is ZHA}^*\text{-good} \\ \mathsf{ECube}_\to & \text{is ZHA}^*\text{-good} \end{array}$$

ZHAs with J-operators do not distinguish all sets of expressions, but they distinguish some sets, like $\mathsf{ECube}_\vee$, $\mathsf{ECube}_\wedge$, $\mathsf{ECube}_\to$, that are very useful.

12 Good valuations

If $(\vee)_i \leq (\vee)_j$ is true in $\mathsf{FCube}^*_\vee$ then it is a theorem, and it holds in every $\mathsf{ECube}_\vee$-valuation (H, J, v) — so $\mathsf{FCube}^*_\vee \subseteq \mathsf{VCube}_\vee(H, J, v)$. The important information that a ZHA*-valuation carries is in its '$\not\leq$'s, as they say that something *cannot* be a theorem and that (H, J, v) is a countermodel showing that. For example, in $(H_\vee, J_\vee, v_\vee)$ we had $(\vee)_7 \not\leq (\vee)_3$; if we could prove, using new derived rules like the ones in sec.10, that $(\vee)_7 \leq (\vee)_3$ is a theorem, then we would have $(\vee)_7 \leq (\vee)_3$ in all valuations, which is incompatible with the $(\vee)_7 \not\leq (\vee)_3$ in $\mathsf{VCube}_\vee(H_\vee, J_\vee, v_\vee)$.

Note that this means that: 1) that if a statement of the form $(\vee)_i \leq (\vee)_j$ is not in $\mathsf{FCube}^*_\vee$ then it cannot be proved, i.e., all attempts to find a tree-proof for that $(\vee)_i \leq (\vee)_j$ using the HA rules and J1, J2, J3 are bound to fail; 2) the theorems of the form $(\vee)_i \leq (\vee)_j$ are exactly the ones that are true in $\mathsf{VCube}_\vee(H_\vee, J_\vee, v_\vee)$, so we can use $(H_\vee, J_\vee, v_\vee)$ as a *reminder* for which sentences of the form $(\vee)_i \leq (\vee)_j$ are theorems — and the same for '$\wedge$' and '$\to$'.

13 Polynomial J-operators

It is not hard to check that for any Heyting Algebra H and any $Q, R \in H$ the operations $(\neg\neg)$, ..., $(\vee Q \wedge {\to} R)$ below are J-operators:

$$\begin{array}{rcl} (\neg\neg)(P) & = & \neg\neg P \\ ({\to}{\to}R)(P) & = & (P{\to}R){\to}R \\ (\vee Q)(P) & = & P \vee Q \\ ({\to}R)(P) & = & P{\to}R \\ (\vee Q \wedge {\to}R)(P) & = & (P{\vee}Q) \wedge (P{\to}R) \end{array}$$

Checking that they are J-operators means checking that each of them obeys J1, J2, J3. Let's define formally what are J1, J2 and J3 "for a given $F : H \to H$":

$$\begin{array}{rcc} \mathsf{J1}_F & := & (P \le F(P)) \\ \mathsf{J2}_F & := & (F(P) = F(F(P)) \\ \mathsf{J3}_F & := & (F(P \wedge P') = F(P) \wedge F(P')) \end{array}$$

and:

$$\mathsf{J123}_F \quad := \quad \mathsf{J1}_F \wedge \mathsf{J2}_F \wedge \mathsf{J3}_F.$$

Checking that $(\neg\neg)$ obeys J1, J2, J3 means proving $\mathsf{J123}_{(\neg\neg)}$ using only the rules from intuitionist logic from sec.**??**; we will leave the proof of this, of and $\mathsf{J123}_{({\to}{\to}R)}$, $\mathsf{J123}_{(\vee Q)}$, and so on, to the reader.

The J-operator $(\vee Q \wedge {\to}R)$ is a particular case of building more complex J-operators from simpler ones. If $J, K : H \to H$, we define:

$$(J \wedge K) := \lambda P{:}H.(J(P){\wedge}K(P))$$

it not hard to prove $\mathsf{J123}_{(J \wedge K)}$ from $\mathsf{J123}_J$ and $\mathsf{J123}_K$ using only the rules from intuitionistic logic.

The J-operators above are the first examples of J-operators in Fourman and Scott's "Sheaves and Logic" ([**?**]); they appear in pages 329–331, but with these names (our notation for them is at the right):

(i) *The closed quotient,*

$$J_a p = a \vee p \qquad J_Q = (\vee Q).$$

(ii) *The open quotient,*

$$J^a p = a \to p \qquad J^R = ({\to}R).$$

(iii) *The Boolean quotient.*

$$B_a p = (p \to a) \to a \qquad B_R = ({\to}{\to}R).$$

(iv) *The forcing quotient.*

$$(J_a \wedge J^b)p = (a \vee p) \wedge (b \to p) \qquad (J_Q \wedge J^R) = (\vee Q \wedge {\to}R).$$

(vi) *A mixed quotient.*

$$(B_a \wedge J^a)p = (p \to a) \to p \qquad (B_Q \wedge J^Q) = ({\to}{\to}Q \wedge {\to}Q).$$

The last one is tricky. From the definition of B_a and J^a what we have is

$$(B_a \wedge J^a)p = ((p \to a) \to a) \wedge (a \to p),$$

but it is possible to prove

$$((p \to a) \to a) \wedge (a \to p) \quad \leftrightarrow \quad ((p \to a) \to p)$$

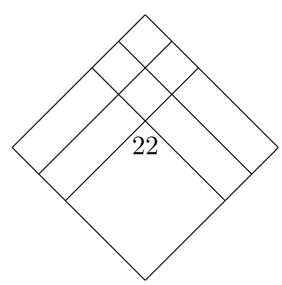
intuitionistically.

The operators above are "polynomials on $P, Q, R, \to, \wedge, \vee, \bot$" in the terminology of Fourman/Scott: "If we take a polynomial in $\to, \wedge, \vee, \bot$, say, $f(p, a, b, \ldots)$, it is a decidable question whether for all $a, b, \ldots$ it defines a J-operator" (p.331).

When I started studying sheaves I spent several years without any visual intuition about the J-operators above. I was saved by ZHAs and brute force — and the brute force method also helps in testing if a polynomial (in the sense above) is a J-operator in a particular case. For example, take the operators $\lambda P{:}H.(P \wedge 22)$ and $(\vee 22)$ on $H = [00, 44]$:

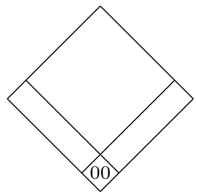
$$\lambda P{:}H.(P \wedge 22) \quad = \quad \begin{array}{ccccccccc}
 & & & & 22 & & & & \\
 & & & 22 & & 22 & & & \\
 & & 22 & & 22 & & 22 & & \\
 & 21 & & 22 & & 22 & & 12 & \\
20 & & 21 & & 22 & & 12 & & 02 \\
 & 20 & & 21 & & 12 & & 02 & \\
 & & 20 & & 11 & & 02 & & \\
 & & & 10 & & 01 & & & \\
 & & & & 00 & & & &
\end{array}$$

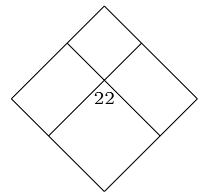
$$(\vee 22) \quad = \quad \begin{array}{ccccccccc}
 & & & & 44 & & & & \\
 & & & 43 & & 34 & & & \\
 & & 42 & & 33 & & 24 & & \\
 & 42 & & 32 & & 23 & & 24 & \\
42 & & 32 & & 22 & & 23 & & 24 \\
 & 32 & & 22 & & 22 & & 23 & \\
 & & 22 & & 22 & & 22 & & \\
 & & & 22 & & 22 & & & \\
 & & & & 22 & & & &
\end{array} \quad =$$

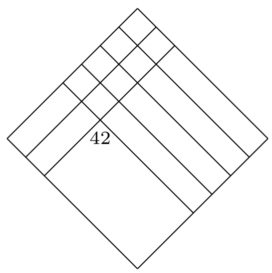


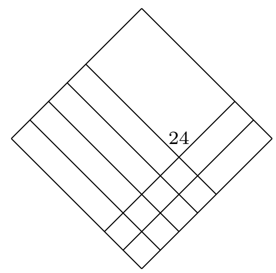
The first one, $\lambda P{:}H.(P \wedge 22)$, is not a J-operator; one easy way to see that is to look at the region in which the result is 22 — its top element is 44, and this violates the conditions on slash-operators in sec.5. The second operator, $(\vee 22)$, is a slash operator and a J-operator; at the right we introduce a convenient notation for visualizing the action of a polynomial slash-operator, in which we draw only the contours of the equivalence classes and the constants that appear in the polynomial.

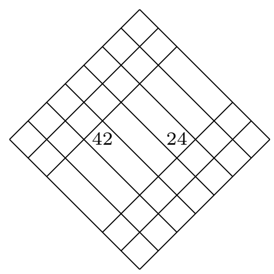
Using this new notation, we have:

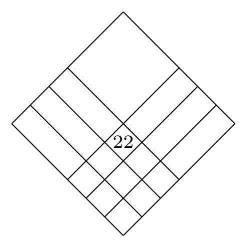
$(\neg\neg) \;=\; (\to\to 00) \;=$ 

$(\to\to 22) \;=$ 

$(\vee 42) \;=$ 

$(\to 24) \;=$ 

$(\vee 42 \wedge \to 24) \;=$ 

$(\to\to 22 \wedge \to 22) \;=$ 

Note that the slashing for $(\vee 42 \wedge \to 24)$ has all the cuts for $(\vee 42)$ plus all the cuts for $(\to 24)$, and $(\vee 42 \wedge \to 24)$ "forces $42 \le 24$" in the following sense: if $P^* = (\vee 42 \wedge \to 24)(P)$ then $42^* \le 24^*$.

14 An algebra of piccs

We saw in the last section a case in which $(J \wedge K)$ has all the cuts from J plus all the cuts from K; this suggests that we *may* have an operation dual to that, that behaves as this: $(J \vee K)$ has exactly the cuts that are both in J and in K:

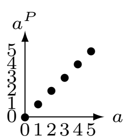
$$\begin{array}{rcl} \mathsf{Cuts}(J \wedge K) & = & \mathsf{Cuts}(J) \cup \mathsf{Cuts}(K) \\ \mathsf{Cuts}(J \vee K) & = & \mathsf{Cuts}(J) \cap \mathsf{Cuts}(K) \end{array}$$

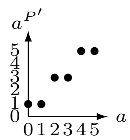
And it $J_1, \ldots, J_n$ are all the slash-operators on a given ZHA, then

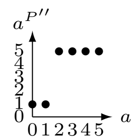
$$\begin{array}{rclcl} \mathsf{Cuts}(J_1 \wedge \ldots \wedge J_n) & = & \mathsf{Cuts}(J_1) \cup \ldots \cup \mathsf{Cuts}(J_k) & = & \text{(all cuts)} \\ \mathsf{Cuts}(J_1 \vee \ldots \vee J_n) & = & \mathsf{Cuts}(J_1) \cap \ldots \cap \mathsf{Cuts}(J_k) & = & \text{(no cuts)} \end{array}$$

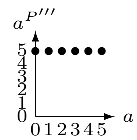
yield the minimal element and the maximal element, respectively, of an algebra of slash-operators; note that the slash-operator with "all cuts" is the identity map $\lambda P\colon H.P$, and the slash-operator with "no cuts" is the one that takes all elements to $\top$: $\lambda P\colon H.\top$. This yields a lattice of slash-operators, in which the partial order is $J \leq K$ iff $\mathsf{Cuts}(J) \supseteq \mathsf{Cuts}(K)$. This is somewhat counterintuitive if we think in terms of cuts — the order seems to be reversed — but it makes a lot of sense if we think in terms of piccs (sec.1) instead.

Each picc P on $\{0, \ldots, n\}$ has an associated function $\cdot^P$ that takes each element to the top element of its equivalence class. If we define $P \leq P'$ to mean $\forall a \in \{0, \ldots, n\}.\, a^P \leq a^{P'}$, then we have this:



$\leq$


$\leq$


$\leq$


$$\begin{array}{ccccccc} 0|1|2|3|4|5 & \leq & 01|23|45 & \leq & 01|2345 & \leq & 012345 \\ P & \leq & P' & \leq & P'' & \leq & P''' \end{array}$$

This yields a partial order on piccs, whose bottom element is the identity function $0|1|2|\ldots|n$, and the top element is $012\ldots n$, that takes all elements to n.

The piccs on $\{0, \ldots, n\}$ form a Heyting Algebra, where $\top = 01\ldots n$, $\bot = 0|1|\ldots|n$, and '$\wedge$' and '$\vee$' are the operations that we have discussed above; it is possible to define a '$\to$' there, but this '$\to$' is not going to be useful for us and we are mentioning it just as a curiosity. We have, for example:

$$\begin{tikzcd}[column sep=small] & 01234 & \\ & 01|234 \arrow[u] & \\ 0|1|234 \arrow[ur] & & 01|2|34 \arrow[ul] \\ & 0|1|2|34 \arrow[ul] \arrow[ur] & \\ & 0|1|2|3|4 \arrow[u] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \top & \\ & P \vee Q \arrow[u] & \\ P \arrow[ur] & & Q \arrow[ul] \\ & P \wedge Q \arrow[ul] \arrow[ur] & \\ & \bot \arrow[u] & \end{tikzcd}$$

15 An algebra of J-operators

Fourman and Scott define the operations $\wedge$ and $\vee$ on J-operators in pages 325 and 329 ([**?**]), and in page 331 they list ten properties of the algebra of J-operators:

$$\begin{array}{rrcll}
\text{(i)} & J_a \vee J_b & = & J_{a\vee b} & (\vee 21) \vee (\vee 12) = (\vee 22) \\
\text{(ii)} & J^a \vee J^b & = & J^{a\wedge b} & (\to 32) \vee (\to 23) = (\to 22) \\
\text{(iii)} & J_a \wedge J_b & = & J_{a\wedge b} & (\vee 21) \wedge (\vee 12) = (\vee 11) \\
\text{(iv)} & J^a \wedge J^b & = & J^{a\vee b} & (\to 32) \wedge (\to 23) = (\to 33) \\
\text{(v)} & J_a \wedge J^a & = & \bot & (\vee 22) \wedge (\to 22) = (\bot) \\
\text{(vi)} & J_a \vee J^a & = & \top & (\vee 22) \vee (\to 22) = (\top) \\
\text{(vii)} & J_a \vee K & = & K \circ J_a & \\
\text{(viii)} & J^a \vee K & = & J^a \circ K & \\
\text{(ix)} & J_a \vee B_a & = & B_a & \\
\text{(x)} & J^a \vee B_b & = & B_{a\to b} &
\end{array}$$

The first six are easy to visualize; we won't treat the four last ones. In the right column of the table above we've put a particular case of (i), …, (vi) in our notation, and the figures below put all together.

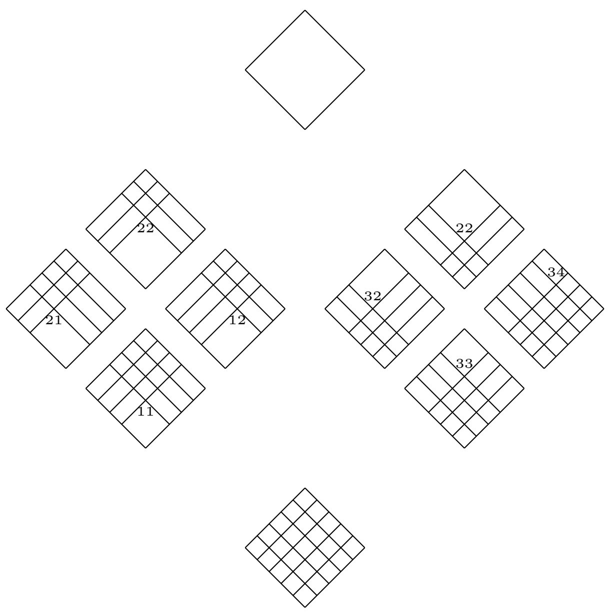
In Fourman and Scott's notation,

$$\begin{tikzcd}
 & & & J_\top = \top = J^\bot & & & \\
 & J_{22} \arrow[urr] & & & & J^{22} \arrow[ull] & \\
J_{21} \arrow[ur] & & J_{12} \arrow[ul] & & J^{32} \arrow[ur] & & J^{23} \arrow[ul] \\
 & J_{11} \arrow[ul] \arrow[ur] & & & & J^{11} \arrow[ul] \arrow[ur] & \\
 & & & J_\bot = \bot = J^\top \arrow[ull] \arrow[urr] & & &
\end{tikzcd}$$

in our notation,

$$\begin{tikzcd}[column sep=tiny]
 & & & (\top\vee) = (\lambda P.\top) = (\bot\to) & & & \\
 & (22\vee) \arrow[urr] & & & & (22\to) \arrow[ull] & \\
(21\vee) \arrow[ur] & & (12\vee) \arrow[ul] & & (32\to) \arrow[ur] & & (23\to) \arrow[ul] \\
 & (11\vee) \arrow[ul] \arrow[ur] & & & & (33\to) \arrow[ul] \arrow[ur] & \\
 & & & (\bot\vee) = (\lambda P.P) = (\top\to) \arrow[ull] \arrow[urr] & & &
\end{tikzcd}$$

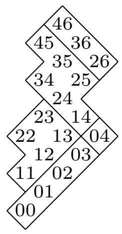
and drawing the polynomial J-operators as in sec.13:

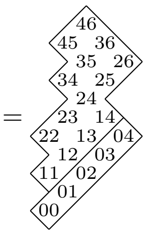


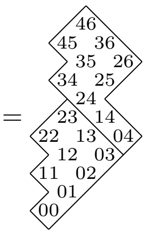
16 All slash-operators are polynomial

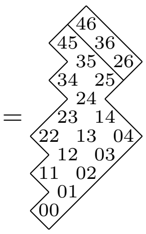
Here is an easy way to see that all slashings — i.e., J-operators on ZHAs — are polynomial. Every slashing J has only a finite number of cuts; call them

$J_1, \ldots, J_n$. For example:

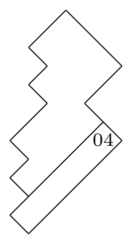
$J =$ 

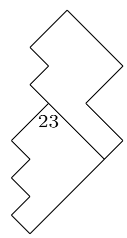
$J_1 =$ 

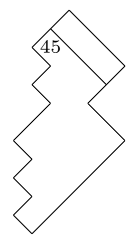
$J_2 =$ 

$J_3 =$ 

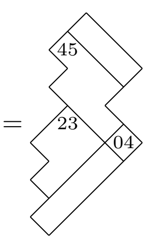
Each cut J_i divides the ZHA into an upper region and a lower region, and $J_i(00)$ yields the top element of the lower region. Also, $(\to\to J_i(00))$ is a polynomial way of expressing that cut:

$J_1 =$
$(\to\to 04) =$ 

$J_2 =$
$(\to\to 23) =$ 

$J_3 =$
$(\to\to 45) =$ 

The conjunction of these '$(\to\to J_i(00))$'s yields the original slashing:

$(\to\to 04) \wedge (\to\to 23) \wedge (\to\to 45) =$  $= J$

17 Question marks

Every ZHA H is equivalent — by the constructions explained in sections 14–17 of [PH1] — to a 2-column graph (P, A). To be more precise, each ZHA H has an associated 2CG (P, A), such that this holds: the partial order $(H, \leq)$ is equivalent to $(\mathcal{O}_A(P), \subseteq)$, where $\mathcal{O}_A(P)$ is the "order topology" on P (sections 12–13 of [PH1]). We will use squiggly arrows to mean "is associated to". For example:

$$
\begin{array}{ccccc}
 & 46 & & & \\
45 & & 36 & & \\
 & 35 & & 26 & \\
34 & & 25 & & \\
 & 24 & & & \\
23 & & 14 & & \\
22 & & 13 & & 04 \\
 & 12 & & 03 & \\
11 & & 02 & & \\
 & 01 & & & \\
00 & & & &
\end{array}
\quad \leftsquigarrow\!\rightsquigarrow \quad
\left(
\begin{tikzcd}
 & \_6 \arrow[d] \\
 & \_5 \arrow[d] \arrow[ddl] \\
4\_ \arrow[d] \arrow[ur] & \_4 \arrow[d] \\
3\_ \arrow[d] \arrow[ur] & \_3 \arrow[d] \\
2\_ \arrow[d] \arrow[r] & \_2 \arrow[d] \\
1\_ \arrow[r] & \_1
\end{tikzcd}
\right)
$$

Each truth-value $cd \in H$ corresponds to the open set $\mathsf{pile}(cd) \in \mathcal{O}_A(P)$; the function $\mathsf{pile} : H \to \mathcal{O}_A(P)$ is a bijection. We will also use squiggly arrows to indicate this correspondence:

$$
23 \quad \leftsquigarrow\!\rightsquigarrow \quad \mathsf{pile}(23) = \{2\_, 1\_, \_3, \_2, \_1\} =
\left(
\begin{tikzcd}
 & 0 \\
 & 0 \arrow[ddl] \\
0 \arrow[ur] & 0 \\
0 \arrow[ur] & 1 \\
1 \arrow[r] & 1 \\
1 \arrow[r] & 1
\end{tikzcd}
\right)
$$

Note that we are used the positional notation from sec.15 of [PH1] to draw $\mathsf{pile}(23) \subseteq P$ as an open set of $\mathcal{O}_A(P)$.

Choose any partition of P into two disjoint subsets $P_! \cup P_?$; we will call $P_!$ the "set of relevant points" and $P_?$ the "set of question marks" of the partition; we will use the letter Q to indicate a certain choice of question marks. Any partition $P = P_! \cup P_?$ induces:

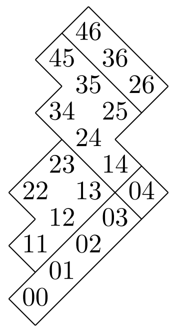
- an equivalence relation '$\sim_{Q'}$' on $\mathcal{O}_A(P)$; if $R', S' \in \mathcal{O}_A(P)$ then $R' \sim_{Q'} S'$ iff $R' \cap P_! = S' \cap P_!$, or, equivalently, iff $R' \backslash P_? = S' \backslash P_?$ ("R' and S' are equivalent modulo question marks"),
- an equivalence relation '$\sim_Q$', on H: $cd \sim_Q ef$ iff $\mathsf{pile}(cd) \sim_{Q'} \mathsf{pile}(ef)$,
- Q-equivalence classes on H: for each $cd \in H$ we write

$$[cd]^Q := \{\, ef \in H \mid cd \sim_Q ef \,\}$$

- a "Q-partition" of H: $\{\, [cd]^Q \mid cd \in H \,\}$.

18 Q-partitions are slash-partitions and vice-versa

Take a slash-partition S on a ZHA H. For example, this one:



Let's start with a naïve idea: how can we build a Q-partition on H that mimics the behavior of S? We have for example $22 \sim_S 13$, so we need to have $\mathsf{pile}(22) \sim_S \mathsf{pile}(13)$, which means that the sets $\{2\_, 1\_, \ \_2, \_1\}$ and $\{1\_, \ \_3, \_2, \_1\}$ should be equal "modulo question marks"; writing

$$\begin{array}{rcl} \mathsf{diff}(A,B) & := & (A\backslash B) \cup (B\backslash A), \\ \mathsf{QM}(cd,ef) & := & \mathsf{diff}(\mathsf{pile}(cd), \mathsf{pile}(ef)) \end{array}$$

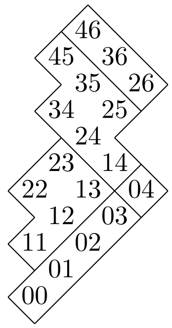
we have this: $cd \sim_S ef$ implies $\mathsf{QM}(cd,ef) \subseteq P_?$, and $22 \sim_S 13$ implies $\mathsf{QM}(22,13) = \{2\_, \ \_3\} \subseteq P_?$. In the slash-partition above we have $22 \sim_S 13$, $14 \sim_S 45$, $00 \sim_S 03$, so $\mathsf{QM}(22,13) \cup \mathsf{QM}(14,45) \cup \mathsf{QM}(00,03) \subseteq P_?$; each set of pairs of S-equivalent elements of H yields a subset of $P_?$, and by starting with a set of S-equivalent pairs that is big enough we *may* be able to discover all points in $P_?$...

In the other direction, what are the consequences of a certain point belonging or not to $P_?$? It is easy to see that in the ZHA above we have:

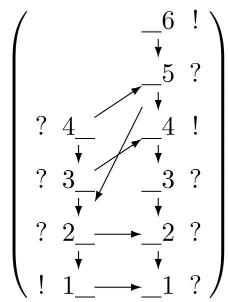
$$\begin{array}{rcl} 1\_ \in P_? & \Longrightarrow & (01 \sim_Q 11) \wedge (02 \sim_Q 12) \wedge (03 \sim_Q 13) \wedge (04 \sim_Q 14) \\ 1\_ \notin P_? & \Longrightarrow & (01 \not\sim_Q 11) \wedge (02 \not\sim_Q 12) \wedge (03 \not\sim_Q 13) \wedge (04 \not\sim_Q 14) \\ \_4 \in P_? & \Longrightarrow & (03 \sim_Q 04) \wedge (13 \sim_Q 14) \wedge (23 \sim_Q 24) \\ \_4 \notin P_? & \Longrightarrow & (03 \not\sim_Q 04) \wedge (13 \not\sim_Q 14) \wedge (23 \not\sim_Q 24) \end{array}$$

so $1\_ \in P_?$ means that we do have a cut going northeast between 0 and 1 — "1/0" in the notation of sec.1 — and $1\_ \notin P_?$ means that we don't have that cut; $\_4 \in P_?$ means that we do have a cut going northwest between 3 and 4 — "3\4" in the notation of sec.1 — and $\_4 \notin P_?$ means that we don't have that cut. The presence or not of a given point of P in $P_?$ means the presence or not of a given cut, and vice-versa.

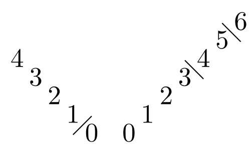
We will skip the low-level details and just show two examples of conversions. This is a ZHA H with a slash-partition $S = (L,R)$ on it, and the 2CG (P,A) associated to H with the partition $P = P_? \cup P_!$ associated to S; we write on the side of each element of P a '?' or '!' to indicate whether it belongs to $P_?$ or $P_!$:



$\rightsquigarrow$



It turns out that we can ignore the exact shape of H and the intercolumn arrows in A. For any ZHA H with top element 46, the slashing $S = (L, R)$ on H drawn below at the left corresponds to the partition of $P = \mathsf{LC}(4) \cup \mathsf{RC}(6)$ into $P_? \cup P_!$ shown at the right:



$\rightsquigarrow$

$$\begin{array}{l} P_! = \{1\_, \ \_4, \_6\} \\ P_? = \{2\_, 3\_, 4\_, \ \_1, \_2, \_3, \_5\} \end{array}$$

[Note that we don't have cuts stopping midway]

19 Q-equivalence classes are intervals

We will see in the next section that each partition into Q-equivalence classes is a slash-partition and vice-versa. Also, each Q-equivalence class is an interval, so each $[cd]^Q$ has a sup and and inf; we will see nice ways to visualize, and to compute, the operations $cd \mapsto \sup([cd]^Q)$ and $cd \mapsto \inf([cd]^Q)$.

20 Open sets of a certain form

A 2-column graph with question marks (a "2CGQ") is a triple $((P, A), B, D)$, where (P, A) is a 2CG and $B \subseteq D \subseteq P$; let $G = ((P, A), B, D)$. We represent G graphically like (P, A), but with '0's, '?'s and '1's on the points of P, and the expression "C is of the form G" means $B \subseteq C \subseteq D$. For example:

$$\left(\begin{tikzcd} & 0 \arrow[d] \\ 1 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 & 1 \end{tikzcd}\right) \text{ is of the form } \left(\begin{tikzcd} & 0 \arrow[d] \\ ? \arrow[r] \arrow[d] & ? \arrow[d] \\ 0 & 1 \end{tikzcd}\right)$$

Informally, a '0' in the graphical representation of a 2CGQ Q means "all 'C's of the form G have a '0' here", a '1' means "all 'C's of the form G have a

'1' here", and a '?' means "some 'C's of the form G have '0's there and some have '1's". More formally, a 2CGQ G corresponds to a partition of P into P_0, P_1 and $P_?$ — the sets of '0's, '1's and '?'s of the graphical representation of G — and we have $P_1 = B$, $P_? = D \backslash B$, $P_0 = P \backslash D$, $D = P_1 \cup P_?$.

Our main use for 2CGQs will be for giving us a nice notation for "the set of open sets of (P, A) betwen B and D":

$$\mathsf{Opens}((P, A), B, D) \quad = \quad \{\, U \subseteq P \mid B \subseteq U \subseteq D \text{ and } U \in \mathcal{O}_A(P) \,\}$$

Note that adding intercolumn arrows reduce sets of opens sets,

$$\mathsf{Opens}\begin{pmatrix} ? & & ? \\ 0 & & 0 \\ ? & & ? \\ ? & & ? \\ 1 & & 1 \\ ? & & ? \end{pmatrix} \supseteq \mathsf{Opens}\begin{pmatrix} ? & & ? \\ 0 & & 0 \\ ? & \searrow & ? \\ ? & & ? \\ 1 & & 1 \\ ? & & ? \end{pmatrix} \supseteq \mathsf{Opens}\begin{pmatrix} ? & & ? \\ 0 & \nwarrow & 0 \\ ? & \searrow & ? \\ ? & & ? \\ 1 & & 1 \\ ? & & ? \end{pmatrix}$$

because each arrow is a "restriction" (sec.**??**) on what is considered an open set. We can propagate '1's forward along arrows like '$1 \to ?$' and '0's backward along arrows like '$? \to 0$' without changing the result of '$\mathsf{Opens}(\ldots)$':

$$\mathsf{Opens}\begin{pmatrix} ? & ? \\ 0 & 0 \\ ? & ? \\ ? & ? \\ 1 & 1 \\ ? & ? \end{pmatrix} = \mathsf{Opens}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ ? & ? \\ ? & ? \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mathsf{Opens}\begin{pmatrix} & & 0 \\ & & ? \\ ? & \nearrow \swarrow & 0 \\ ? & & ? \\ ? & \to & ? \\ 1 & \to & ? \end{pmatrix} = \mathsf{Opens}\begin{pmatrix} & & 0 \\ & & 0 \\ 0 & \nearrow \swarrow & 0 \\ 0 & & ? \\ ? & \to & ? \\ 1 & \to & 1 \end{pmatrix}$$

21 Propagation

Fix a 2CG (P, A). There are two good, natural ways to get rid of all arrows '$1 \to 0$' in a subset $C \subseteq P$: one, called '$\mathsf{prp}_{1,(P,A)}$' or 'prp_1', "propagates the '1's forward", and the other one, called 'prp_0' or '$\mathsf{prp}_{1,(P,A)}$', "propagates the '0's backward". An example:

$$\mathsf{prp}_0 \begin{pmatrix} & & 0 \\ 1 & \to & 0 \\ 0 & & 1 \end{pmatrix} = \begin{pmatrix} & & 0 \\ 0 & \to & 0 \\ 0 & & 1 \end{pmatrix} \qquad \mathsf{prp}_1 \begin{pmatrix} & & 0 \\ 1 & \to & 0 \\ 0 & & 1 \end{pmatrix} = \begin{pmatrix} & & 0 \\ 1 & \to & 1 \\ 1 & & 1 \end{pmatrix}$$

It easy to see that $\mathsf{prp}_1(C)$ returns the smallest open set containing C, and $\mathsf{prp}_0(C)$ returns the largest open set contained in C,

The *interior* of a set S in a topology $\mathcal{U}$ on P is the biggest open set in $\mathcal{U}$ contained in S, and, dually, the *cointerior* of a set S is the smallest open set in $\mathcal{U}$ containing S. In finite topologies cointeriors always exist.

Theorem 1. For any 2CG (P, A) and $S \subseteq P$ we have

$$\mathsf{int}(S) = \mathsf{prp}_0(S) \subseteq S \subseteq \mathsf{prp}_1(S) = \mathsf{coint}(S).$$

We can define propagations for 2CGQs in a way that changes only the '?'s. If $G = ((P, A), B, D)$ is a 2CGQ, then $\mathsf{prp}_1(G)$ propagates forward only the '1's in arrows like ‘$1 \to ?$’, and $\mathsf{prp}_0(G)$ propagates backward only the ‘0’s in arrows like ‘$? \to 0$’.

The operations ‘prp_1’ and ‘prp_0’ on 2CGQs need not commute:

$$\begin{aligned}
\mathsf{prp}_1\left(\mathsf{prp}_0\begin{pmatrix} 1 & & ? \\ ? & \nearrow & 0 \end{pmatrix}\right) &= \begin{pmatrix} 1 & & 0 \\ 0 & \nearrow & 0 \end{pmatrix} \\
\mathsf{prp}_0\left(\mathsf{prp}_1\begin{pmatrix} 1 & & ? \\ ? & \nearrow & 0 \end{pmatrix}\right) &= \begin{pmatrix} 1 & & 1 \\ 1 & \nearrow & 0 \end{pmatrix}
\end{aligned}$$

but they can only fail to commute when $\mathsf{Opens}(G) = \emptyset$. When they commute we will write their composite as ‘prp’.

Theorem 2. Let $G = ((P, A), B, D)$ be a 2CGQ with $\mathsf{Opens}(G) \neq \emptyset$ and let $G' = \mathsf{prp}(G) = \mathsf{Opens}((P, A), B', D')$, $P'_1 = B'$, $P'_? = D' \backslash B'$, $P'_1 = P \backslash D'$. Then:
a) In G' everything below a ‘1’ is also ‘1’,
b) In G' everything above a ‘0’ is also ‘0’,
c) $B' = P'_1$ is an open set,
d) $D' = P'_1 \cup P'_? = P \backslash P'_0$ is an open set,
e) $B' = \mathsf{prp}_1(B) = \mathsf{coint}(B)$,
f) $D' = \mathsf{prp}_0(D) = \mathsf{int}(D)$,
g) $B' = \mathsf{pile}(ab)$ for some ab,
h) $D' = \mathsf{pile}(ef)$ for some ef,
i) $B' \in \mathsf{Opens}(G) = \mathsf{Opens}(G')$,
j) $D' \in \mathsf{Opens}(G) = \mathsf{Opens}(G')$.

An example:

$$G = \begin{pmatrix} & & 0 \\ & & ? \\ ? & \nearrow & 0 \\ ? & \nearrow\!\!\!\!\swarrow & ? \\ ? & \to & ? \\ 1 & \to & ? \end{pmatrix} \qquad G' = \mathsf{prp}(G) = \begin{pmatrix} & & 0 \\ & & 0 \\ 0 & \nearrow & 0 \\ 0 & \nearrow\!\!\!\!\swarrow & ? \\ ? & \to & ? \\ 1 & \to & 1 \end{pmatrix} = ((P, A), \mathsf{pile}(11), \mathsf{pile}(23))$$

The next theorem translates this to ZHAs, and shows that when $\mathsf{Opens}(G) \neq \emptyset$ then it returns an interval in a ZHA (in the sense of sec.1),

Theorem 3. Let $G = ((P, A), B, D)$ be a 2CGQ with $\mathsf{Opens}(G) \neq \emptyset$ and let $G' = \mathsf{prp}(G) = \mathsf{Opens}((P, A), B', D')$, $ab = \mathsf{pile}^{-1}(B')$, $ef = \mathsf{pile}^{-1}(D')$, $I = \mathsf{pile}^{-1}(\mathsf{Opens}(G)) = \mathsf{pile}^{-1}(\mathsf{Opens}(G'))$, and let H be the ZHA generated by (P, A), i.e., $H = \mathsf{pile}^{-1}(\mathcal{O}_A(P))$. Then:
a) ab is the minimal point of I,
b) ef is the maximal point of I,
c) $I \subseteq H$,
d) $I = [ab, ef] \cap H$,
e) if A has no intercolumn arrows then $I = [ab, ef]$.

With Theorem 3 we can extend the last example to:

$$G = \begin{pmatrix} & & 0 \\ & & ? \\ ? & \nearrow & 0 \\ ? & \swarrow & ? \\ ? & \to & ? \\ 1 & \to & ? \end{pmatrix} \qquad G' = \mathsf{prp}(G) = \begin{pmatrix} & & 0 \\ & & 0 \\ 0 & \nearrow & 0 \\ 0 & \swarrow & ? \\ ? & \to & ? \\ 1 & \to & 1 \end{pmatrix} = ((P,A), \mathsf{pile}(11), \mathsf{pile}(23))$$

$$\mathsf{Opens}(G) = \mathsf{Opens}(G') = I \quad \underset{\mathsf{pile}^{-1}}{\overset{\mathsf{pile}}{\rightleftarrows}} \quad [11,23] \cap \begin{array}{ccccc} & & 46 & & \\ & 45 & & 36 & \\ & & 35 & & 26 \\ & 34 & & 25 & \\ & & 24 & & \\ & 23 & & 14 & \\ 22 & & 13 & & 04 \\ & 12 & & 03 & \\ 11 & & 02 & & \\ & 01 & & & \\ 00 & & & & \end{array}$$

In the next sections we will see that in some important cases the results of $\mathsf{Opens}(\ldots)$ coincide with J-equivalence classes.

22 The set of relevant points of a slashing

We saw in sec.1 that a slashing on a ZHA H can be represented a pair (L,R) of piccs, that we drew in a V-shaped diagram; let's write S for the set of numbers above the cuts in the V-shaped diagram, converting them to the notation for elements of the left and the right columns of 2-column graphs:

$$J = \begin{array}{ccccc} & & 46 & & \\ & 45 & & 36 & \\ & & 35 & & 26 \\ & 34 & & 25 & \\ & & 24 & & \\ & 23 & & 14 & \\ 22 & & 13 & & 04 \\ & 12 & & 03 & \\ 11 & & 02 & & \\ & 01 & & & \\ 00 & & & & \end{array} \qquad (L,R) = \begin{array}{ccccccccccc} & & & & & & & & & & 6 \\ & & & & & & & & & 5 & \\ 4 & & & & & & & & 4 & & \\ & 3 & & & & & & 3 & & & \\ & & 2 & & & & 2 & & & & \\ & & & 1 & & 1 & & & & & \\ & & & & 0 \;\; 0 & & & & & & \end{array} \qquad S = \{1\_, \ \_4, \_6\}$$

We also saw (sec.7) that on ZHAs there is a bijection between slashings and J-operators. Let $\mathsf{relev}(J)$ be the operation that obtains the set S for a J-operator J: $\mathsf{relev}(J) = \{1\_, \ \_4, \_6\}$ for the J above. We will call $S \subseteq P$ the *set of relevant points* of the J-operator J, and $Q = \mathsf{qmarks}(J) = P \backslash S$ will be the *set of (points that will be replaced by) question marks* by J. Note that we can also go from a set $Q \subseteq P$ to a slashing and a J-operator, but we will not need a notation for that.

We can define the operation that receives a $C \subseteq P$ and "forgets the information on the points of Q" on C, returning a 2CGQ, as:

$$\mathsf{forget}_{(P,A),Q}(C) \quad = \quad ((P,A), C \backslash Q, C \cup Q)$$

for example:

$$\mathsf{forget}_{(P,A),Q}(\mathsf{pile}(12)) = \begin{pmatrix} & & 0 \\ & & ? \\ ? & \nearrow & 0 \\ ? & \swarrow & ? \\ ? & \to & ? \\ 1 & \to & ? \end{pmatrix}$$

Note that

$$\begin{array}{rcl} \mathsf{prp}(\mathsf{forget}_{(P,A),Q}(\mathsf{pile}(12))) & = & \begin{pmatrix} & & 0 \\ & & 0 \\ 0 & \nearrow & 0 \\ 0 & \swarrow & ? \\ ? & \to & ? \\ 1 & \to & 1 \end{pmatrix} \\ & = & ((P,A), \mathsf{pile}(11), \mathsf{pile}(23)) \end{array}$$

and that:

$$\begin{array}{rcl} \mathsf{pile}^{-1}(\mathsf{Opens}(\mathsf{prp}(\mathsf{forget}_{(P,A),Q}(\mathsf{pile}(12))))) & = & [11, 23] \cap H \\ & = & [\mathrm{co}J(12), J(12)] \cap H \\ & = & [12]^J \end{array}$$

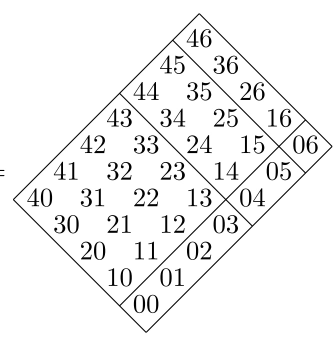
this holds in general, as we will see soon.

23 Rectangular versions

The "rectangular version" of a 2CG, a ZHA and a J-operator are defined as this. Let (P, A) be a 2CG and H its associated ZHA, and $J : H \to H$ a J-operator on H; then A' is A minus its intercolumn arrows, H' is the (rectangular) ZHA associated to (P, A'), and $J' : H' \to H'$ is J-operator on H' that has the same cuts as J. The primes on A', H' and J' will always mean from here on that we are on the rectangular versions. Let $Q = \mathsf{qmarks}(J) = \mathsf{qmarks}(J')$.

The rectangular versions for the (P, A) and the J that we are using in our examples are:

$$(P, A') = \begin{pmatrix} & \_6 \\ & \downarrow \\ & \_5 \\ & \downarrow \\ 4\_ & \_4 \\ \downarrow & \downarrow \\ 3\_ & \_3 \\ \downarrow & \downarrow \\ 2\_ & \_2 \\ \downarrow & \downarrow \\ 1\_ & \_1 \end{pmatrix} \qquad J' =$$



.

Take any $C \subseteq P$, The result of $\mathsf{forget}_{(P,A'),Q}(C)$ is always of this form,

$$\mathsf{forget}_{(P,A'),Q}(C) = \begin{pmatrix} & c \\ & ? \\ ? & b \\ ? & ? \\ ? & ? \\ a & ? \end{pmatrix}$$

for some $a, b, c \in \{0, 1\}$; moreover, if C is open then $\mathsf{forget}_{(P,A'),Q}(C)$ doesn't have '1's above '0's. Take any $C \subseteq P$ open in (P, A); C will be of the form $\mathsf{pile}(cd)$ for some $cd \in H'$. Let $G = \mathsf{forget}_{(P,A'),Q}(C)$. The action of prp on 'G's of this form is particularly simple: each column of G is made of blocks of consecutive '?'s separated by '0's or '1's, and prp acts homogeneously on each block, leaving '?'s in at most one block in each column. For example, if $a = b = 1$ and $c = 0$ then

$$\mathsf{prp}(\mathsf{forget}_{(P,A'),Q}(C)) = \begin{pmatrix} & 0 \\ & ? \\ ? & 1 \\ ? & 1 \\ ? & 1 \\ 1 & 1 \end{pmatrix}$$

It is easy to see that:

Theorem 1. If $C = \mathsf{pile}(cd)$ then $\mathsf{pile}^{-1}(\mathsf{Opens}(\mathsf{prp}(\mathsf{forget}_{(P,A'),Q}(C))))$ is a J'-equivalence class.

Theorem 2. If $C = \mathsf{pile}(cd)$ then $\mathsf{pile}^{-1}(\mathsf{Opens}(\mathsf{prp}(\mathsf{forget}_{(P,A'),Q}(C))))$ is $[\mathrm{co}J'(cd), J'(cd)]$.

Theorem 3. Suppose that $cd \in H$ (instead of $cd \in H'$) and let:

$$\begin{array}{rcl} C & = & \mathsf{pile}(cd) \\ G & = & \mathsf{forget}_{(P,A'),Q}(C) \\ G' & = & \mathsf{prp}(\mathsf{forget}_{(P,A'),Q}(C)) \\ G'' & = & \mathsf{prp}(\mathsf{forget}_{(P,A),Q}(C)) \\ I' & = & \mathsf{pile}^{-1}(\mathsf{Opens}(G')) \\ I'' & = & \mathsf{pile}^{-1}(\mathsf{Opens}(G'')) \end{array}$$

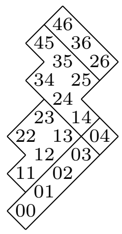
then G' is a "rectangular" (and "propagated") 2CGQ, and $I' = [\mathrm{co}J'(cd), J'(cd)]$ is a "rectangular interval"; G'' is G' plus the intercolumn arrows, and with the propagations having been done through the intercolumn arrows too. It is not hard to see that:

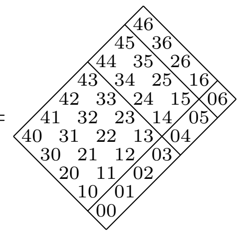
a) $\mathsf{Opens}(G) = \mathsf{Opens}(G') \supseteq \mathsf{Opens}(G'')$
b) $I'' = I' \cap H$
c) $cd \in I''$
d) $I'' = [\mathrm{co}J(cd), J(cd)] \cap H$
e) $\mathsf{pile}(\mathrm{co}J(cd)), \mathsf{pile}(J(cd)) \in I''$
f) $G'' = ((P, A), \mathsf{pile}(\mathrm{co}J(cd)), \mathsf{pile}(J(cd)))$
g) $G'' = ((P, A), \mathsf{coint}(C \backslash Q), \mathsf{int}(C \cup Q))$, so:
h) $\mathsf{pile}(\mathrm{co}J(cd)) = \mathsf{coint}(C \backslash Q) = \mathsf{prp}_1(C \backslash Q)$ and
i) $\mathsf{pile}(J(cd)) = \mathsf{int}(C \cup Q) = \mathsf{prp}_0(C \cup Q)$,
j) $\mathrm{co}J(cd)) = \mathsf{pile}^{-1}(\mathsf{coint}(C \backslash Q)) = \mathsf{pile}^{-1}(\mathsf{prp}_1(C \backslash Q))$ and
k) $J(cd) = \mathsf{pile}^{-1}(\mathsf{int}(C \cup Q)) = \mathsf{pile}^{-1}(\mathsf{prp}_0(C \cup Q))$.

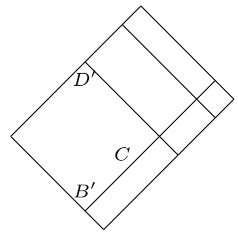
A way to visualize what Theorem 3 means is to define $B, B', B'', D, D'D''$ like this:

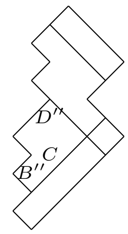
$$\begin{array}{rcl} (B, D) & = & (C \backslash Q, C \cup Q) \\ G' & = & ((P, A'), B', D') \\ G'' & = & ((P, A), B'', D'') \end{array}$$

then, in the example we are using, omitting some 'pile's and 'pile^{-1}'s, we have:

$J =$ 

$J' =$ 





Theorem 3 shows several ways to calculate B', C', B'', C''.

24 Sub-2-column graphs

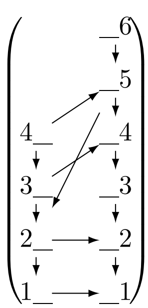
Another way to understand the properties of the operation $\mathsf{forget}_{(P,A),Q}$ is to think that it relates two topologies, $\mathcal{O}_A(P)$ and $\mathcal{O}_{A|_S}(S)$ (mnemonic: S is a "smaller set", and $S = \mathsf{relev}(J) = P\backslash Q$). We will sometimes denote $\mathcal{O}_A(P)$ and $\mathcal{O}_{A|_S}(S)$ as just $\mathcal{O}(P)$ and $\mathcal{O}(S)$; $\mathcal{O}(S)$ is a restriction of $\mathcal{O}(P)$ to S in the following sense: the open sets of $\mathcal{O}(S)$ are exactly the sets of the form $U \cap S$, where $U \in \mathcal{O}_A(P)$.

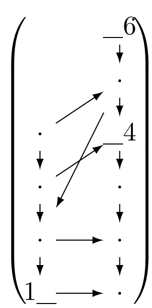
The topology $\mathcal{O}(S) = \mathcal{O}_{A|_S}(S)$ comes from a "sub-2-column graph" $(S, A|_S)$ of (P, A), where the set of arrows $A|_S$ can be obtained from A and S by

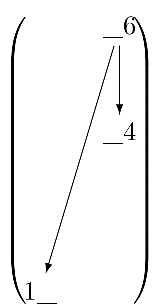
$$A|_S \quad := \quad (A^* \cap (S \times S))^{\mathsf{ess}},$$

which means: take the transitive-reflexive closure A^* of A, which yields a partial order on P, and restrict that order to S by taking $A^* \cap (S \times S)$; then (note: this last step is optional!) drop the redundant arrows in $A^* \cap (S \times S)$ and keep only the "essential" ones, which are the ones that can't be deleted without changing the order.

For clarity, we will draw the arrows in $(S, A|_S)$ as in the original 2CG (P, A), even though some arrows may look as coming from or going to nonexistent points; a really honest drawing of $(S, A|_S)$ in the example below would be the one at the right, that has only one intercolumn arrow, $1\_ \leftarrow \_6$, and only one vertical arrow, $\_6 \to \_4$.

$(P, A) =$ 

$(S, A|_S) =$ 

$=$ 

A *sub-2-column graph* is a graph $(S, A|_S)$ generated by a 2CG (P, A) and an $S \subseteq P$ by the procedure above: $A|_S = (A^* \cap (S \times S))^{\mathsf{ess}}$.

Theorem 1. Fix a ZHA H and a J-operator J on it, and from that produce (P,A), $\mathcal{U} = \mathcal{O}_A(P)$, S, and Q. We clearly have bijections between:
1) the set of fixed points of J, $\{\, ef \in H \mid J(ef) = ef \,\}$,
2) the set of fixed points of co$J$, $\{\, ab \in H \mid \mathrm{co}J(ab) = ab \,\}$,
3) the image of J, $J(H) = \{\, J(cd) \mid cd \in H \,\}$,
4) the image of co$J$, $\mathrm{co}J(H) = \{\, \mathrm{co}J(cd) \mid cd \in H \,\}$,
5) the set of J-equivalence classes of H, $H/J = \{\, [cd]^J \mid cd \in H \,\}$,
6) the elements $ef \in H$ such that $\mathsf{pile}(ef) = \mathsf{int}(\mathsf{pile}(ef) \cup Q)$,
7) the elements $ab \in H$ such that $\mathsf{pile}(ab) = \mathsf{coint}(\mathsf{pile}(ab) \backslash Q)$,
8) the sets $U \subseteq \mathcal{O}(P)$ such that $U = \mathsf{int}(U \cup Q)$,
9) the sets $W \subseteq \mathcal{O}(P)$ such that $W = \mathsf{coint}(W \backslash Q)$,
10) the sets $U \subseteq P$ such that $U = \mathsf{int}(U \cup Q)$,
11) the sets $W \subseteq P$ such that $W = \mathsf{coint}(W \backslash Q)$,
12) the opens sets in $\mathcal{O}(S)$.

The partial order on $\mathcal{O}(S)$ is given by inclusion; some of the corresponding partial orders on the other sets of Theorem 1 are not so obvious.

Theorem 2. Let $ab, cd \in H$, $A = \mathsf{pile}(ab)$, $B = \mathsf{pile}(cd)$, $A' = A \cap S$, $B' = B \cap S$. The following are all equivalent:
1) $A' \subseteq B'$,
2) $A \backslash Q \subseteq B \backslash Q$,
2') $A \cup Q \subseteq B \cup Q$,
3) $\mathsf{coint}(A \backslash Q) \subseteq \mathsf{coint}(B \backslash Q)$,
3') $\mathsf{int}(A \cup Q) \subseteq \mathsf{int}(B \cup Q)$,
4) $\mathsf{prp}_1(A \backslash Q) \subseteq \mathsf{prp}_1(B \backslash Q)$
4') $\mathsf{prp}_0(A \cup Q) \subseteq \mathsf{prp}_0(B \cup Q)$
5) $\mathrm{co}J(ab) \leq \mathrm{co}J(cd)$,
5') $J(ab) \leq J(cd)$,
6) $\inf([ab]^J) \leq \inf([cd]^J)$,
6') $\sup([ab]^J) \leq \sup([cd]^J)$.

Items 6 and 6' give us a way to endow H/J with a partial order. Remember that $\sup([ab]^J) = J(ab)$ and $\inf([ab]^J) = \mathrm{co}J(ab)$; we say that $[ab]^J \leq [cd]^J$ when $J(ab) \leq J(cd)$, or, equivalently, $\mathrm{co}J(ab) \leq \mathrm{co}J(cd)$.

Theorem 3. For any $ab, cd, ef \in H$ we have:
1) $[cd]^J \leq [ef]^J$ iff $cd \leq J(ef)$,
2) $[ab]^J \leq [cd]^J$ iff $\mathrm{co}J(ab) \leq cd$.

We can put that in a diagram,

$$\begin{tikzcd}
{[ef]^J} \arrow[r, mapsto, "\sup"] & J(ef) \\
{[cd]^J} \arrow[u] \arrow[ur, phantom, "\leftrightarrow"] & cd \arrow[l, mapsto] \arrow[u] \\
{[ab]^J} \arrow[u] \arrow[r, mapsto, "\inf"'] \arrow[ur, phantom, "\leftrightarrow"] & \mathrm{co}J(ab) \arrow[u]
\end{tikzcd}$$

that can be read as a categorical statement: the functor $[\cdot]^J : H \to H/J$ has a left adjoint $\inf : H/J \to H$ and a right adjoint $\sup : H/J \to H$, where inf returns the smallest element of a J-equivalence class, and sup returns the biggest.

25 J-operators as adjunctions

The last diagram of the last section can be translated to topological language:

$$
\begin{tikzcd}
 & & U \arrow[r, mapsto, "f_*"] & \mathsf{int}(U \cup Q) \\
 & & & \\
\mathcal{O}(S) \arrow[r, shift left=2, "f_*"] \arrow[r, shift right=2, "f^!"'] & \mathcal{O}(P) \arrow[l, "f^*" description] & V \cap S \arrow[uu] \arrow[r, phantom, "\leftrightarrow"] & V \arrow[l, mapsto, "f^*"'] \arrow[uu] \\
 & & & \\
 & & W \arrow[uu] \arrow[r, mapsto, "f^!"] & \mathsf{coint}(U \backslash Q) \arrow[uu] \\
S \arrow[r, "f"] & P & S \arrow[r, "f"] & P
\end{tikzcd}
$$

The notation used in the diagram above is essentially the one from figures 6 and 7 in [?]; the "external view" is at the left,"internal view" is at the right, the adjunction is $f^! \dashv f^* \dashv f_*$, and the diagram shows that $f_*(U) = \mathsf{int}(U \cup Q)$, $f^*(V) = V \cap S$ and $f^!(W) = \mathsf{coint}(U \backslash Q)$ (where int and coint use the topology $\mathcal{O}(P)$).

The order in which things are constructed in the diagram above is different from last section, though. Now we start with a finite set P, a topology $\mathcal{O}(P)$, and a subset $S \subseteq P$, and we define $\mathcal{O}(S)$ by restriction:

$$\mathcal{O}(S) \quad = \quad \{\, V \cap S \mid V \in \mathcal{O}(P) \,\}$$

we define Q as $P \backslash S$, we let $f : S \to P$ be the inclusion and $f^*(V)$ be $V \cap S$; then *it turns out* (theorem!) that the $f^!$ and f_* as defined above are the left and the right adjoints of f^* — and J and co$J$ are built from $f^!$, f^* and f_*: the definitions

$$
\begin{array}{rcl}
J(V) & = & f_*(f^*(V)) \\
\mathrm{co}J(V) & = & f^!(f^*(V))
\end{array}
$$

yield a J-operator $J : \mathcal{O}(P) \to \mathcal{O}(P)$ and its 'co' version, that returns the smallest element in each equivalence class; and if $\mathcal{O}(P) = \mathcal{O}_A(P)$ for some 2CG (P, A), then we can define J and co$J$ in this other way,

$$
\begin{array}{rcl}
J(cd) & = & \mathsf{pile}^{-1}(f_*(f^*(\mathsf{pile}(cd)))) \\
\mathrm{co}J(cd) & = & \mathsf{pile}^{-1}(f^!(f^*(\mathsf{pile}(cd))))
\end{array}
$$

that yields a J-operator (and its 'co' version) on the ZHA H generated by the 2CG (P, A).

This "topological version" of the adjunction is a nice concrete starting point for understanding toposes and geometric morphisms between them — or, actually, for introducing geometric morphisms to "children" who prefer to start with finite examples in which everything can be calculated explicitly. The toposes involved are $\mathbf{Set}^{\mathcal{O}(S)^{\mathrm{op}}}$ and $\mathbf{Set}^{\mathcal{O}(P)^{\mathrm{op}}}$, and the adjunction $f^! \dashv f^* \dashv f_*$ presented above acts only on the subobjects of the terminal of each topos — it needs to be extended to an (essential) geometric morphism between these toposes. This, and several concepts from section A4 of [**?**], will be treated in a sequel of this paper, in a joint work with Peter Arndt.

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