Planar Heyting Algebras for Children 3: Visualizing Geometric Morphisms

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Warning: this is a *very preliminary* working draft!!!
Links:
Parts 1, 2, and 3 (this one) in this series of papers:
http://angg.twu.net/math-b.html#zhas-for-children-2
A workshop very much related to this:
http://angg.twu.net/logic-for-children-2018.html
Notes on notation: Elephant
http://angg.twu.net/LATEX/2017elephant.pdf
http://angg.twu.net/math-b.html#notes-on-notation

1 "Children" and "adults"

Different people have different ways of remembering theorems. A person with a very visual mind may remember a theorem in Category Theory mainly by the shape of a diagram and the order in which its objects are constructed. For such a person most books on Category Theory feel as if they have lots of missing diagrams, that she has to reconstruct if she wants to understand the subject.

The shape of a categorical diagram remains the same if we specialize it to a particular case — and this means that we can sometimes remember a general diagram, and the theorems associated to it, from the diagram of a particular case. A diagram for a particular case becomes a skeleton, or a scaffold, from which we reconstruct mentally a general statement when we need it.

In this paper we will use "missing diagrams", "particular cases", and a third technique called "internal views", to show how to obtain a version "for children" of some theorems about topos (actually about factorizations of geometrical morphisms between toposes). Our aim on a higher level, though, is to sketch some techniques for doing "maths for children" and "maths for adults" in parallel, with tools to creating a — somewhat faulty, but still useful — bridge between the two languages.

Let's *define* what we mean by "children" and "adults". "Children" means (for us!!!) "people without mathematical maturity", which in its turn means people who:

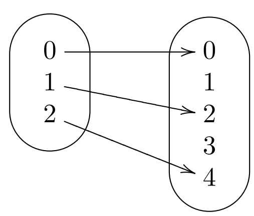
- have trouble with very abstract definitions,
- prefer starting from particular cases (and then generalize),
- handle diagrams better than algebraic notations,
- like to use diagrams and analogies,
- like to work with finite cases where all calculations can be done explicitly "by brute force".

People who prefer to always work on a very high-level abstract language, and who frown on working out examples in details, are "adults", or "mathematicians".

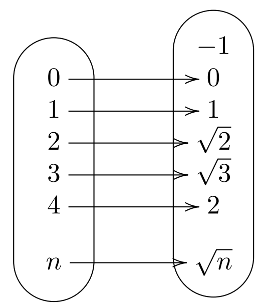
With these terms, categorical definitions are "for adults", because they may be very abstract, and particular cases, diagrams, and analogies are "for children". "Children" are willing to use "tools for children" to do mathematics, *even if they will have to translate everything to a language "for adults" to make their results dependable and publishable*, and even if the bridge between their tools "for children" and "for adults" is somewhat defective, i.e., if the translation only works on simple cases...

2 Internal view

When I was a kid my first exposure to functions was through diagrams like this:



after a while — actually years — the blob-sets got names, like A, B, $\mathbb{N}$, $\mathbb{R}$, the functions got names like f, g, $\sqrt{\ }$, and several conventions were established: we didn't have to draw all elements in the blob-sets; we could draw a "generic element", n, and indicate that it goes to $\sqrt{n}$; and we could draw an "external view” of the function above or below the “internal view” given by the blobs:



$$\mathbb{N} \xrightarrow{\sqrt{\ }} \mathbb{R}$$

Then the internal view gradually disappeared from our mathematical practice, and we started to write functions like this,

$$\begin{array}{rccc} \sqrt{\ }: & \mathbb{N} & \to & \mathbb{R} \\ & n & \mapsto & \sqrt{n} \end{array}, \qquad \begin{array}{rccc} f: & A & \to & B \\ & a & \mapsto & f(a) \end{array},$$

$$\begin{array}{rccc} \cdot: & C \times D & \to & E \\ & (c,d) & \mapsto & c \cdot d \end{array}$$

which inspires a distinction between the tailless arrow,‘$\to$’, and the arrow with tail,‘$\mapsto$’: $f : A \to B$ is a function that takes elements (plural!) from A to elements of B, and $n \mapsto \sqrt{(n)}$ is an element (in the singular) being taken to another. Rewriting our diagram for the internal and the external views of “$\sqrt{\ }$” without blobs, it becomes:

$$\begin{array}{c} 4 \xmapsto{\sqrt{\ }} 2 \\ n \xmapsto{\sqrt{\ }} \sqrt{n} \\ \\ \mathbb{N} \xrightarrow{\sqrt{\ }} \mathbb{R} \end{array}, \quad \text{or simply:} \qquad \begin{array}{c} n \longmapsto \sqrt{n} \\ \\ \mathbb{N} \xrightarrow{\sqrt{\ }} \mathbb{R} \end{array}$$

We will often use the convention that $f : A \to B$ is a function from A to B, but $A \to B$ is the set of all functions from A to B — i.e., $(A \to B) = B^A$ and $f : A \to B$ means $f \in (A \to B)$ — on ‘$\mapsto$’s this doesn’t hold, and the names on ‘$\mapsto$’s can be omitted.

The internal view of a functor $F : \mathbf{A} \to \mathbf{C}$ is more complex. The category $\mathbf{A}$ has not only “points” (the objects of $\mathbf{A}$) but also ‘arrows” (the morphisms of $\mathbf{A}$). The functor F takes a morphism $g : A \to B$ in $\mathbf{A}$ to a morphism $Fg : FA \to FB$ in $\mathbf{B}$; and sometimes we will denote the action of F on objects by F_0 and its action on morphisms by F_1, so a diagram with the internal and the external views of F may be drawn, for example, as:

$$\begin{tikzcd} A \arrow[r, mapsto, "F_0"] \arrow[d, "g"'] & F_0A \arrow[d, "F_1g"] \\ B \arrow[r, mapsto, "F_0"] & F_0B \\ \mathbf{A} \arrow[r, "F"] & \mathbf{C} \end{tikzcd} \quad \text{or as:} \quad \begin{tikzcd} A \arrow[r, mapsto] \arrow[d, "g"'] & FA \arrow[d, "Fg"] \\ B \arrow[r, mapsto] & FB \\ \mathbf{A} \arrow[r, "F"] & \mathbf{C} \end{tikzcd}$$

Note that the diagram at the right above is very similar to the one in p.17 of Emily Riehl's *Category Theory in Context*:

`http://www.math.jhu.edu/~eriehl/context.pdf`

I took the idea of an "internal diagram" from Lawvere and Schanuel's *Conceptual Mathematics* (p.13), and used it a lot in my *Internal Diagrams and Archetypal Reasoning in Category Theory*, available at:

`http://angg.twu.net/math-b.html#idarct`

`http://angg.twu.net/LATEX/idarct-preprint.pdf`

3 Particular cases

4 ZCategories

5 ZFunctors