A skeleton for the proof of the Yoneda Lemma (working draft)

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Abstract

These notes consists of five parts.

The first part explains how to draw the "internal view" of a diagram (or of a function, functor, natural transformation, etc).

The second part shows that a certain diagram, that we call diagram Y0, is the "skeleton" of the proof of the Yoneda Lemma in the following sense. In order to interpret that diagram formally we have to infer the types of all its entities, and then infer (by "term inference", as in [Och13], obtaining untyped λ-terms) the actions on morphisms of the four functors in Y0, and also the actions of the four natural transformations and the actions of three bijections. The bijections are called B_1, B_2 and B_3, where B_1 is easy to construct, B_2 is obtained from B_1 by substituing a generic functor and a generic object that appear in B_1 by specific ones, and B_3 is B_2 composed with two trivial bijections, one at each side. The statement of the Yoneda Lemma is essentially just "B_3 is a bijection". In Category Theory texts above a certain level most term inferences are treated as "obvious", so a (skeleton of a) proof of the Yoneda Lemma is just diagram Y0 plus "do the obvious type inferences and term inferences".

The third part discusses a gap in the second part. The "bijection" B_3 converts a map $f \in \mathrm{Hom}_{\mathbf{C}}(B, C)$ into a natural transformation $T'' \in \mathrm{Nat}((C, -), (B, -))$ and a T'' into an f, but what we got in the second part is just a pair of λ-terms of the right types, (B_3, B_3^{-1}), without the proofs that $B_3^{-1}(B_3(f)) = f$ and that $B_3(B_3^{-1}(T'')) = T''$. In the language of [Och13] what we did was to drop, or erase, a lot of information (mainly the "equational parts") and then work in the "syntactical world"; we obtained a "skeleton of a proof" that must now must be "lifted" to the "real world" by completing some missing parts. It turns out that $B_3^{-1}(B_3(f)) = f$ is trivial, but $B_3(B_3^{-1}(T'')) = T''$ only holds if T'' obeys the "naturality condition" that comes from it being a natural transformation. The moral of the story so far is that 90% of the proof of the Yoneda Lemma can be extracted from diagram Y0 if we do the "obvious" type and term inferences on it ("for some value of 90%", of course); only a tiny part of the proof needs things that get erased in the passage to the skeleton.

The fourth part uses these tools to state and prove three other "Yoneda Lemmas" and to define universal arrows, universal elements, representable

functors, and to show how some of these ideas are motivated by adjunctions.

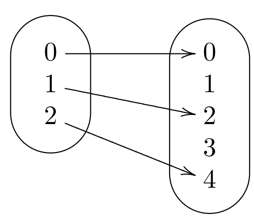
The fifth part uses all this to build "bridges" between several notations. The less trivial case is how to translate between our notation and the one in Reyes, Reyes and Zolfaghari's *Generic Figures and Their Glueings*; the translations between our notation and MacLane's, Riehl's and Awodey's are easy (but only RRZ has been written in details at the moment).

Index of sections:

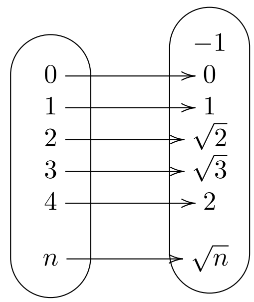
1 Internal views

Note: this section is an *introduction* to the idea of "internal views" of categorical diagrams. I first saw this idea in [LS97], p.13, but it is used in other places too — for example in p.17 of [Rie16]. I used it a lot in [Och13], but there I insisted on a notion of "downcasing" that I've since abandoned.

When I was a kid my first exposure to functions was through diagrams like this:



after a while — actually years — the blob-sets got names, like A, B, $\mathbb{N}$, $\mathbb{R}$, the functions got names like f, g, $\sqrt{\ }$, and several conventions were established: we didn't have to draw all elements in the blob-sets; we could draw a "generic element", n, and indicate that it goes to $\sqrt{n}$; and we could draw an "external view" of the function above or below the "internal view" given by the blobs:



$$\mathbb{N} \xrightarrow{\sqrt{\ }} \mathbb{R}$$

Then the internal view gradually disappeared from our mathematical practice, and we started to write functions like this,

$$\begin{array}{rrcl} \sqrt{\ }: & \mathbb{N} & \to & \mathbb{R} \\ & n & \mapsto & \sqrt{n} \end{array}, \qquad \begin{array}{rrcl} f: & A & \to & B \\ & a & \mapsto & f(a) \end{array},$$

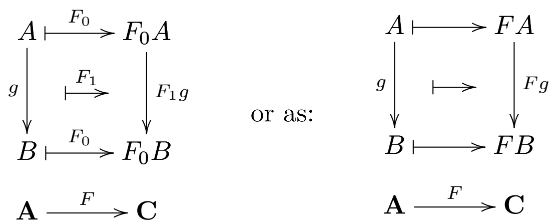
$$\begin{array}{rrcl} \cdot: & C \times D & \to & E \\ & (c,d) & \mapsto & c \cdot d \end{array}$$

which makes a clear distinction between the tailless arrow, '$\to$', and the arrow with tail, '$\mapsto$': $f : A \to B$ is a function that takes elements (plural!) from A to elements of B, and $n \mapsto \sqrt{(n)}$ is an element (in the singular) being taken to another. Rewriting our diagram for the internal and the external views of "$\sqrt{\ }$" without blobs, it becomes:

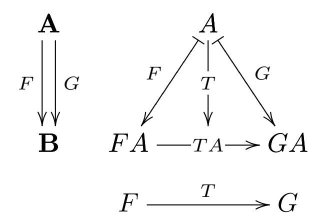
$$\begin{array}{c} 4 \overset{\sqrt{\ }}{\longmapsto} 2 \\ n \overset{\sqrt{\ }}{\longmapsto} \sqrt{n} \\ \\ \mathbb{N} \xrightarrow{\sqrt{\ }} \mathbb{R} \end{array}, \quad \text{or simply:} \quad \begin{array}{c} n \longmapsto \sqrt{n} \\ \\ \mathbb{N} \xrightarrow{\sqrt{\ }} \mathbb{R} \end{array}$$

We will often use the convention that $f : A \to B$ is a function from A to B, but $A \to B$ is the set of all functions from A to B — i.e., $(A \to B) = B^A$ and $f : A \to B$ means $f \in (A \to B)$ — on '$\mapsto$'s this doesn't hold, and the names on '$\mapsto$'s can be omitted.

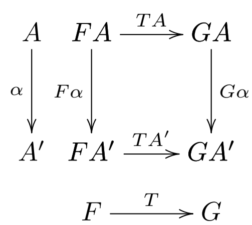
The internal view of a functor $F : \mathbf{A} \to \mathbf{C}$ is more complex. The category $\mathbf{A}$ has not only "points" (the objects of $\mathbf{A}$) but also "arrows" (the morphisms of **A**). The functor F takes a morphism $g : A \to B$ in **A** to a morphism $Fg : FA \to FB$ in **C**; and sometimes we will denote the action of F on objects by F_0 and its action on morphisms by F_1, so a diagram with the internal and the external views of F may be drawn, for example, as:



The "action" of a natural transformation $T : F \to G$, where $F, G : \mathbf{A} \to \mathbf{B}$ are functors, consists of a single operation — not two as in functors — that expects an object $A \in \mathbf{A}$ and returns a morphism $TA : FA \to GA$ in **B**. We can represent that action as $A \mapsto (TA : FA \to GA)$ or $A \mapsto (FA \overset{TA}{\to} GA)$, or as a diagram:

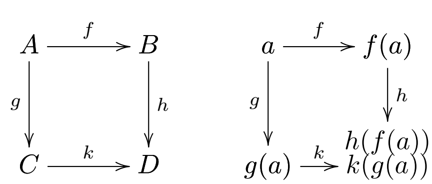


The "naturality condition" of a natural transformation $T : F \to G$ is the assurance that for every arrow $\alpha : A \to A'$ in **A** this square commutes:



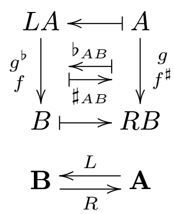
Diagrams like the one above will be our favorite ways to draw internal views of natural transformations. Note that the arrows for the functors F and G are left implicit.

We will sometimes use diagrams like this to show the internal view of a commutative diagram, especially when it is in **Set**:



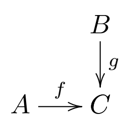
the internal view shows that $h(f(a)) = k(g(a))$ for every $a \in A$.

Our favorite way to choose names for the components of an adjunction and to draw its internal view is this:

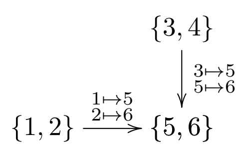


The adjunction $L \dashv R$ "is" a bijection $\mathrm{Hom}_{\mathbf{B}}(LA, B) \underset{\sharp_{AB}}{\overset{\flat_{AB}}{\rightleftarrows}} \mathrm{Hom}_{\mathbf{A}}(A, RB)$ for each A in $\mathbf{A}$ and each B in $\mathbf{B}$. Note that the functor L appears at the left of the '$\dashv$' and of the ',', and it goes left; the functor R appears at the right of the '$\dashv$' and of the ',', and it goes right; the direction '$\flat$' of the bijection goes left in the diagram, and it pulls the functor in $g : A \to RB$ to the left of the '$\to$'; the direction '$\sharp$' of the bijection goes right in the diagram and pushes the functor in $f : LA \to B$ to the right of the '$\to$'.

When $\mathbf{C}$ is a finite category that can be drawn explicitly, like this,



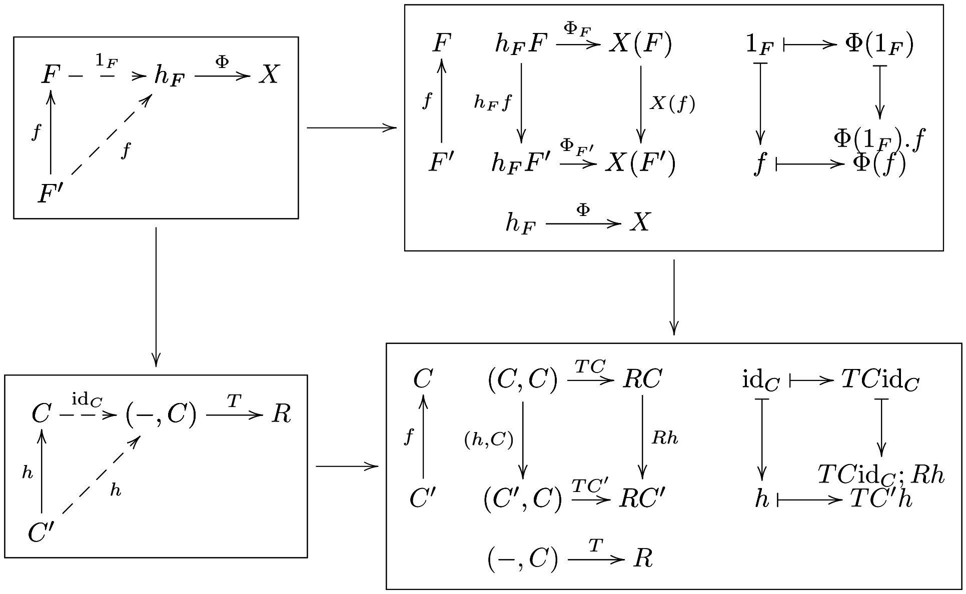
we can represent functors from $\mathbf{C}$ to other categories very compactly using a positional notation similar to the ones in sec.1 of [Och17]. For example, this diagram



can be intepreted as a functor $F : \mathbf{C} \to \mathbf{Set}$ with $F(A) = \{1, 2\}$, $F(f) = \{(1, 5), (2, 6)\}$ and so on — we *define* F by the internal view of its image.

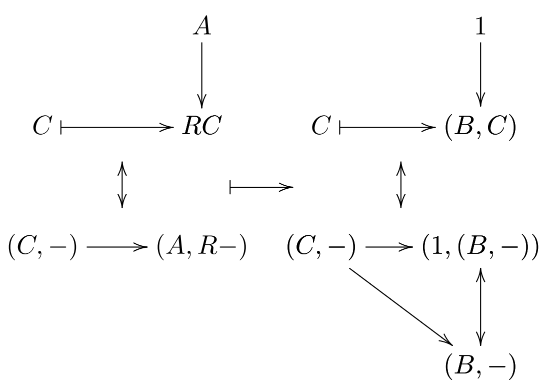
2 Changing shape, changing notation

The translation between two languages for diagrams can be done in two steps — changing shape and changing notation — that are independent and can be applied in any order. In the example below we start with a diagram from [RRZ04] (see sec.12.2) at the top left; moving right changes its shape to show the internal view of its natural transformation, and moving down changes its notation to the one in sec.4:

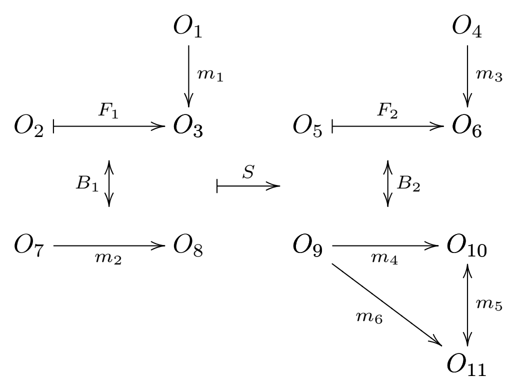


3 Interpreting diagrams Y0, Y1, and Y2

My favorite diagram for remembering the *proof* of (one of the forms of) the Yoneda Lemma is this one ("diagram Y0"):



It is made of 11 objects in different categories, 6 morphisms, two functors, two bijections, and a middle arrow that performs some substitutions on the first bijection to obtain the second one. Let's name (or "number") all of them:



The existence of a morphism $O_1 \overset{m_1}{\to} O_3$ tells us that O_1 and O_3 belong to the same category; as $O_1 = A$ let's call that category $\mathbf{A}$. Similarly, $O_2 = O_5 = C$, so O_2 and O_5 belong to a category that we will call $\mathbf{C}$. O_4 and O_6 belong to the same category, and $O_4 = 1$, which is an object of $\mathbf{Set}$, so O_4 and O_6 are objects of $\mathbf{Set}$. Similarly, $O_7 = O_9 = (C, -)$, so O_7, O_8, O_9, O_{10}, O_{11} all belong to the same category. The functor $F_1 = R$ goes from $\mathbf{C}$ to $\mathbf{A}$ and the functor F_2, that will turn out to be $(B, -)$, goes from $\mathbf{C}$ to $\mathbf{Set}$.

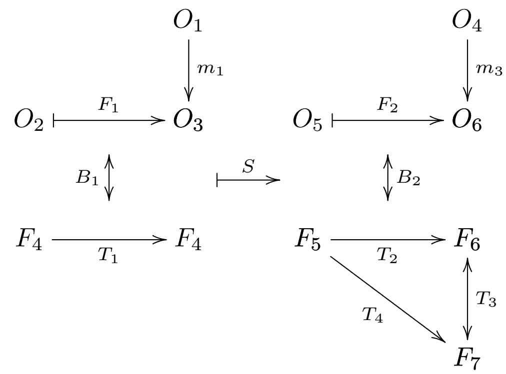
(B, C) is a shorthand for $\mathrm{Hom}_{\mathbf{C}}(B, C)$; the two objects B and C have to belong to the same category so B is an object of $\mathbf{C}$. $(C, -)$ is a shorthand for the functor $\mathrm{Hom}_{\mathbf{C}}(C, -)$, which goes from $\mathbf{C}$ to $\mathbf{Set}$ (obs: $\mathbf{C}$ has to be locally small). $(C, -)$ is an object of the category of functors from $\mathbf{C}$ to $\mathbf{Set}$, and O_7 to O_{11}, so:

$$\begin{array}{rrrr}
C : \mathbf{C} & A : \mathbf{A} & 1 : \mathbf{Set} & (C, -) : \mathbf{C} \to \mathbf{Set} \\
B : \mathbf{C} & RC : \mathbf{A} & (B, C) : \mathbf{Set} & (B, -) : \mathbf{C} \to \mathbf{Set} \\
 & R : \mathbf{C} \to \mathbf{A} & & (A, R-) : \mathbf{C} \to \mathbf{Set} \\
 & & & (1, (B, -)) : \mathbf{C} \to \mathbf{Set}
\end{array}$$

$(A, R-)$ is a shorthand for $\mathrm{Hom}_{\mathbf{A}}(A, R-) : \mathbf{C} \to \mathbf{Set}$, $(B, -)$ for $\mathrm{Hom}_{\mathbf{C}}(B, -) : \mathbf{C} \to \mathbf{Set}$, and $(1, (B, -))$ for $\mathrm{Hom}_{\mathbf{Set}}(1, \mathrm{Hom}(\mathbf{C}(B, -)))$. O_7 to O_{11} are all functors from $\mathbf{C}$ to $\mathbf{Set}$ and so objects of the category $\mathbf{Set}^{\mathbf{C}}$, and the morphisms m_3, m_4, m_5, m_6, are natural transformations; m_5 is a natural isomorphism.

If we indicate in the diagram that O_7 to O_{11} are functors and m_3 to m_6 are

NTs, we get:

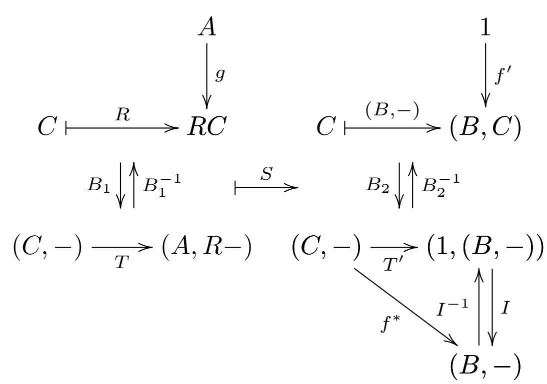


Warning: the bijection B_1 is between m_1 and T_1, not between F_1 and T_1, even though we draw it vertically; similarly, the bijection B_2 is between m_2 and T_2. The reason for drawing the diagram in this way instead of making O_1 and O_2 switch places with one another and doing the same with O_4 and O_5 will be explained later.

The arrow S is a substitution that produces B_2 from B_1. It's better to write it in a notation for simultaneous substitutions, not in λ-calculus notation:

$$S = \begin{bmatrix} R := (B, -) \\ \mathbf{A} := \mathbf{Set} \\ A := 1 \end{bmatrix}$$

Now that we have typed most objects in diagram Y0 let's go back to the original notation, and give names to some arrows. This is diagram Y1:



The next step is to define precisely how the four functors work. We can do that by drawing internal views:

$$\begin{array}{ccc} C' & \longmapsto & (C,C') \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \lambda h.(h;k)} \\ C'' & \longmapsto & (C,C'') \end{array} \qquad \begin{array}{ccc} C' & \mapsto & (A,RC')) \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \lambda g'.(g';Rk)} \\ C'' & \mapsto & (A,RC'')) \end{array}$$

$$\mathbf{C} \xrightarrow{(C,-)} \mathbf{Set} \qquad \mathbf{C} \xrightarrow{(A,R-)} \mathbf{Set}$$

$$\begin{array}{ccc} C & \longmapsto & (B,C) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \lambda f.(f;g)} \\ C' & \longmapsto & (B,C') \end{array} \qquad \begin{array}{ccc} C & \mapsto & (1,(B,C)) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \lambda f'.\lambda e.(f'(e);g)} \\ C' & \mapsto & (1,(B,C')) \end{array}$$

$$\mathbf{C} \xrightarrow{(B,-)} \mathbf{Set} \qquad \mathbf{C} \xrightarrow{(1,(B,-))} \mathbf{Set}$$

The actions of the functors $(C,-)$, $(B,-)$, and $(A,R-)$ can be inferred by term inference or by looking at the diagrams below:

$$\begin{array}{c} C \\ {\scriptstyle h}\downarrow \\ C' \\ {\scriptstyle k}\downarrow \\ C'' \end{array} \Bigg\downarrow{\scriptstyle h;k} \qquad \begin{array}{c} B \\ {\scriptstyle h}\downarrow \\ C \\ {\scriptstyle k}\downarrow \\ C' \end{array} \Bigg\downarrow{\scriptstyle h;k} \qquad \begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle g'} \\ C & \longmapsto & RC' \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle Rk} \\ C' & \longmapsto & RC'' \end{array} \Bigg\downarrow{\scriptstyle g';Rk}$$

and the action of $(1,(B,-))$ is a variant of $(B,-)$. We get:

$$\begin{array}{rcl} (C,-)_0 & = & \lambda C'.\,\mathrm{Hom}_{\mathbf{C}}(C,C') \\ (C,-)_1 & = & \lambda k.\lambda h.(h;k) \\ (B,-)_0 & = & \lambda C.\,\mathrm{Hom}_{\mathbf{C}}(B,C) \\ (B,-)_1 & = & \lambda g.\lambda f.(f;g) \\ (A,R-)_0 & = & \lambda C'.\,\mathrm{Hom}_{\mathbf{A}}(A,RC') \\ (A,R-)_1 & = & \lambda k.\lambda g'.(g';Rk) \\ (1,(B,-))_0 & = & \lambda C'.\,\mathrm{Hom}_{\mathbf{Set}}(1,\mathrm{Hom}_{\mathbf{C}}(B,C')) \\ (1,(B,-))_1 & = & \lambda k.\lambda g'.(g';Rk) \end{array}$$

We can do the same for the natural transformations.

$$\begin{array}{c} (C,C') \xrightarrow{\lambda h.(g;Rh)} (A,RC') \\ (C,-) \xrightarrow{\quad T \quad} (A,R-) \end{array} \qquad \begin{array}{c} (C,C') \xrightarrow{\lambda h.\lambda e.(f;h)} (1,(B,C')) \\ (C,-) \xrightarrow{\quad T' \quad} (1,(B,-)) \end{array} \qquad \begin{array}{c} (1,(B,C')) \xrightarrow{\lambda f'.f'(e)} (B,C') \\ (1,(B,-)) \xrightarrow{\quad I \quad} (B,-) \end{array}$$

$$\begin{array}{c} (C,C') \xrightarrow{\lambda h.(f;h)} (B,C') \\ (C,-) \xrightarrow{\quad f^* \quad} (B,-) \end{array} \qquad \begin{array}{c} (B,C') \xrightarrow{\lambda f.\lambda e.f} (1,(B,C')) \\ (B,-) \xrightarrow{\quad I^{-1} \quad} (1,(B,-)) \end{array}$$

We get:

$$\begin{array}{rcl} T &=& \lambda C'.\lambda h.(g;Rh) \\ T' &=& \lambda C'.\lambda h.\lambda e.(f;h) \\ f^* &=& \lambda C'.\lambda h.(f;h) \\ I &=& \lambda C'.\lambda f'.f'(e) \\ I^{-1} &=& \lambda C'.\lambda f.\lambda e.f \end{array}$$

And we can also do the same for the bijections.

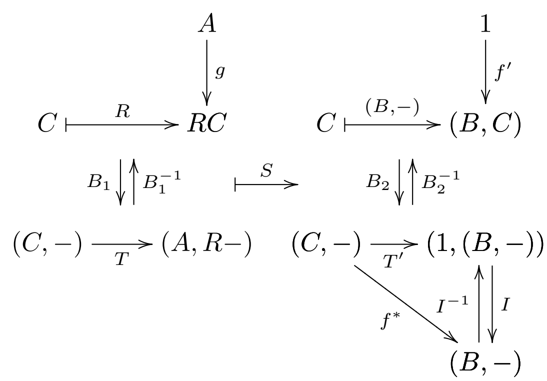
$$\begin{array}{ccc} g : A \to RC & \qquad & f' : 1 \to (B,C) \\ {\scriptstyle T := \lambda C'.\lambda h.(g;Rh)} \downarrow \uparrow {\scriptstyle g := TC(\mathrm{id}_C)} & & {\scriptstyle T := \lambda C'.\lambda h.(g;(B,-)(h))\ (?)} \downarrow \uparrow {\scriptstyle f' := T'C(\mathrm{id}_C)} \\ T : (C,-) \to (A,R-) & & T' : (C,-) \to (1,(B,-)) \end{array}$$

so:

$$\begin{array}{rcl} B_1 &=& \lambda g.\lambda C'.\lambda h.(g;Rh) \\ B_1^{-1} &=& \lambda T.TC(\mathrm{id}_C) \\ B_2 &=& \lambda f'.\lambda C'.\lambda h.(g;(B,-)(h))\ \ (?) \\ B_2^{-1} &=& \lambda T'.T'C(\mathrm{id}_C). \end{array}$$

Note that we used only type inference and term inference — which is not little, but most books and articles on CT pretend that simple type inferences and term inferences like these are "obvious" — and now have the types and the terms for everything in diagram Y1. Let's call the diagram below "diagram

Y2”; it is Y1 plus lots of information.



$$\begin{array}{rrrr} C:\mathbf{C} & A:\mathbf{A} & 1:\mathbf{Set} & (C,-):\mathbf{C}\to\mathbf{Set} \\ B:\mathbf{C} & RC:\mathbf{A} & (B,C):\mathbf{Set} & (B,-):\mathbf{C}\to\mathbf{Set} \\ & R:\mathbf{C}\to\mathbf{A} & & (A,R-):\mathbf{C}\to\mathbf{Set} \\ & & & (1,(B,-)):\mathbf{C}\to\mathbf{Set} \end{array}$$

$$\begin{array}{rcl} (C,-)_0 &=& \lambda C'.\,\mathrm{Hom}_{\mathbf{C}}(C,C') \\ (C,-)_1 &=& \lambda k.\lambda h.(h;k) \\ (B,-)_0 &=& \lambda C.\,\mathrm{Hom}_{\mathbf{C}}(B,C) \\ (B,-)_1 &=& \lambda g.\lambda f.(f;g) \\ (A,R-)_0 &=& \lambda C'.\,\mathrm{Hom}_{\mathbf{A}}(A,RC') \\ (A,R-)_1 &=& \lambda k.\lambda g'.(g';Rk) \\ (1,(B,-))_0 &=& \lambda C'.\,\mathrm{Hom}_{\mathbf{Set}}(1,\mathrm{Hom}_{\mathbf{C}}(B,C')) \\ (1,(B,-))_1 &=& \lambda k.\lambda g'.(g';Rk) \end{array}$$

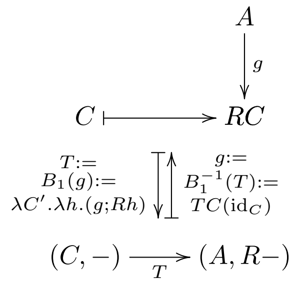
$$\begin{array}{rcl} T &=& \lambda C'.\lambda h.(g;Rh) \\ T' &=& \lambda C'.\lambda h.\lambda e.(f;h) \\ f^* &=& \lambda C'.\lambda h.(f;h) \\ I &=& \lambda C'.\lambda f'.f'(e) \\ I^{-1} &=& \lambda C'.\lambda f.\lambda e.f \end{array}$$

$$S = \left[\begin{smallmatrix} R:=(B,-) \\ \mathbf{A}:=\mathbf{Set} \\ A:=1 \end{smallmatrix}\right]$$

$$\begin{array}{rcl} B_1 &=& \lambda g.\lambda C'.\lambda h.(g;Rh) \\ B_1^{-1} &=& \lambda T.TC(\mathrm{id}_C) \\ B_2 &=& \lambda f'.\lambda C'.\lambda h.(g;(B,-)(h)) \ \ (?) \\ B_2^{-1} &=& \lambda T'.T'C(\mathrm{id}_C) \end{array}$$

4 B_1 is really a bijection

In this diagram, that is just a part of diagram Y1 with the bijection B_1 made more explicit,



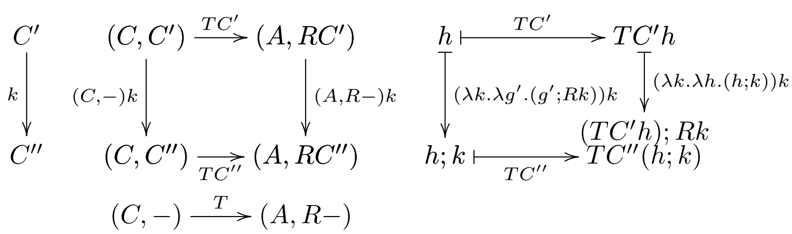
it is easy to see that $B_1^{-1}(B_1(g)) = g$:

$$\begin{array}{rcl} B_1^{-1}(B_1(g)) & = & B_1^{-1}(\lambda C'.\lambda h.(g;Rh)) \\ & = & (\lambda C'.\lambda h.(g;Rh))C(\mathrm{id}_C) \\ & = & (\lambda h.(g;Rh))(\mathrm{id}_C) \\ & = & g;R(\mathrm{id}_C) \\ & = & g;\mathrm{id}_{RC} \\ & = & g \end{array}$$

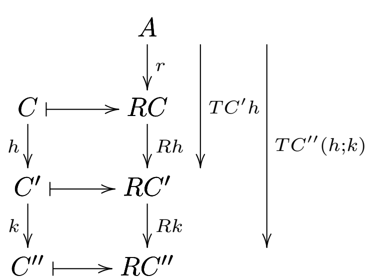
Let's try to calculate $B_1(B_1^{-1}(T))$:

$$\begin{array}{rcl} B_1(B_1^{-1}(T)) & = & B_1(TC(\mathrm{id}_C)) \\ & = & \lambda C'.\lambda h.(TC(\mathrm{id}_C);Rh) \end{array}$$

This is not necessarily equal to T... but note that if T is a natural transformation then its naturality condition means that for every $k : C' \to C''$ this square commutes,



i.e., $(TC'h);Rk = TC''(h;k)$; this diagram helps understanding the types:



If we replace $k : C' \to C''$ by $h : C \to C'$ and h by id_C we get:

$$((TC'h); Rk = TC''(h;k)) \begin{bmatrix} \scriptstyle C':=C \\ \scriptstyle C'':=C' \\ \scriptstyle k:=h \\ \scriptstyle h:=\mathrm{id}_C \end{bmatrix} \quad = \quad (TC\mathrm{id}_C; Rh = TC'(\mathrm{id}_C; h))$$

which lets us continue the calculation of $B_1(B_1^{-1}(T))$:

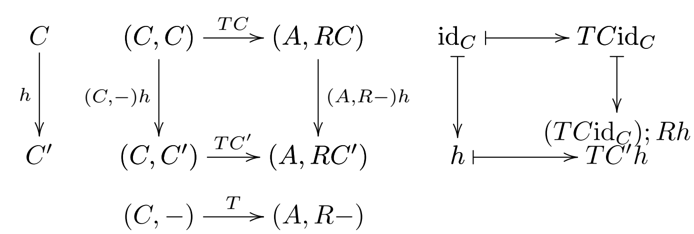
$$\begin{array}{rcl} B_1(B_1^{-1}(T)) & = & B_1(TC(\mathrm{id}_C)) \\ & = & \lambda C'.\lambda h.(TC(\mathrm{id}_C); Rh) \\ & = & \lambda C'.\lambda h.(TC'(\mathrm{id}_C; h)) \\ & = & \lambda C'.\lambda h.TC'h \end{array}$$

this means that for all C' and h we have

$$\begin{array}{rcl} B_1(B_1^{-1}(T))C'h & = & (\lambda C'.\lambda h.TC'h)C'h \\ & = & (\lambda h.TC'h)h \\ & = & TC'h \end{array}$$

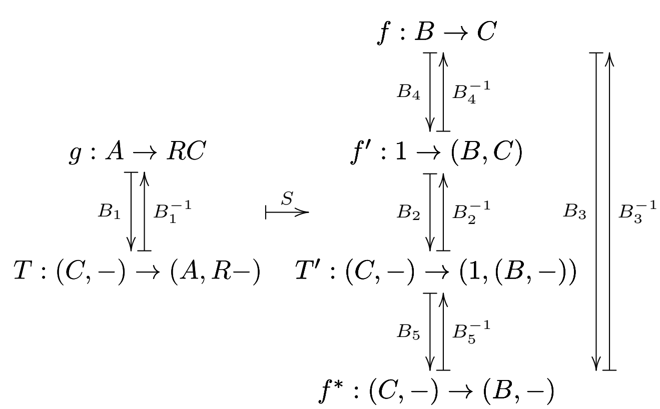
so by η-reduction $B_1(B_1^{-1}(T))C' = TC'$ and $B_1(B_1^{-1}(T)) = T$.

Note that the proof of $TC\mathrm{id}_C; Rh = TC'h$ can be represented as a diagram:

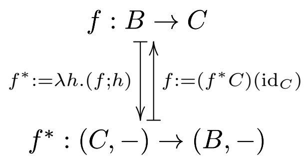


5 Making the bijections more explicit

Let's introduce a new diagram that stresses the bijections — and names a few bijections that were unnamed before. This is diagram Y3:



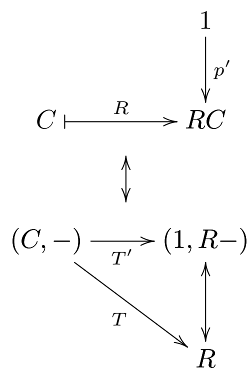
The statement of the Yoneda Lemma is just this: "B_3 is a bijection". If we build B_4 and B_5, define B_3 as $B_5 \circ B_2 \circ B_4$ and simplify the λ-terms we obtain that B_3 is just this:



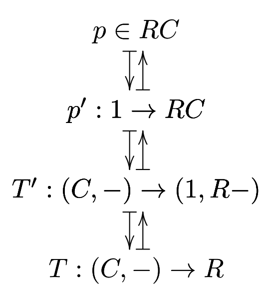
A direct proof that B_3 and B_3^{-1} are inverses to one another requires naturality like we did in section 4 (trust me!), and less direct proof can be structured like this: B_1 is a bijection implies that B_2 is a bijection, that implies that B_3 is a bijection.

6 A stronger Yoneda Lemma

If we don't replace the functor R by $(B, -)$ in Y0 and we make $\mathbf{A} := \mathbf{Set}$ and $A := 1$ we can build this diagram here ("diagram Y4"),



that yields a bijection between points of RC and natural transformations from $(C, -)$ to R ("diagram Y5"):

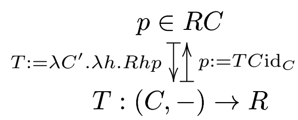


This bijection feels much more abstract than the one that we were looking at before.

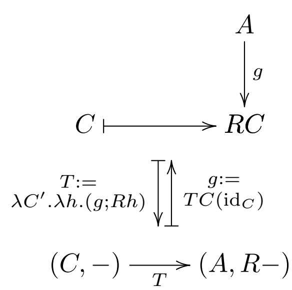
7 Representable functors

8 Universal elements and universal arrows

We say that an element $p \in RC$ is a *universal element* when the natural transformation T associated to it by diagram Y4 is a natural isomorphism, i.e., when for every C' the map $TC' = \lambda h.Rhp$ is an iso:



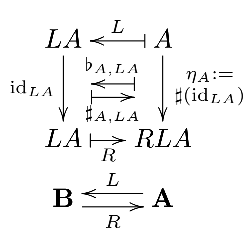
A *universal arrow* is an arrow $g : A \to RC$ such that the associated T $(= \lambda C'.\lambda h.(g; Rh))$ is a natural isomorphism:



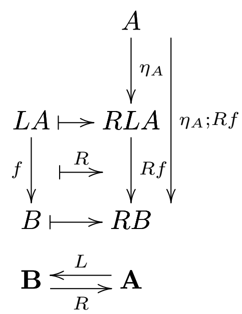
9 Adjunctions

At the end of sec.1 we presented a convention for naming the components of an adjunction and drawing its internal view, but we didn't include units or counits.

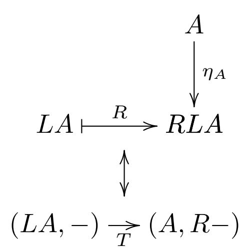
For any $A \in \mathbf{A}$ the unit map η_A, defined like this,



induces a map $(f \mapsto (\eta_A; Rf)) : (LA, B) \to (A, RB)$ that is equal to the bijection $\sharp_{AB} : (LA, B) \to (A, RB)$:

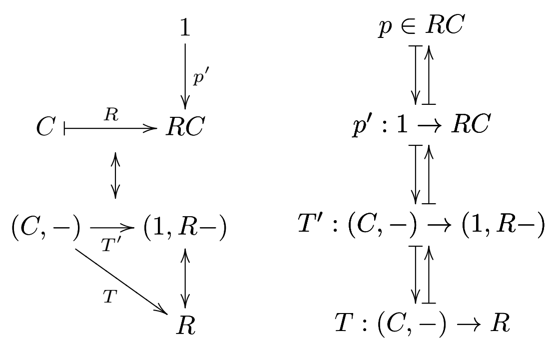


the map $(f \mapsto (\eta_A; Rf))$ is natural in B, and we can see (I'm omitting the details now) that it induces a natural transformation $T : (LA, -) \to (A, R-)$:



We are now in a situation similar to diagram Y0 — we can see that any natural transformation $T : (LA, -) \to (A, R-)$ yields a map $\eta_A : A \to RLA$ (that is not necessarily the unit of the adjuction, of course).

Now make the category $\mathbf{A}$ be $\mathbf{Set}$, and make $A := 1$ and $C := LA = L1$. Then $RLA = RL1 = RC$, and we get these diagrams:



We have a bijection between $RC = RL1$ and the set of natural transformations from $(C, -)$ to R, but we also have more: when $p' : 1 \to RL1 = RC$ is a unit map of the adjunction then the corresponding $T : (C, -) \to R$ is a *natural isomorphism*, so this functor R is representable and represented by C, the map $p' : 1 \to RC$ is a universal arrow, $p \in RC$ is a universal element. Most (or all?)

items in Examples 2.1.5 in pp.51–53 of [Rie16] are applications of this idea using adjunctions of the form $F \dashv U$ — e.g., in item (ii) the functor $F : \mathsf{Set} \to fGroup$ takes each set A to the free group FA having A as its set of generators.

(To do: debug the ideas above!)

10 Two contravariant Yoneda Lemmas

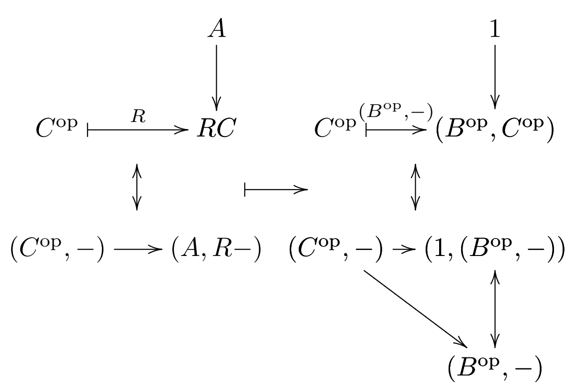
Let's introduce some notations for dealing with opposite categories. If B and C are objects of $\mathbf{C}$ then B^{op} and C^{op} are objects of $\mathbf{C}^{\text{op}}$; a morphism $f : B \to C$ in $\mathbf{C}$ is written as $f^{\text{op}} : C^{\text{op}} \to B^{\text{op}}$ when regarded as a morphism in $\mathbf{C}^{\text{op}}$. Looking at hom-sets, we have that $f \in \text{Hom}_{\mathbf{C}}(B, C)$ iff $f^{\text{op}} \in \text{Hom}_{\mathbf{C}^{\text{op}}}(C^{\text{op}}, B^{\text{op}})$, and in the shorthand notation this means that (B, C) and $(C^{\text{op}}, B^{\text{op}})$ are equal except for the 'op's in the elements of $(C^{\text{op}}, B^{\text{op}})$.

Let $G : \mathbf{C}^{\text{op}} \to \mathbf{D}$ be a contravariant functor. We will write its action on objects as $C^{\text{op}} \mapsto GG$, and the internal view of G is:

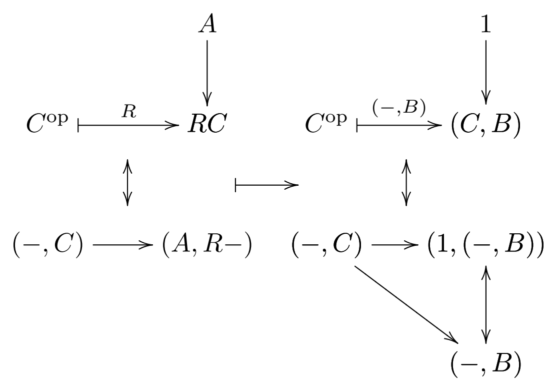
$$\begin{array}{ccccc} B & & B^{\text{op}} & \longmapsto & GB \\ {\scriptstyle f}\uparrow & & {\scriptstyle f^{\text{op}}}\downarrow & & \downarrow{\scriptstyle Gf} \\ C & & C^{\text{op}} & \longmapsto & GC \end{array}$$

$$\mathbf{C}^{\text{op}} \xrightarrow{G} \mathbf{D}$$

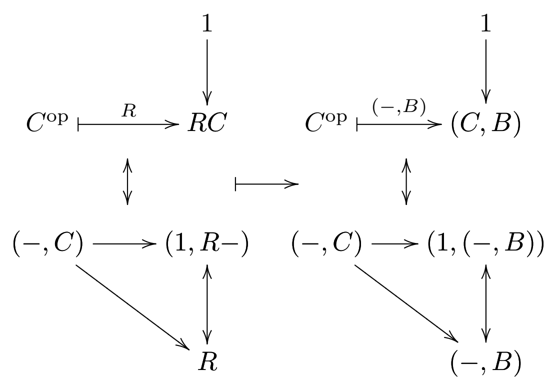
If we take Diagram Y0 and replace the category $\mathbf{C}$ by $\mathbf{C}^{\text{op}}$ we get this; note that $R : \mathbf{C}^{\text{op}} \to \mathbf{A}$:



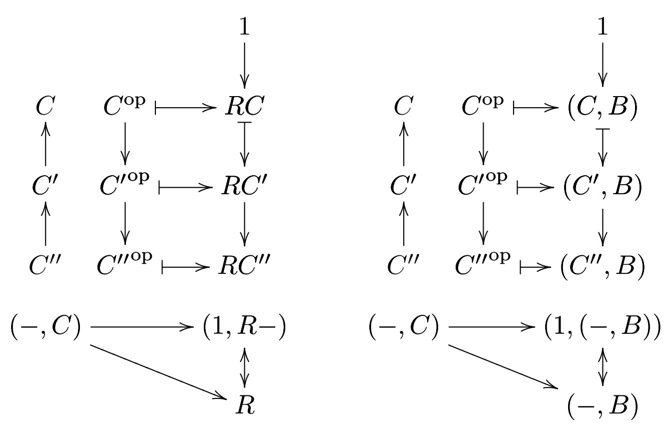
We can simplify this a bit, rewriting it as:



If we replace **A** by **Set** and A by 1 and complete the triangle at the lower left we get a single diagram that states the two contravariant Yoneda Lemmas:



The diagrams that help us understand how the functors and natural transformations above work are:



The statements of these contravariant Yoneda Lemmas are:

$$\begin{array}{ccc} & p : RC & \\ {\scriptstyle T:= \atop \lambda C'.\lambda h.(Rh)(p)} & \downarrow\uparrow & {\scriptstyle p:= \atop TC\mathrm{id}_C} \\ & T : (-,C) \to R & \end{array} \qquad \begin{array}{ccc} & f : C \to B & \\ {\scriptstyle f_*:= \atop \lambda C'.\lambda h.(h;f)} & \downarrow\uparrow & {\scriptstyle f:= \atop f_* C\mathrm{id}_C} \\ & f_* : (-,C) \to (-,B) & \end{array}$$

Note that the action of the (contravariant) functor $(-,C)$ on objects can be written $B^{\mathrm{op}} \mapsto (B,C)$; sometimes by abuse of language we will denote the whole functor $(-,C)$ by $B^{\mathrm{op}} \mapsto (B,C)$, and, similarly, denote the covariant functor $(B,-)$ used in sec.3 by $C \mapsto (B,C)$.

11 The Yoneda Embeddings

Our two "less abstract Yoneda lemmas" can be drawn like this:

$$\begin{array}{ccc} B & & (B,-) \\ {\scriptstyle f}\downarrow & \leftrightarrow & \uparrow{\scriptstyle f^*} \\ C & & (C,-) \end{array} \qquad \begin{array}{ccc} B & & (-,B) \\ {\scriptstyle f}\downarrow & \leftrightarrow & \downarrow{\scriptstyle f_*} \\ C & & (-,C) \end{array}$$

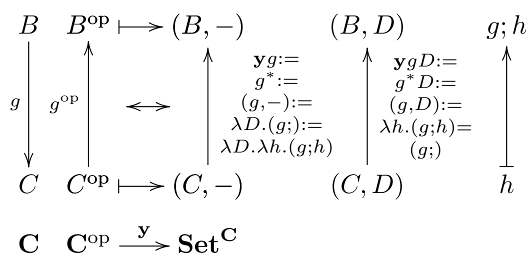
They are usually presented at a slightly higher level, as:

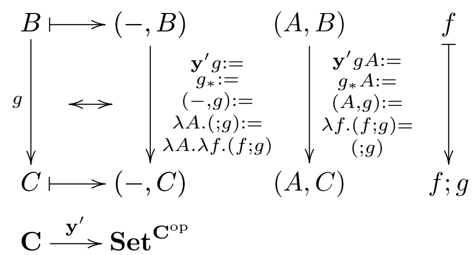
$$\begin{array}{cccc} B & B^{\mathrm{op}} & \longmapsto & (B,-) \\ {\scriptstyle f}\downarrow & {\scriptstyle f^{\mathrm{op}}}\uparrow & \leftrightarrow & \uparrow{\scriptstyle f^*} \\ C & C^{\mathrm{op}} & \longmapsto & (C,-) \\ \mathbf{C} & \mathbf{C}^{\mathrm{op}} & \xrightarrow{\mathbf{y}} & \mathbf{Set}^{\mathbf{C}} \end{array} \qquad \begin{array}{ccc} B & \longmapsto & (-,B) \\ {\scriptstyle f}\downarrow & \leftrightarrow & \downarrow{\scriptstyle f_*} \\ C & \longmapsto & (-,C) \\ \mathbf{C} & \xrightarrow{\mathbf{y}'} & \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \end{array}$$

The Yoneda Lemma says that the functors $B^{\mathrm{op}} \mapsto (B,-)$ and $B \mapsto (-,B)$ — whose short names are $\mathbf{y}$ and $\mathbf{y}'$ — are full and faithful. These functors are usually called the *Yoneda Embeddings*

If we expand the '$(B,-)$' and the '$(-,B)$' in $B^{\mathrm{op}} \mapsto (B,-)$ and $B \mapsto (-,B)$ we get $B^{\mathrm{op}} \mapsto (C \mapsto (B,C))$ and $B \mapsto (A^{\mathrm{op}} \mapsto (A,B))$, and this makes the actions of $\mathbf{y}$ and $\mathbf{y}'$ on objects very easy to understand. A trick to figure out how $\mathbf{y}$ and $\mathbf{y}'$ act on morphisms is to draw the internal views of the natural transformations g^* and g_* at the right of the diagram, and rewrite $\mathbf{y}g$ and $\mathbf{y}'g$

in several equivalent notations:

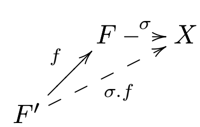




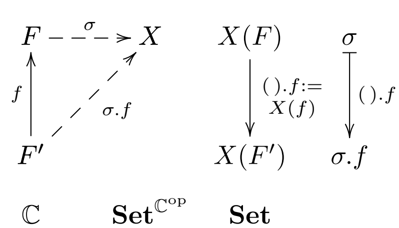
12 Reading "Generic Figures and ther Glueings"

When I first tried to read Reyes, Reyes and Zolfaghari's [RRZ04] ("RRZ" from here on) I got *very* stuck, as I didn't have any good methods to work on its notation bit by bit to make it make sense to me...

Take this diagram from page 11 of the book:



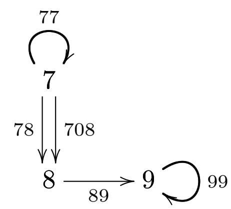
We can type its entities:



$\mathbb{C}$ is a category
$F, F' \in \mathbb{C}$
$f : F' \to F$
$X : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$
$X(F), X(F') \in \mathbf{Set}$
$\sigma \in X(F)$
$\sigma.f \in X(F')$ $\qquad \sigma.f = X(f)(\sigma)$

In sec.1 we said that we would sometimes write $A \to B$ for B^A or $\mathrm{Hom}(A, B)$; we can do something similar for '$-\!\!>$'. In RRZ $F \overset{\sigma}{-\!\!>} X$ means $\sigma \in X(F)$, so we will interpret $F -\!\!> X$ as $X(F)$ and $F \overset{\sigma}{-\!\!>} X$ as $\sigma : F -\!\!> X$.

We can make the examples in RRZ more elementary if we work with finite mathematical objects built from integers, pairs, and sets, as done in [Och17] (sec.2 and onwards). Let $\mathbf{M}$ be the directed (multi-)graph with labeled arrows ("DGLA") below:



$$\mathbf{M} = \left(\{7,8,9\}, \left\{ \begin{array}{c} (7,7,77), \\ (7,8,78), \\ (7,8,708), \\ (8,9,89), \\ (9,9,99) \end{array} \right\} \right)$$

We can set $\mathbb{C}$ to this category (the identity arrows are omitted),

$$\begin{array}{c} A \\ {\scriptstyle s}\big\uparrow\big\uparrow{\scriptstyle t} \\ V \end{array}$$

to define figures made of vertices and arrows. This functor $M : \mathbb{C}^{\text{op}} \to \mathbf{Set}$

$$\begin{array}{ccc} M(A) & & \{77, 78, 708, 89, 99\} \\ {\scriptstyle M(s)}\big\downarrow\big\downarrow{\scriptstyle M(t)} & & {\scriptstyle \begin{array}{c} 77\mapsto 7 \\ 78\mapsto 7 \\ 708\mapsto 7 \\ 89\mapsto 8 \\ 99\mapsto 9 \end{array}}\big\downarrow\big\downarrow{\scriptstyle \begin{array}{c} 77\mapsto 7 \\ 78\mapsto 8 \\ 708\mapsto 8 \\ 89\mapsto 9 \\ 99\mapsto 9 \end{array}} \\ M(V) & & \{7, 8, 9\} \end{array}$$

"is" the DGLA $\mathbf{M}$ above.

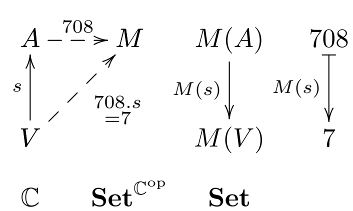
I am not sure what this notation means when it appears in RRZ:

$$\frac{V - \succ X}{a, b, c, d, e}$$

It may be either "$a, b, c, d, e : V - \succ X$" or "$(V - \succ X) = \{a, b, c, d, e\}$"... anyway, in M we have:

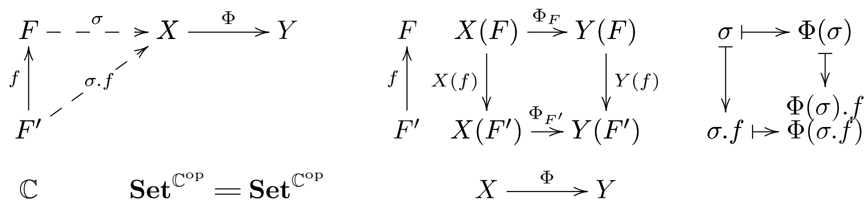
$$\frac{V - \succ M}{7, 8, 9} \qquad \frac{A - \succ M}{77, 78, 708, 89, 99}$$

And this is a change of figure:



12.1 Morphisms of $\mathbb{C}$-sets

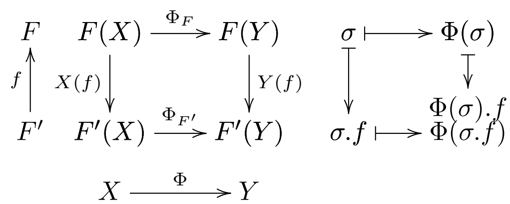
A *Morphisms of* $\mathbb{C}$*-sets* $\Phi : X \to Y$ (see p.11 of RZZ) is a natural transformation from X to Y, where both X and Y are $\mathbb{C}$-sets, i.e., $X, Y : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$. The figure at left below appears in p.11 of RRZ except for the last line with the the category annotations; at the right of it is an internal view, in RRZ's notation, of the natural transformation Φ:



Here is a type inference for the subexpressions of the "naturality condition" $\Phi(\sigma).f = \Phi(\sigma.f)$:

$$\underbrace{\underbrace{\underbrace{\Phi}_{:X\to Y}(\underbrace{\sigma}_{\in X(F)})}_{\in Y(f)} \;.\; \underbrace{\underbrace{f}_{:F'\to F}}_{:Y(F)\to Y(F')}}_{\in Y(F')} = \underbrace{\underbrace{\Phi}_{:X\to Y}(\underbrace{\underbrace{\sigma}_{\in X(F)} \;.\; \underbrace{f}_{:F'\to F}}_{\in X(F')})}_{\in Y(F')}$$

One difficulty in translating $\Phi(\sigma).f = \Phi(\sigma.f)$ to standard notation is that the arguments to the dot operation are "in the wrong order". If we rewrite $\sigma.f$ as $(.f)(\sigma)$ then the naturality condition becomes $(.f)(\Phi(\sigma)) = \Phi((.f)(\sigma))$, i.e., $(.f) \circ \Phi = \Phi \circ (.f)$, and the easiest way (for me) to understand the types is to write first $f : F' \to F$ and $\Phi : X \to Y$ and then all the rest in the diagram below:



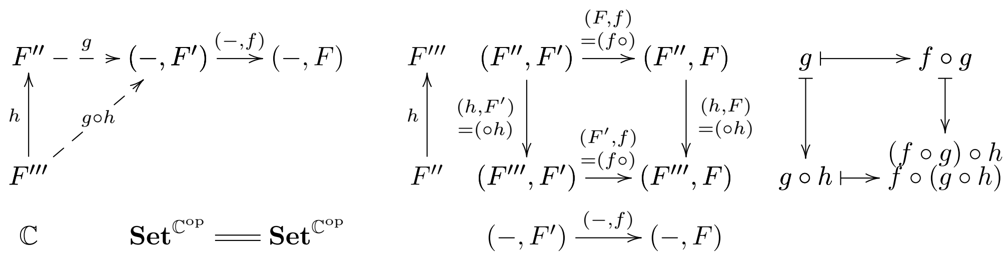
and $(.f)(\Phi(\sigma)) = \Phi((.f)(\sigma))$ becomes $Y(f)(\Phi_F(\sigma)) = \Phi_{F'}(X(f)(\sigma))$, and $(.f) \circ \Phi = \Phi \circ (.f)$ becomes $Y(f) \circ \Phi_F = \Phi_{F'} \circ X(f)$.

12.2 The Yoneda Lemma in RRZ

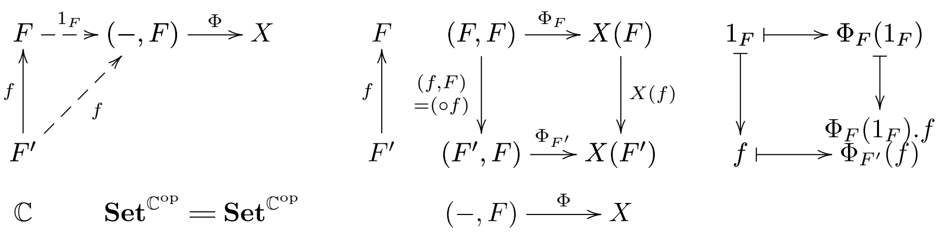
The Yoneda Lemma appears in pages 22–23 and again at pages 29–30 of the book. Let's examine the enlarged versions — drawn with internal views — of some of the figures used in the proof. Our enlarged versions will be called diagrams YR1, YR2, and YR3.

Important: we will make one change in RRZ's notation — where the book writes h_F we will write $(-, F)$, and where it writes h_f we will write $(-, f)$; we saw in sec.11 that the action of the natural transformation $(-, f)$ (a.k.a. f_*) is essentially $(f\circ)$.

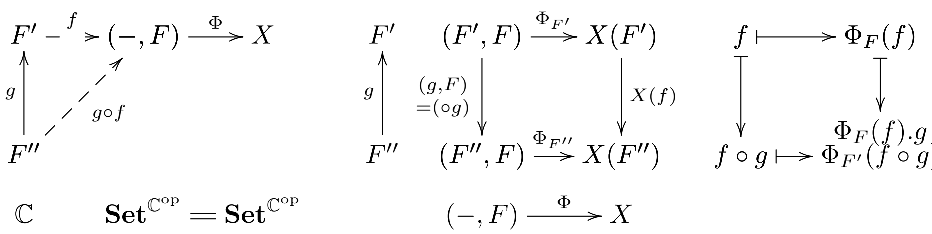
This ("YR1") is from p.23:



This ("YR2") is from p.29:



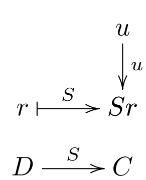
This ("YR3") is also from p.29:



(YR1 is used to prove that $(-, f)$ is morphism of $\mathbb{C}$-sets)
(YR2 is used to prove that $\Phi_{F'}(f)$ can be calculated as $\Phi_F(1_F).f = X(f)(\Phi_F(1_F)))$
(YR3 is used to prove that (???))
(To do: debug this, compare with sections 4 and 10...)

13 Reading MacLane's CWM

MacLane (in [Mac98], section III.2, pages 59–62) starts by fixing a functor $S : D \to C$ and showing that for any pair $\langle r, u : c \to Sr\rangle$, that we draw like this,



any choice of an object $d \in D$ induces a map $\varphi_d : D(r,d) \to C(c,Sd)$,

$$
\begin{array}{ccccc}
 & & c & & \\
 & & \downarrow{\scriptstyle u} & & \\
r & \overset{S}{\longmapsto} & Sr & & \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle Sf'} & & \\
d & \overset{S}{\longmapsto} & Sd & \qquad f' \overset{\varphi_d}{\longmapsto} Sf' \circ u & \\
\\
D(r,d) & \overset{\varphi_d}{\longrightarrow} & C(c,Sd) & \qquad D(r,d) \overset{\varphi_d}{\longrightarrow} C(c,Sd) &
\end{array}
$$

It turns out that $D(r,-)$ and $C(c,S-)$ are functors,

$$
\begin{array}{ccccccc}
d & \longmapsto & D(r,d) & \qquad & d & \longmapsto & C(c,Sd) \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle (D(r,-))(h):= \lambda f'.(h \circ f')} & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle (C(s,S-))(h):= \lambda k.(Sh \circ k)} \\
d' & \longmapsto & D(r,d') & & d' & \longmapsto & C(c,Sd') \\
\\
D & \overset{D(r,-)}{\longrightarrow} & \mathbf{Set} & & D & \overset{C(c,S-)}{\longrightarrow} & \mathbf{Set}
\end{array}
$$

and $\varphi : D(r,-) \to C(c,S-)$ is a natural transformation between them:

$$
\begin{array}{cccccccc}
d & D(r,d) & \overset{\varphi_d}{\longrightarrow} & C(c,Sd) & & f' & \longmapsto & Sf' \circ u \\
{\scriptstyle h}\downarrow & {\scriptstyle D(r,h)}\downarrow & & \downarrow{\scriptstyle C(c,Sh)} & & \downarrow & & \downarrow \\
 & & & & & & & Sh \circ (Sf' \circ u) \\
d' & D(r,d') & \overset{\varphi_{d'}}{\longrightarrow} & C(c,Sd') & & h \circ f' & \longmapsto & S(h \circ f') \circ u \\
\\
 & D(r,-) & \overset{\varphi}{\longrightarrow} & C(c,S-) & & & &
\end{array}
$$

However, MacLane introduces, *right in the beginning*, a concept that I feel that should better be left to a second moment...

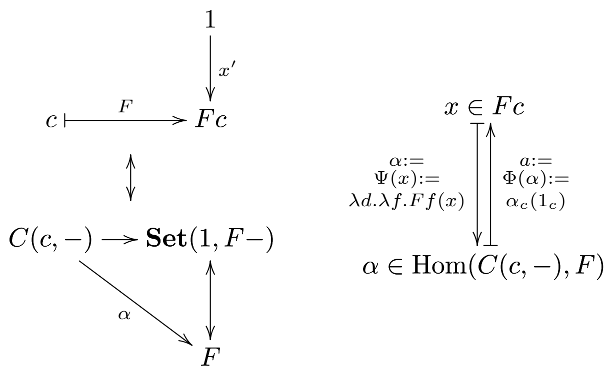
(To be continued!!!)

14 Reading Emily Riehl's "CT in Context"

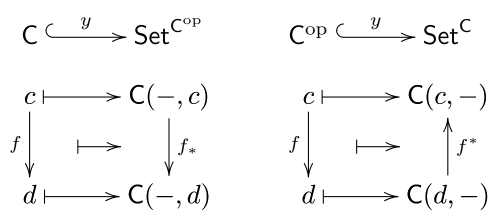
The Yoneda Lemma is proved in [Rie16] in section 2.2, pages 55–61.

Here's Riehl's formula 2.2.5 from pages 57–58 in the shape of our diagram

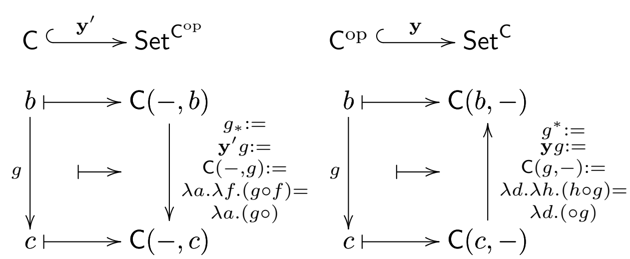
Y4:



The Yoneda embeddings appear as Corollary 2.2.8 in [Rie16], in p.60. She makes this diagram:



She uses the 'y' with two different meanings, one in the left half and another in the right half of her diagram. If we modify her diagram to add some of the information from our diagrams in sec.11 and change her letters just a little bit, we get this:



15 Reading Awodey's "Category Theory"

(To do: show internal views etc of sections 8.2–8.4 of [Awo06] (pp.160–167))

16 Related projects

These notes are related to three, ahem, *things*: a workshop called "Logic for Children", a series of papers on "Planar Heyting Algebras for Children" (these notes prepare the ground for the material on presheaves, sheaves and geometric morphisms that will be presented in the third paper), and a *very* introductory course on λ-calculus, logics and Categories that I am giving every semester in my university since 2016. Links:

`http://angg.twu.net/logic-for-children-2018.html`
`http://angg.twu.net/math-b.html#zhas-for-children-2`
`http://angg.twu.net/math-b.html#lclt`

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