Cálculo 2
PURO-UFF - 2018.2
P1 - 12/nov/2018 - Eduardo Ochs
Respostas sem justificativas não serão aceitas.
Proibido usar quaisquer aparelhos eletrônicos.

1) **(Total: 2.0)** Calcule
$$\int (\operatorname{sen} x)^4 (\cos x)^2 \, dx.$$

2) **(Total: 2.0)** Calcule
$$\int \frac{x^2}{\sqrt{4x^2-9}} \, dx.$$

3) **(Total: 2.0)** Calcule
$$\int \frac{x^3}{x^2+7x+12} \, dx.$$

4) **(Total: 2.0)** Calcule
$$\int \frac{x^4}{(x-3)^2} \, dx.$$

5) **(Total: 2.0)** Calcule por integração por partes:
a) **(1.0 pts)** $\int 1 \cdot \ln x \, dx$,
b) **(1.0 pts)** $\int x \cdot \ln x \, dx$.

Algumas definições, fórmulas e substituições:

$$\begin{array}{llll}
c = \cos\theta & c^2+s^2=1 & \frac{ds}{d\theta} = c & E = c+is \\
s = \operatorname{sen}\theta & z^2 = t^2+1 & \frac{dc}{d\theta} = -s & c = \frac{E+E^{-1}}{2} \\
t = \tan\theta & \sqrt{1-s^2} = c & \frac{dt}{d\theta} = z^2 & s = \frac{E-E^{-1}}{2i} \\
z = \sec\theta & \sqrt{1-t^2} = z & \frac{dz}{d\theta} = zt & e^{ik\theta}+e^{-ik\theta} = 2\cos k\theta \\
E = e^{i\theta} & \sqrt{z^2-1} = t & & e^{ik\theta}-e^{-ik\theta} = 2i\operatorname{sen} k\theta
\end{array}$$

Gabarito

1) $(\operatorname{sen}\theta)^4 = \left(\frac{E-E^{-1}}{2i}\right)^4 = \frac{1}{16}(E^4 - 4E^2 + 6 - 4E^{-2} + E^4)$

$(\cos\theta)^2 = \left(\frac{E+E^{-1}}{2}\right)^2 = \frac{1}{4}(E^2 + 2 + E^{-2})$

$$\begin{array}{rlllllll}
(\operatorname{sen}\theta)^4(\cos\theta)^2 = \frac{1}{64} & (& E^6 & -4E^4 & +6E^2 & -4 & +E^{-2} & & & \\
& & & +2E^4 & -8E^2 & +12 & -8E^{-2} & +2E^{-4} & & \\
& & & & +E^2 & -4 & +6E^{-2} & -4E^{-4} & +E^{-6} &)
\end{array}$$

$$\begin{aligned}
&= \tfrac{1}{64}(E^6 - 2E^4 - E^2 + 4 - E^{-2} - 2E^{-4} + E^{-6}) \\
&= \tfrac{1}{64}((E^6 + E^{-6}) - 2(E^4 + E^{-4}) - (E^2 + E^{-2}) + 4) \\
&= \tfrac{1}{64}(2\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 4)
\end{aligned}$$

$\int(\operatorname{sen}\theta)^4(\cos\theta)^2\,d\theta = \frac{2}{64\cdot 6}\operatorname{sen}6\theta + \frac{4}{64\cdot 4}\operatorname{sen}4\theta - \frac{2}{64\cdot 2}\operatorname{sen}2\theta + \frac{4}{64}\theta$

2)
$$\begin{aligned}
\int \frac{x^2}{\sqrt{4x^2-9}}\,dx &= \int \frac{x^2}{3\sqrt{\frac{4}{9}x^2-1}}\,dx \\
&= \int \frac{x^2}{3\sqrt{(\frac{2}{3}x)^2-1}}\,dx \qquad \left[\begin{smallmatrix} \frac{2}{3}x=z \\ x=\frac{3}{2}z \\ dx=\frac{3}{2}dz \end{smallmatrix}\right] \\
&= \int \frac{(\frac{3}{2}z)^2}{3\sqrt{z^2-1}}\frac{3}{2}\,dz \\
&= \frac{9}{8}\int \frac{z^2}{\sqrt{z^2-1}}\,dz \qquad \left[\begin{smallmatrix} z=\sec\theta=\frac{1}{c} \\ \sqrt{z^2-1}=\tan\theta=\frac{s}{c} \\ dz=zt d\theta=\frac{s}{c^2}d\theta \end{smallmatrix}\right] \\
&= \frac{9}{8}\int \frac{c^{-2}}{sc^{-1}}sc^{-2}\,d\theta \\
&= \frac{9}{8}\int c^{-3}\,d\theta \\
&= \frac{9}{8}\int c^{-4}\,c\,d\theta \qquad \left[\begin{smallmatrix} c^2=1-s^2 \\ c\,d\theta=ds \end{smallmatrix}\right] \\
&= \frac{9}{8}\int \frac{1}{(1-s^2)^2}\,ds \\
&= \frac{9}{8}\int \frac{1}{(s^2-1)^2}\,ds \\
&= \frac{9}{8}\int \frac{1}{(s+1)^2(s-1)^2}\,ds \\
&= \frac{9}{8}\int \frac{1}{4}\left(\frac{1}{s+1} + \frac{1}{(s+1)^2} + \frac{1}{s-1} + \frac{1}{(s-1)^2}\right)\,ds \\
&= \frac{9}{32}\left(\ln|s+1| - \frac{1}{s+1} + \ln|s-1| - \frac{1}{s-1}\right) \\
&= \frac{9}{32}\left(\ln|\tfrac{1}{z}+1| - \frac{1}{\frac{1}{z}+1} + \ln|\tfrac{1}{z}-1| - \frac{1}{\frac{1}{z}-1}\right) \\
&= \frac{9}{32}\left(\ln|\tfrac{3}{2x}+1| - \frac{1}{\frac{3}{2x}+1} + \ln|\tfrac{3}{2x}-1| - \frac{1}{\frac{3}{2x}-1}\right)
\end{aligned}$$

3) $$\int \frac{x^3}{x^2+7x+12}\,dx = \int x-7+\frac{37x+84}{(x+3)(x+4)}\,dx = \int x-7-\frac{27}{x+3}+\frac{64}{x+4}\,dx = \frac{x^2}{2}-7x-27\ln|x+3|+64\ln|x+4|$$

4) $$\int \frac{x^4}{(x-3)^2}\,dx \left[\begin{smallmatrix} u=x-3 \\ x=u+3 \\ dx=du \end{smallmatrix}\right]$$
$$= \int \frac{(u+3)^4}{u^2}\,du$$
$$= \int \frac{u^4+4\cdot u^3\cdot 3+6\cdot u^2\cdot 9+4\cdot u\cdot 27+81}{u^2}\,du$$
$$= \int \frac{u^4+12u^3+54u^2+108u+81}{u^2}\,du$$
$$= \int u^2+12u+54+\frac{108}{u}+\frac{81}{u^2}\,du$$
$$= \frac{u^2}{3}+6u^2+54u+108\ln|u|-\frac{81}{u}$$
$$= \frac{(x-3)^2}{3}+6(x-3)^2+54(x-3)+108\ln|x-3|-\frac{81}{x-3}$$

5a) $\int \underbrace{1}_{f'}\cdot\underbrace{\ln x}_{g}\,dx = \underbrace{x}_{f}\cdot\underbrace{\ln x}_{g} - \int \underbrace{x}_{f}\cdot\underbrace{\tfrac{1}{x}}_{g'}\,dx = x\ln x - \int 1\,dx = x\ln x - x$

5b) $\int \underbrace{x}_{f'}\cdot\underbrace{\ln x}_{g}\,dx = \underbrace{\tfrac{x^2}{2}}_{f}\cdot\underbrace{\ln x}_{g} - \int \underbrace{\tfrac{x^2}{2}}_{f}\cdot\underbrace{\tfrac{1}{x}}_{g'}\,dx = \tfrac{1}{2}x^2\ln x - \tfrac{1}{2}\int x\,dx$
$= \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2$