Visualizing Geometric Morphisms

An application of the "Logic for Children" project to Category Theory

(talk @ "Logic and Categories" workshop, UniLog 2018)

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2018vichy-vgms-slides June 22, 2018 11:28

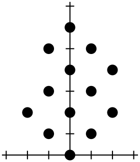
Logic / categories / toposes for children
(*Very* short version; for the long version see
the resources for the "Logic for Children" workshop)

Many years ago...
Non-Standard Analysis
$\to$ Johnstone's "Topos Theory"
$\to$ FAR too abstract for me
$\to$ **I NEED A VERSION FOR CHILDREN OF THIS**

For Children: using "internal views" and
examples with finite objects that are easy to draw
Heyting Algebras that are subset of $\mathbb{Z}^2$ (paper)
Presheaves that can be drawn on a subset of $\mathbb{Z}^2$ (new)

Planar Heyting Algebras for Children
(↑ paper submitted in 2017 —
http://angg.twu.net/math-b.html)

Main definition:
A ZHA is a finite subset of $\mathbb{Z}^2$
made of all even points $(x + y = 2k)$
between $(0, 0)$ and $\top$
between a "left" and a "right wall".
(The "Z" in ZHA means "$\subset \mathbb{Z}^2$")



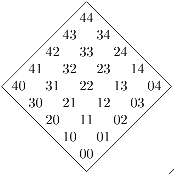
↑ a ZHA

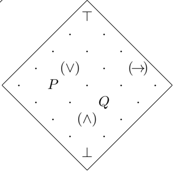
Main theorems:
every ZHA is a Heyting Algebra
every ZHA is a topology in disguise

Planar Heyting Algebras for Children
(↑ Very good paper! No prerequisites!
Lots of fun! Go read it!)

Most toposes have more than two truth-values and an intuitionistic logic.

The paper PHAfC shows how to visualize this (on ZHAs). It uses LR-coordinates and shows how the '→' on ZHAs can be calculated quickly using a formula with four cases.





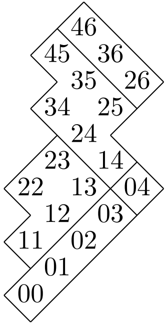
Planar Heyting Algebras for Children 2: Local Operators

The second paper in the series.

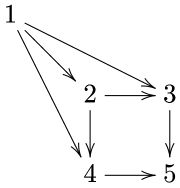
Sheaves *correspond* to local operators on HAs.

A local operator on a ZHA corresponds to slashing the ZHA by diagonal cuts and blurring the distinction between the truth-values in each region.

PHAfC doesn't mention categories.
PHAfC2 doesn't mention categories yet.

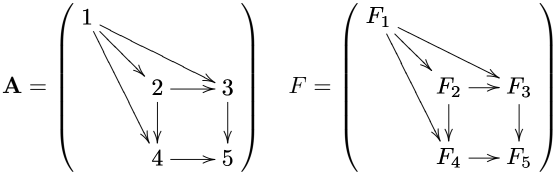


ZCategories
Choose a finite subset of $\mathbb{Z}^2$.
(Optional step: rename its points.)
Use this set as the set of objects
of a category.
Add a finite set of arrows.
This is a ZCategory.
The $\mathbb{Z}^2$-coordinates tell
how to draw it.



ZPresheaves and ZToposes

A ZPresheaf is a functor $F : \mathbf{A} \to \mathbf{Set}$,
where $\mathbf{A}$ is a ZCategory.
(Obs: not $F : \mathbf{A}^{\mathrm{op}} \to \mathbf{Set}$!)
A ZPresheaf F inherits its drawing instructions from $\mathbf{A}$.
("Positional notations")

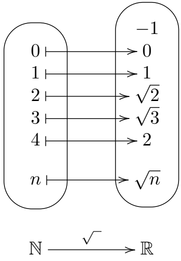


A ZTopos is a category $\mathbf{Set}^{\mathbf{A}}$ where $\mathbf{A}$ is a ZCategory.

Internal views

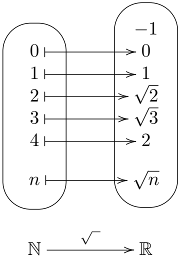
(Part 1: functions)

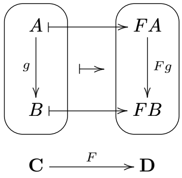
The internal view of the function $\sqrt{\ }: \mathbb{N} \to \mathbb{R}$ is:



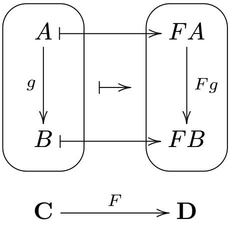
('$\mapsto$'s take elements of a blob-set to another blob-set)

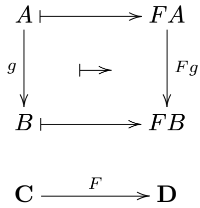
Internal views
(Part 2: functors)
Internal views of functors have blob-categories instead of blob-sets. Compare:





Internal views
(Part 3: omitting the blobs)





Internal views
(Part 4: adjunctions)
Left: generic adjunction $L \dashv R$
Middle: generic geometric morphism $f^* \dashv f_*$
Right: g.m. between toposes $\mathbf{Set}^{\mathbf{A}}$ and $\mathbf{Set}^{\mathbf{B}}$

$$
\begin{array}{ccc}
LC & \longleftarrow\!\dashv & C \\
\downarrow & \longleftrightarrow & \downarrow \\
D & \longmapsto & RD
\end{array}
\qquad
\begin{array}{ccc}
f^*F & \longleftarrow\!\dashv & F \\
\downarrow & \longleftrightarrow & \downarrow \\
G & \longmapsto & f_*G
\end{array}
\qquad
\begin{array}{ccc}
f^*F & \longleftarrow\!\dashv & F \\
\downarrow & \longleftrightarrow & \downarrow \\
G & \longmapsto & f_*G
\end{array}
$$

$$
\mathbf{D} \underset{R}{\overset{L}{\rightleftarrows}} \mathbf{C}
\qquad
\mathcal{E} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{F}
\qquad
\mathbf{Set}^{\mathbf{A}} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Set}^{\mathbf{B}}
$$

$$
\mathbf{A} \xrightarrow{\ f\ } \mathbf{B}
$$

Working in two languages in parallel

Ideas: do things "for children" and "for adults" in parallel, find ways to *transfer knowledge* between the two approaches...

$$\begin{pmatrix} \text{particular} \\ \text{case} \\ \text{“for children”} \end{pmatrix} \underset{\substack{\text{generalize} \\ \text{(hard)}}}{\overset{\substack{\text{particularize} \\ \text{(easy)}}}{\rightleftarrows}} \begin{pmatrix} \text{general} \\ \text{case} \\ \text{“for adults”} \end{pmatrix}$$

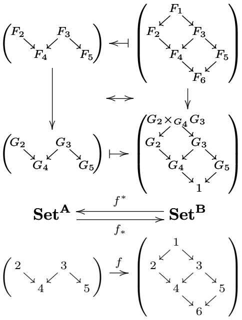
The diagrams for the general case and for a particular case *have the same shape!!!*

Working in two universes in parallel

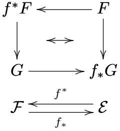
In Non-Standard Analysis we have *transfer theorems*

$$\begin{pmatrix} \text{Standard} \\ \text{universe} \end{pmatrix} \overset{\longleftarrow}{\longrightarrow} \begin{pmatrix} \text{Non-Standard} \\ \text{universe} \\ \text{(ultrapower)} \end{pmatrix}$$

Our first geometric morphism



(for children; inclusion, sheaf)



(for adults)

A factorization
Elephant = Bible
Section A4: Geometric Morphisms
Each '$\longrightarrow$' below is a g.m. (an adjunction)
Any g.m. factors as a surjection followed by an inclusion.
Any inclusion factors as a dense g.m.
followed by a closed g.m..

$$\mathcal{A} \xrightarrow{\text{any}} \mathcal{D}$$

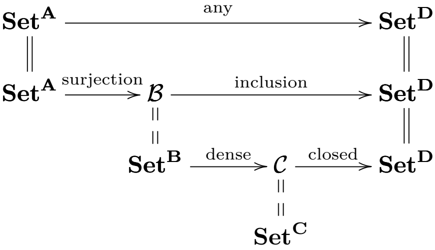
$$\mathcal{A} \xrightarrow{\text{surjection}} \mathcal{B} \xrightarrow{\text{inclusion}} \mathcal{D}$$

$$\mathcal{B} \xrightarrow{\text{dense}} \mathcal{C} \xrightarrow{\text{close}} \mathcal{D}$$

The Elephant *constructs* the toposes $\mathcal{B}$, $\mathcal{C}$ and the maps.

A factorization: version using ZPresheaves

This would be a nicer theorem — that if we start with ZToposes $\mathbf{Set}^{\mathbf{A}}$ and $\mathbf{Set}^{\mathbf{D}}$ the factorization can be through ZToposes...

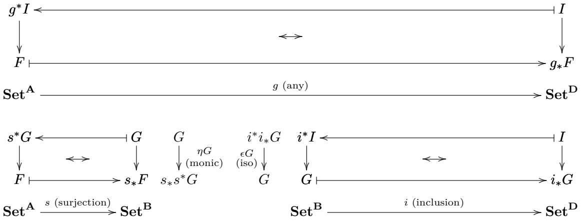


That factorization, for children

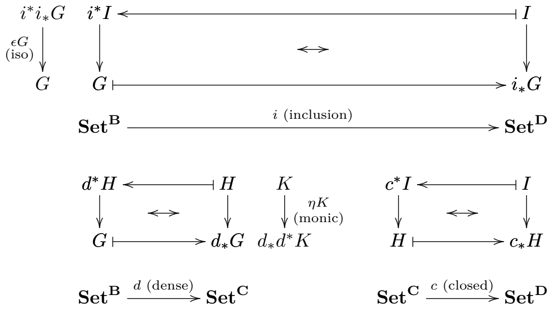
We start with a particular case, with a factorization that only has ZToposes, and we use it to understand how the Elephant defines sujection, inclusion, etc...
(s is not an inclusion, i is not a surjection, and so on)

$$
\begin{array}{ccccccc}
F & & G & & H & & I \\
\mathbf{Set}^{\mathbf{A}} & & & \xrightarrow{g\ \text{(any)}} & & & \mathbf{Set}^{\mathbf{D}} \\
\| & & & & & & \| \\
\mathbf{Set}^{\mathbf{A}} & \xrightarrow{s\ \text{(surjection)}} & \mathbf{Set}^{\mathbf{B}} & & \xrightarrow{i\ \text{(inclusion)}} & & \mathbf{Set}^{\mathbf{D}} \\
 & & \| & & & & \| \\
 & & \mathbf{Set}^{\mathbf{B}} & \xrightarrow{d\ \text{(dense)}} & \mathbf{Set}^{\mathbf{C}} & \xrightarrow{c\ \text{(closed)}} & \mathbf{Set}^{\mathbf{D}}
\end{array}
$$

The surjection-inclusion factorization for children



The dense-closed factorization for children

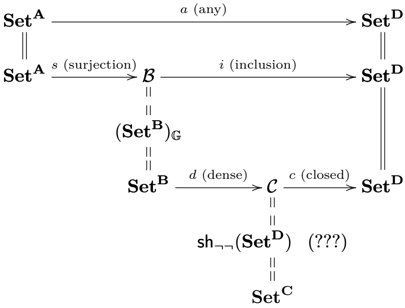


(K is a constant ZPresheaf in $\mathbf{Set}^{\mathbf{C}}$)

Acoording to the Elephant...

A4.2.7, 4.2.10:
to build $\mathcal{B}$ we need comonads and coalgebras

A4.5.9, A4.5.20:
$\mathcal{C} = \mathsf{sh}_{\neg\neg}(\mathbf{Set}^{\mathbf{D}})$
(can't be!)



Another strategy
Start with a functor $g : \mathbf{A} \to \mathbf{D}$.
It induces a geometric morphism $g^* \dashv g_*$.
g^* is trivial to build.
g_* can be found by guess-and-test.
(or by Kan extensions)

The functor g can:
collapse objects, $(1 \quad 2) \to (1)$
create objects, $(\ ) \to (3)$
collapse arrows, $(4 \rightrightarrows 5) \to (4 \to 5)$
create arrows, $(6 \quad 7) \to (6 \to 7)$

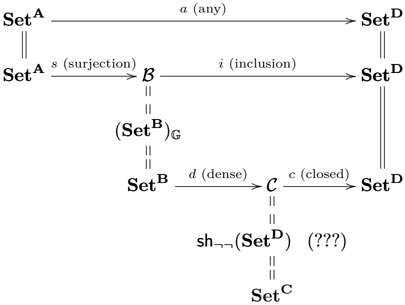
Try to factor it. Example: if g just collapses objects...

Another strategy

The functor g can do several things:
collapse objects, $(1 \quad 2) \to (1)$
create objects, $(\,) \to (3)$
collapse arrows, $(4 \rightrightarrows 5) \to (4 \to 5)$
create arrows, $(6 \quad 7) \to (6 \to 7)$
refine the order, $(2 \to 4) \to (1 \to 2 \to 3 \to 4 \to 5)$...

Try to factor it.
Example: if g just collapses objects,
then it factor as $s = g$ (surj. part), $i = \mathrm{id}$ (inclusion part)...
The factorization filters the *things* that the functor can do,
collapsing objects go to the surjective part.

Another strategy
Choose a functor
$f : \mathbf{A} \to \mathbf{B}$
that does all things.
Factorize it.
$\mathcal{B}$ and $\mathcal{C}$ will
be non-trivial.
They tell us how
$\mathcal{B}$ and $\mathcal{C}$ will be
modulo
isomorphism.



For more information:
http://angg.twu.net/logic-for-children-2018.html
http://angg.twu.net/math-b.html