Logic for Children

(i.e., for people without mathematical maturity —
a workshop at UniLog 2018)

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$\rightarrow$ Ultrapowers
$\rightarrow$ Filter-powers
$\rightarrow$ Toposes
$\rightarrow$ Johnstone's "Topos Theory"

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A MUCH BIGGER
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Some subtasks:

1. Find the right definition of "children"
   (inspired by how I function)
2. Develop a basic toolbox
   (and name its tools)
3. Make these things publishable
   (make them look formal and non-trivial)

The opposite of “children”

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Example: finding a right adjoint by guesswork / trial and error...

One expression that I love is: "*this step (or argument) offends adults*".

Task 1: The right definition of "children":
They prefer to start from particular cases
and then generalize —
They like diagrams and finite objects
drawn very explicitly —
They become familiar with mathematical ideas
by calculating / checking several cases
(rather than by proving theorems)

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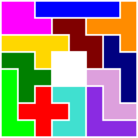
Example: pentominos.
Let "children" play
with pentominos for a while
before showing to them
theorems and game trees!



Task 2: Develop a basic toolbox
I'm starting with "Category Theory for children"
because I am a categorist, and
because CT uses diagrams and generalizations *a lot*...

Basic tools:
Use parallel diagrams,
positional notations,
internal views,
archetypal cases...

(I'll show some diagrams soon)

Task 3: Find ways to publish this
CT books treat examples very briefly,
as if they were trivial exercises... =(
Ideas: do things "for children" and "for adults"
in parallel, find ways to *transfer knowledge*
between the two approaches...

(Non-standard Analysis has transfer theorems
between the standard universe, **Set**, and $\mathbf{Set}^{\mathcal{I}}/\mathcal{U}$)

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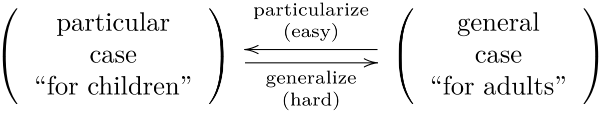


The diagrams for the general case and for a particular case
have the same shape!!!

In the rest of these slides...
...we will show an example:
Geometric Morphisms for children!
($\uparrow$ a thing from Topos Theory)

Visualizing Geometric Morphisms

An application:

Sheaves and Geometric Morphisms

↑ two parts of Topos Theory that look *incredibly abstract* at first

(Btw, I'll give a talk at the "Logic and Categories" workshop about that)

Trick:

Start with presheaves *that are easy to visualize;*

Start with a very small, planar category like this...

Visualizing Geometric Morphisms

Trick: positional notations

Start with presheaves *that are easy to visualize;*

Start with a very small, planar category like this,

$$\mathbf{B} = \left(\begin{array}{ccccccc} & & 1 & & & & \\ & \swarrow & & \searrow & & & \\ 2 & & & & 3 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & 4 & & & & 5 \\ & & & \searrow & & \swarrow & \\ & & & & 6 & & \end{array} \right)$$

Technicalities:

B is a preorder

Visualizing Geometric Morphisms
...and now a presheaf F on $\mathbf{B}$
can be drawn like this...

$$\mathbf{B} = \left(\begin{array}{ccccccc} & & 1 & & & & \\ & \swarrow & & \searrow & & & \\ 2 & & & & 3 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & 4 & & & & 5 \\ & & & \searrow & & \swarrow & \\ & & & & 6 & & \end{array} \right) \qquad F = \left(\begin{array}{ccccccc} & & F_1 & & & & \\ & \swarrow & & \searrow & & & \\ F_2 & & & & F_3 & & \\ & \searrow & & \swarrow & & \searrow & \\ & & F_4 & & & & F_5 \\ & & & \searrow & & \swarrow & \\ & & & & F_6 & & \end{array} \right)$$

Technicalities:
A "real" presheaf would have an 'op'.
This is a *ZPresheaf*!!!
See the slides for "Visualizing Geometric Morphisms"!

Visualizing Geometric Morphisms

...choose a subcategory **A** of **B**, e.g., the one below.
Then a presheaf G on **A** can be drawn as:

$$\mathbf{B} = \left(\begin{array}{ccccc} & 1 & & & \\ \swarrow & & \searrow & & \\ 2 & & 3 & & \\ \searrow & \swarrow & & \searrow & \\ & 4 & & & 5 \\ & & \searrow & \swarrow & \\ & & 6 & & \end{array} \right) \qquad F = \left(\begin{array}{ccccc} & F_1 & & & \\ \swarrow & & \searrow & & \\ F_2 & & F_3 & & \\ \searrow & \swarrow & & \searrow & \\ & F_4 & & & F_5 \\ & & \searrow & \swarrow & \\ & & F_6 & & \end{array} \right)$$

$$\mathbf{A} = \left(\begin{array}{ccccc} 2 & & 3 & & \\ \searrow & \swarrow & & \searrow & \\ & 4 & & & 5 \end{array} \right) \qquad G = \left(\begin{array}{ccccc} G_2 & & G_3 & & \\ \searrow & \swarrow & & \searrow & \\ & G_4 & & & G_5 \end{array} \right)$$

Technicalities: too many =(

Visualizing Geometric Morphisms

...and the inclusion $f : \mathbf{A} \to \mathbf{B}$
induces a geometric morphism $f : \mathbf{Set}^{\mathbf{A}} \to \mathbf{Set}^{\mathbf{B}}$,
that "is" an adjunction $f^* \dashv f_*$:

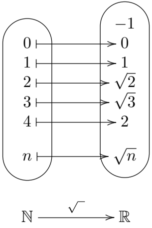
$$\mathbf{Set}^{\mathbf{A}} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Set}^{\mathbf{B}}$$

...where f^* is "obvious" (for some value of "obvious")
and f_* can be obtained by trial and error if we don't understand Kan Extensions...
Kan Extensions: for adults
Trial and error: for children

Interlude: internal views

The best way to explain the adjunction of the previous slide to children is through *internal views*.

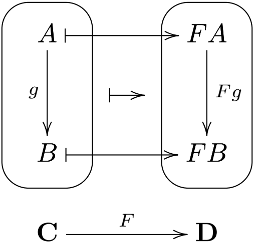
The internal view of the function $\sqrt{\ }: \mathbb{N} \to \mathbb{R}$ is:



('$\mapsto$'s take elements of a blob-set to another blob-set)

Interlude: internal views

Internal views of functors have blob-categories instead of blob-sets, like this:



Interlude: internal views

We draw the internal view of $F : \mathbf{C} \to \mathbf{D}$ as this,

$$\begin{array}{ccc} A & \longmapsto & FA \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Fg} \\ B & \longmapsto & FB \end{array}$$

$$\mathbf{C} \xrightarrow{F} \mathbf{D}$$

we omit the blobs (the “▯”s), and we draw the internal view — objects and maps in $\mathbf{C}$ and $\mathbf{D}$ — above the external view ($F : \mathbf{C} \to \mathbf{D}$).

Internal views

Here is the internal view of the geometric morphism $f : \mathbf{Set}^{\mathbf{A}} \to \mathbf{Set}^{\mathbf{B}}$... remember that f is an adjunction $f^* \dashv f_*$.

$$
\begin{array}{ccc}
f^*F & \longleftarrow\!\shortmid & F \\
\downarrow & \longleftrightarrow & \downarrow \\
G & \longmapsto & f_*G
\end{array}
\qquad\qquad
\begin{array}{ccc}
f^*F & \longleftarrow\!\shortmid & F \\
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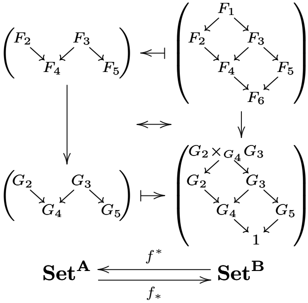
\end{array}
$$

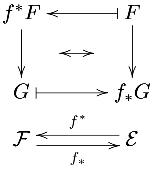
$$
\mathbf{Set}^{\mathbf{A}} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Set}^{\mathbf{B}}
\qquad\qquad
\mathcal{F} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{E}
$$

(particular case) (general case)

A geometric morphism (for children)



(for children)



(for adults)

Resources about the workshop

Here:

http://angg.twu.net/logic-for-children-2018.html

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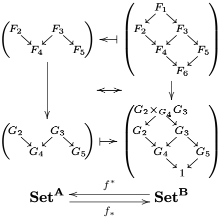

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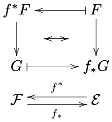

(for adults)