Notes about classifiers and local operators in a $\mathbf{Set}^{(P,A)}$

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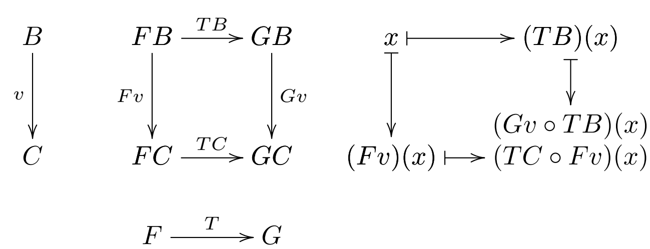
Abstract

The last section of the paper [PH2] shows, quite briefly, how to translate slashings on Planar Heyting Algebras to local operators on toposes, but it omits some details and calculations and says that they are "routine". These notes are an attempt to fill those gaps.

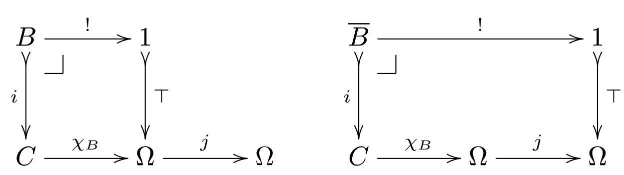
Warning: this is currently 1) a mess 2) a work in progress!

If F and G are functors from $\mathbf{A}$ to $\mathbf{Set}$ and $T : F \to G$ then we can draw the two internal views of the square condition of T as:



Formally, this is: $\forall(v : B \to C).\, \forall x \in FB.\, (Gv \circ TB)(x) = (TC \circ Fv)(x)$.

In Section 7 of [PH2] I used these two diagrams to discuss Ω and j,



and I defined the objects 1 and Ω by:

$$\begin{array}{rcl} 1(p) & = & \{*\} \\ 1(p \overset{!}{\to} q) & = & \lambda * .* \\ \Omega(p) & = & \mathrm{Sub}(\downarrow p) \\ \Omega(p \overset{!}{\to} q) & = & \lambda R{:}\,\mathrm{Sub}(\downarrow p).R \wedge \downarrow q \end{array}$$

Let's suppose that $B \rightarrowtail C$ is a canonical subobject, i.e., we have $\forall p \in P.B(p) \subseteq C(p)$ and every map $B(p \xrightarrow{!} q)$ is a restriction of the corresponding map $C(p \xrightarrow{!} q)$. This means that:

$$
\begin{array}{ccccccc}
p & & B(p) & \xhookrightarrow{ip} & C(p) & b & \longmapsto & b \\
\downarrow{\scriptstyle !} & & \downarrow{\scriptstyle B(p \xrightarrow{!} q)} & & \downarrow{\scriptstyle C(p \xrightarrow{!} q)} & \downarrow & & \downarrow \\
 & & & & & & & C(p \xrightarrow{!} q)(b) \\
q & & B(q) & \xhookrightarrow{iq} & C(q) & B(p \xrightarrow{!} q)(b) & \longmapsto & B(p \xrightarrow{!} q)(b)
\end{array}
$$

$$
B \xhookrightarrow{i} C
$$

$$\begin{array}{c} p \\ \big\downarrow{\scriptstyle !} \\ q \end{array} \qquad \begin{array}{ccc} B_0(p) & \xrightarrow{!} & \{*\} \\ {\scriptstyle B_1(p \xrightarrow{!} q)}\big\downarrow & & \big\downarrow{\scriptstyle !} \\ B_0(q) & \xrightarrow{!} & \{*\} \end{array} \qquad \begin{array}{ccc} b & \longmapsto & * \\ \big\downarrow & & \big\downarrow \\ & & * \\ B_1(p \xrightarrow{!} q)(b) & \longmapsto & * \end{array}$$

$$B \longrightarrow 1$$

$$\begin{array}{c} p \\ \big\downarrow{\scriptstyle !} \\ q \end{array} \qquad \begin{array}{ccc} B_0(p) & \xhookrightarrow{ip} & C_0(p) \\ {\scriptstyle B_1(p \xrightarrow{!} q)}\big\downarrow & & \big\downarrow{\scriptstyle C_1(p \xrightarrow{!} q)} \\ B_0(q) & \xhookrightarrow{iq} & C_0(q) \end{array} \qquad \begin{array}{ccc} b & \longmapsto & b \\ \big\downarrow & & \big\downarrow \\ & & C_1(p \xrightarrow{!} q)(b) \\ B_1(p \xrightarrow{!} q)(b) & \longmapsto & B_1(p \xrightarrow{!} q)(b) \end{array}$$

$$B \xhookrightarrow{i} C$$

$$\begin{array}{c} p \\ \big\downarrow{\scriptstyle !} \\ q \end{array} \qquad \begin{array}{ccc} \{*\} & \xrightarrow{\top p} & \mathrm{Sub}(\downarrow p) \\ {\scriptstyle !}\big\downarrow & & \big\downarrow{\scriptstyle !} \\ \{*\} & \xrightarrow{\top q} & \mathrm{Sub}(\downarrow q) \end{array} \qquad \begin{array}{ccc} * & \longmapsto & \downarrow p \\ \big\downarrow & & \big\downarrow \\ & & \downarrow p \wedge \downarrow q \\ * & \longmapsto & \downarrow q \end{array}$$

$$1 \xrightarrow{\top} \Omega$$

$$\begin{array}{c} p \\ \big\downarrow{\scriptstyle !} \\ q \end{array} \qquad \begin{array}{ccc} C(p) & \xrightarrow{\chi_B(p)} & \mathrm{Sub}(\downarrow p) \\ {\scriptstyle C(p \xrightarrow{!} q)}\big\downarrow & & \big\downarrow{\scriptstyle \Omega(p \xrightarrow{!} q)} \\ C(q) & \xrightarrow{\chi_B(q)} & \mathrm{Sub}(\downarrow q) \end{array}$$

$$\begin{array}{ccc} c & \longmapsto & \{\, r \in \downarrow p \mid C(p \xrightarrow{!} r)(c) \in B(r) \,\} \\ \big\downarrow & & \big\downarrow \\ & & \{\, r \in \downarrow p \mid C(p \xrightarrow{!} r)(c) \in B(r) \,\} \wedge \downarrow q \\ C(p \xrightarrow{!} q)(c) & \longmapsto & \{\, s \in \downarrow q \mid C(q \xrightarrow{!} s)(C(p \xrightarrow{!} q)(c)) \in B(s) \,\} \end{array}$$

$$C \xrightarrow{\chi_B} \Omega$$

$$\begin{array}{c} p \\ \big\downarrow{\scriptstyle !} \\ q \end{array} \qquad \begin{array}{ccc} \mathrm{Sub}(\downarrow p) & \xrightarrow{j(p)} & \mathrm{Sub}(\downarrow p) \\ {\scriptstyle \Omega(p \xrightarrow{!} q)}\big\downarrow & & \big\downarrow{\scriptstyle \Omega(p \xrightarrow{!} q)} \\ \mathrm{Sub}(\downarrow q) & \xrightarrow{j(q)} & \mathrm{Sub}(\downarrow q) \end{array} \qquad \begin{array}{ccc} R & \longmapsto & R^* \wedge \downarrow p \\ \big\downarrow & & \big\downarrow \\ & & (R^* \wedge \downarrow p) \wedge \downarrow q \\ R \wedge \downarrow q & \longmapsto & (R \wedge \downarrow q)^* \wedge \downarrow q \end{array}$$

$$\Omega \xrightarrow{j} \Omega$$

1 Garbage?

$\Omega(p) = \mathrm{Sub}(\downarrow p)$

$\Omega(p \stackrel{!}{\to} q) = \lambda R{:}\, \mathrm{Sub}(\downarrow p).(R \wedge \downarrow q)$

We need to understand these five morphisms in $\mathbf{Set}^{(P,A)}$.
Each one of them is a natural transformation.
The object $1 \in \mathbf{Set}^{(P,A)}$ is $1_0(p) = \{*\}$, $1_1(p \stackrel{!}{\to} q) = \lambda *.*$.
The object $\Omega \in \mathbf{Set}^{(P,A)}$ is $\Omega_0(p) = \downarrow p$, $\Omega_1(p \stackrel{!}{\to} q) = \lambda r{:}\downarrow p.r \wedge \downarrow q$.

$B \stackrel{!}{\to} 1$ is trivial: $(B \stackrel{!}{\to} 1)(p) = \lambda b{:}B_0(p).*$.

$B \stackrel{i}{\hookrightarrow} C$ is an inclusion:
for every $p \in P$ we have $B_0(p) \subseteq C_0(p)$, and
for every $p \stackrel{!}{\to} q$ the map $(B \stackrel{i}{\hookrightarrow} C)(p \stackrel{!}{\to} q)$ is a restriction of $C_1(p \stackrel{!}{\to} q)$.

$1 \stackrel{\top}{\rightarrowtail} \Omega$ is $\lambda p{:}P.\lambda *{:}1(p).\downarrow p$.

$C \xrightarrow{\chi_B} \Omega$ is $\lambda p{:}P.\lambda c{:}C(p).???$.

$$\begin{array}{ccccc}
B & \xrightarrow{!} & 1 & & \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \top} & & \\
C & \xrightarrow{\chi_B} & \Omega & \xrightarrow{j} & \Omega
\end{array}$$

References

[PH2] E. Ochs. "Planar Heyting Algebras for Children 2: Local Operators, J-Operators, and Slashings". http://angg.twu.net/math-b.html#zhas-for-children-2. 2020.