

Notes on [Kleisli65]:

“Every standard construction is induced by a pair of adjoint functors”

Proc. Amer. Math. Soc. 16 (1965), 544-546

<https://doi.org/10.1090/S0002-9939-1965-0177024-4>

<https://www.ams.org/journals/proc/1965-016-03/S0002-9939-1965-0177024-4/>

<https://www.ams.org/journals/proc/1965-016-03/S0002-9939-1965-0177024-4/S0002-9939-1965-0177024.pdf>

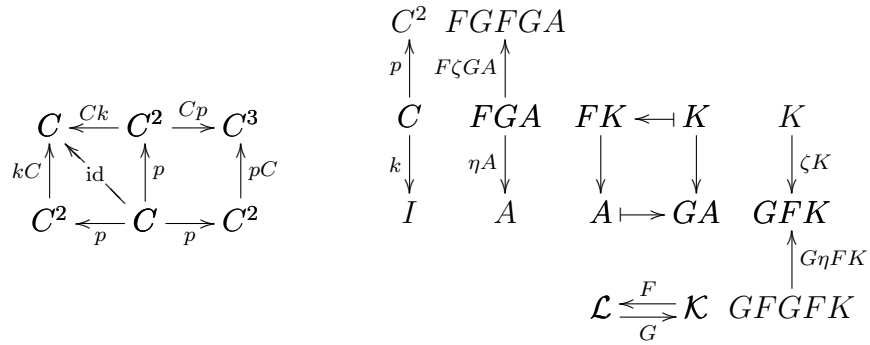
These notes are at:

<http://angg.twu.net/LATEX/2020notes-kleisli.pdf>

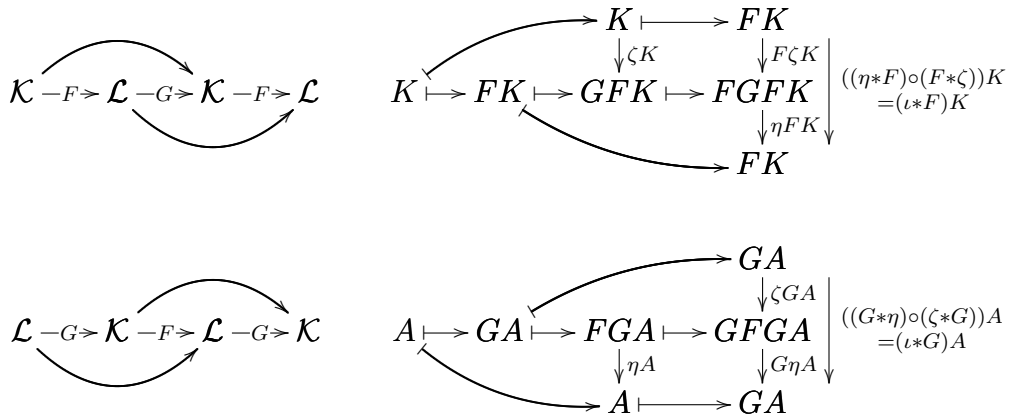
(Page 544):

Left: equations (3) and (4);

Right: notation for the adjunction.



The equations (1) and (2):



The triangular identities for an adjunction, in Kleisli's notation, are:

- (1) $(\eta * F) \circ (F * \zeta) = \iota * F$
- (2) $(G * \eta) \circ (\zeta * G) = \iota * G$

or, in diagrams:

$$\begin{array}{c} \cdot \xrightarrow{-F} \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \\ \downarrow \zeta \quad \downarrow \eta \\ \cdot \xrightarrow{-F} \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \\ \downarrow \zeta \quad \downarrow \eta \\ \cdot \xrightarrow{-F} \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \end{array} = \begin{array}{c} \cdot \xrightarrow{-F} \cdot \\ \downarrow \iota \\ \cdot \xrightarrow{-F} \cdot \end{array}$$

$$\begin{array}{c} \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \xrightarrow{-G} \cdot \\ \downarrow \eta \quad \downarrow \zeta \\ \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \xrightarrow{-G} \cdot \\ \downarrow \eta \quad \downarrow \zeta \\ \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \xrightarrow{-G} \cdot \end{array} = \begin{array}{c} \cdot \xrightarrow{-G} \cdot \\ \downarrow \iota \\ \cdot \xrightarrow{-G} \cdot \end{array}$$

Then he defines a comonad induced by that adjunction

Def: $(FG, \eta, F * \zeta * G) =: (C, k, p)$

$$\begin{array}{c} \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \\ \downarrow \eta \\ \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \\ \downarrow \eta \\ \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \end{array} \quad \begin{array}{c} \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \\ \downarrow \zeta \\ \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \\ \downarrow \zeta \\ \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \xrightarrow{-G} \cdot \xrightarrow{-F} \cdot \end{array}$$

$$\begin{array}{c} \cdot \xrightarrow{-C} \cdot \\ \downarrow k \\ \cdot \xrightarrow{-C} \cdot \\ \downarrow k \\ \cdot \xrightarrow{-C} \cdot \end{array} \quad \begin{array}{c} \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \\ \downarrow p \\ \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \\ \downarrow p \\ \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \end{array}$$

$$\begin{array}{c} \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \\ \downarrow p \quad \downarrow k \\ \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \\ \downarrow p \quad \downarrow k \\ \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \end{array} = \begin{array}{c} \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \\ \downarrow p \quad \downarrow k \\ \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \\ \downarrow p \quad \downarrow k \\ \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \end{array} = \begin{array}{c} \cdot \xrightarrow{-C} \cdot \\ \downarrow \iota \\ \cdot \xrightarrow{-C} \cdot \end{array}$$

$$\begin{array}{c} \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \\ \downarrow p \\ \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \\ \downarrow p \\ \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \end{array} = \begin{array}{c} \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \\ \downarrow p \quad \downarrow p \\ \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \\ \downarrow p \quad \downarrow p \\ \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \xrightarrow{-C} \cdot \end{array}$$