Notes on Monads

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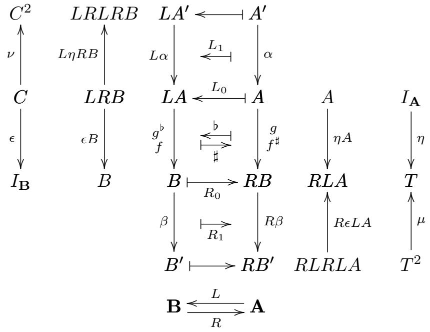
http://angg.twu.net/math-b.html#intro-tys-lfc

An adjunction/monad/comonad: components

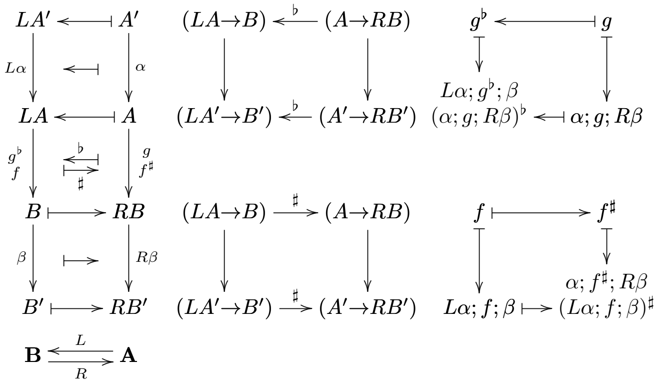


ν	L_1	
C	L_0	
ϵ	$\flat$ $\sharp$	η
	R_0	T
	R_1	μ

$\mathbf{B}$ $\mathbf{A}$

The equations induced by the natural isomorphism

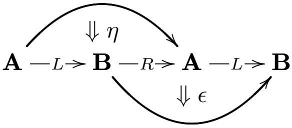
$f^{\sharp\flat} = f$, $g^{\flat\sharp} = g$, plus:

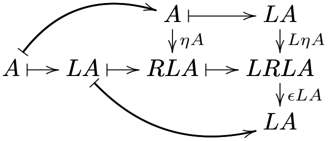


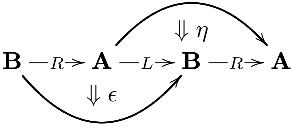
The triangle identities

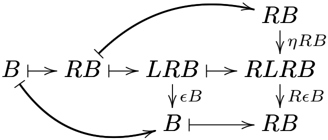
$L\eta A; \epsilon LA = \mathrm{id}_A, \quad \eta RB; R\epsilon B = \mathrm{id}_B.$

$g^\flat := Lg; \epsilon B, \qquad f^\sharp := \eta A; Rf.$









Archetypal case

Our archetypal case is a monoid M,
with elements denoted as $1, x, y, z, \ldots$,
that has a multiplication (associative, with unit 1).

The monoid M acts on sets called $A, B, C, \ldots$,
and action is written as a multiplication.

We want to translate our operations on M
to operations on the functor $(M\times)$. Why? Long story...

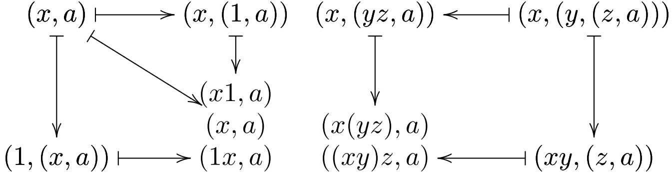
We start with:

$$\begin{array}{c} 1x = x = x1 \\ x(yz) = (xy)z \\ x(ya) = (xy)a \\ 1a = a \end{array} \quad \Rightarrow \quad \begin{array}{c} (1x)a = xa = (x1)a \\ (x(yz))a = ((xy)z)a \\ x(ya) = (xy)a \\ 1a = a \end{array}$$

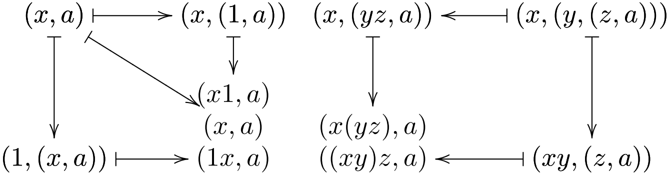
The archetypal monad: the monoid rules

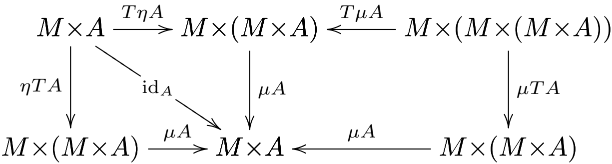
The top two lines are the "monoid rules",
the bottow two lines are the "action rules".
We can draw the monoid rules as a diagram:

$$(1x)a = xa = (x1)a$$
$$(x(yz))a = ((xy)z)a$$

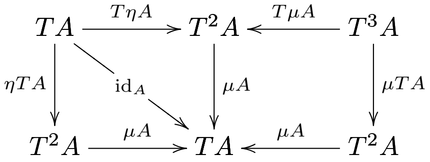


The monoid rules in terms of natural transformations

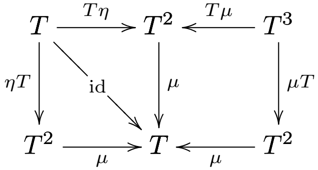




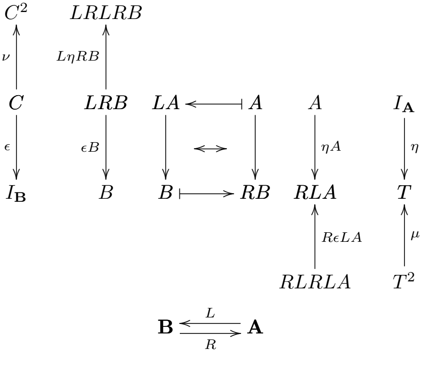
Or, more abstractly...

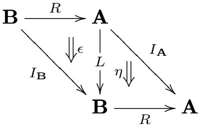


$$1 \xrightarrow{\eta} T \xleftarrow{\mu} T^2$$



Every adjunction induces a monad and a comonad





The internal view of $R\epsilon \circ \eta R = \mathrm{id}_R$