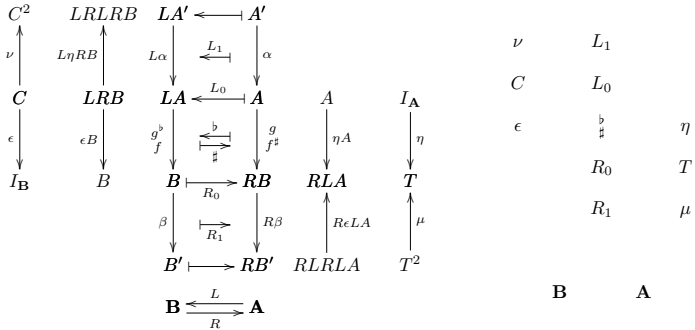


# Notes on Monads

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<http://angg.twu.net/math-b.html#intro-tys-lfc>

# An adjunction/monad/comonad: components



The equations induced by the natural isomorphism  $f^{\sharp b} = f$ ,  $g^{b\sharp} = g$ , plus:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 LA' \longleftarrow A' & (LA \rightarrow B) \xleftarrow{b} (A \rightarrow RB) & g^b \longleftarrow g \\
 \downarrow L\alpha \quad \longleftarrow \quad \downarrow \alpha & \downarrow & \downarrow \\
 LA \longleftarrow A & (LA' \rightarrow B') \xleftarrow{b} (A' \rightarrow RB') & \downarrow L\alpha; g^b; \beta \\
 \downarrow g^b \quad \longleftarrow \quad \downarrow g & \downarrow & \downarrow (\alpha; g; R\beta)^b \longleftarrow \alpha; g; R\beta \\
 f \quad \longleftarrow \quad f^{\sharp} & \downarrow & \\
 B \longleftarrow RB & (LA \rightarrow B) \xrightarrow{\sharp} (A \rightarrow RB) & f \longleftarrow f^{\sharp} \\
 \downarrow \beta \quad \longleftarrow \quad \downarrow R\beta & \downarrow & \downarrow \\
 B' \longleftarrow RB' & (LA' \rightarrow B') \xrightarrow{\sharp} (A' \rightarrow RB') & \downarrow \alpha; f^{\sharp}; R\beta \\
 & & L\alpha; f; \beta \longleftarrow (L\alpha; f; \beta)^{\sharp}
 \end{array} & & \\
 \mathbf{B} \xleftarrow{L} \mathbf{A} & & \\
 \mathbf{A} \xrightarrow{R} \mathbf{B} & & 
 \end{array}$$

### The triangle identities

$$L\eta A; \epsilon LA = \text{id}_A, \quad \eta RB; R\epsilon B = \text{id}_B.$$

$$g^b := Lg; \epsilon B, \quad f^\# := \eta A; Rf.$$

$$\begin{array}{c}
 \text{A} \xrightarrow{-L-} \text{B} \xrightarrow{-R-} \text{A} \xrightarrow{-L-} \text{B} \\
 \downarrow \eta \qquad \qquad \qquad \downarrow \epsilon \\
 \text{A} \xrightarrow{-L-} \text{B} \xrightarrow{-R-} \text{A} \xrightarrow{-L-} \text{B}
 \end{array}$$

$$\begin{array}{c}
 \text{A} \xrightarrow{\quad} \text{LA} \\
 \downarrow \eta A \qquad \downarrow L\eta A \\
 \text{A} \xrightarrow{\quad} \text{LA} \xrightarrow{\quad} \text{RLA} \xrightarrow{\quad} \text{LRLA} \\
 \downarrow \epsilon LA \\
 \text{LA}
 \end{array}$$

$$\begin{array}{c}
 \text{B} \xrightarrow{-R-} \text{A} \xrightarrow{-L-} \text{B} \xrightarrow{-R-} \text{A} \\
 \downarrow \eta \qquad \qquad \qquad \downarrow \epsilon \\
 \text{B} \xrightarrow{-R-} \text{A} \xrightarrow{-L-} \text{B} \xrightarrow{-R-} \text{A}
 \end{array}$$

$$\begin{array}{c}
 \text{B} \xrightarrow{\quad} \text{RB} \\
 \downarrow \eta RB \qquad \downarrow R\epsilon B \\
 \text{B} \xrightarrow{\quad} \text{RB} \xrightarrow{\quad} \text{LRB} \xrightarrow{\quad} \text{RLRB} \\
 \downarrow \epsilon B \\
 \text{RB}
 \end{array}$$

## Archetypal case

Our archetypal case is a monoid  $M$ ,  
with elements denoted as  $1, x, y, z, \dots$ ,  
that has a multiplication (associative, with unit 1).

The monoid  $M$  **acts** on sets called  $A, B, C, \dots$ ,  
and **action** is written as a multiplication.

We want to translate our operations on  $M$   
to operations on the functor  $(M \times)$ . Why? Long story...

We start with:

$$\begin{array}{l}
 1x = x = x1 \\
 x(yz) = (xy)z \\
 x(ya) = (xy)a \\
 1a = a
 \end{array}
 \Rightarrow
 \begin{array}{l}
 (1x)a = xa = (x1)a \\
 (x(yz))a = ((xy)z)a \\
 x(ya) = (xy)a \\
 1a = a
 \end{array}$$

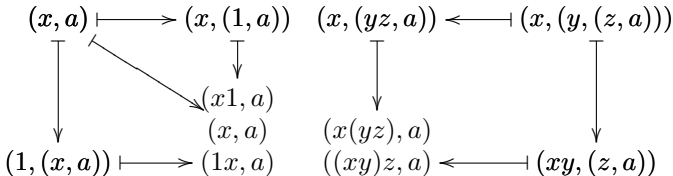
### The archetypal monad: the monoid rules

The top two lines are the “monoid rules”,  
the bottom two lines are the “action rules”.

We can draw the monoid rules as a diagram:

$$(1x)a = xa = (x1)a$$

$$(x(yz))a = ((xy)z)a$$



## The monoid rules in terms of natural transformations

$$\begin{array}{ccc}
 (x, a) \dashv \longrightarrow (x, (1, a)) & (x, (yz, a)) \longleftarrow \dashv (x, (y, (z, a))) \\
 \downarrow & \searrow & \downarrow \\
 & (x1, a) & \\
 & (x, a) & \\
 (1, (x, a)) \dashv \longrightarrow (1x, a) & & (x(yz), a) \\
 & & \downarrow \\
 & & ((xy)z, a) \longleftarrow \dashv (xy, (z, a))
 \end{array}$$

$$\begin{array}{ccccc}
 M \times A & \xrightarrow{T\eta A} & M \times (M \times A) & \xleftarrow{T\mu A} & M \times (M \times (M \times A)) \\
 \eta T A \downarrow & & \searrow \text{id}_A & & \downarrow \mu T A \\
 M \times (M \times A) & & & & M \times (M \times A) \\
 \mu A \downarrow & & \downarrow \mu A & & \downarrow \mu A \\
 M \times (M \times A) & \xrightarrow{\mu A} & M \times A & \xleftarrow{\mu A} & M \times (M \times A)
 \end{array}$$

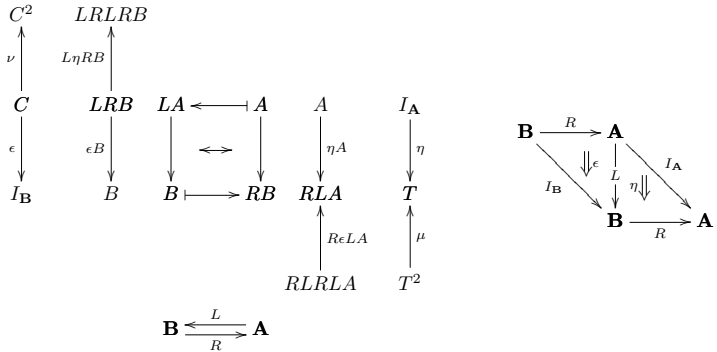
Or, more abstractly...

$$\begin{array}{ccccc}
 TA & \xrightarrow{T\eta A} & T^2 A & \xleftarrow{T\mu A} & T^3 A \\
 \eta TA \downarrow & \searrow \text{id}_A & \downarrow \mu A & & \downarrow \mu TA \\
 T^2 A & \xrightarrow{\mu A} & TA & \xleftarrow{\mu A} & T^2 A
 \end{array}$$

$$\begin{array}{c}
 1 \xrightarrow{\eta} T \xleftarrow{\mu} T^2 \\
 \begin{array}{ccccc}
 T & \xrightarrow{T\eta} & T^2 & \xleftarrow{T\mu} & T^3 \\
 \eta T \downarrow & \searrow \text{id} & \downarrow \mu & & \downarrow \mu T \\
 T^2 & \xrightarrow{\mu} & T & \xleftarrow{\mu} & T^2
 \end{array}
 \end{array}$$



Every adjunction induces a monad and a comonad



The internal view of  $R\epsilon \circ \eta R = \text{id}_R$

