Each closure operator induces a topology and vice-versa ("version for children")

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Abstract

One of the main prerequisites for understanding sheaves on elementary toposes is the proof that a (Lawvere-Tierney) topology on a topos induces a closure operator on it, and vice-versa. That standard theorem is usually presented in a relatively brief way, with most details being left to the reader — see for example [Joh77, section 3.1], [McL92, chapter 21], [LM92, section V.1], [Bel88, chapter 5] — and with no hints on how to visualize some of the hardest axioms and proofs.

These notes are an attempt to present that standard theorem in all details and in a visual way, following the conventions in [FavC]; in particular, some properties, like stability by pullbacks, are always drawn in the same "shape". We also use the idea that "**Set** is the archetypal topos" (from [IDARCT, section 16]) and a variant of the "canonical subobjects" from [LS86, section 2.15] to do a version "for children" of the proof of the correspondence between topologies and closure operators; this proof "for children" can be lifted without much pain to a proof that works on toposes without canonical subobjects.

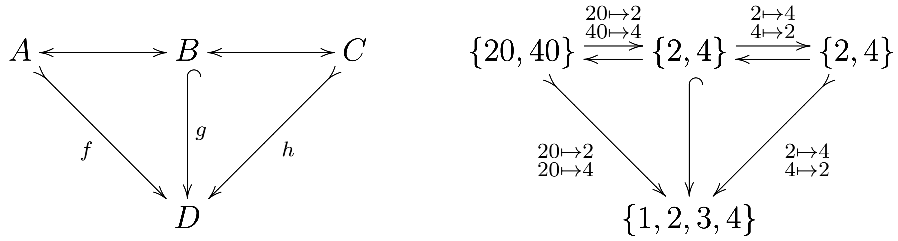
The last sections of these notes show how, for certain toposes, the operation that restricts a closure operation on a topos to its action on Sub(1) — i.e., to a "modal operator on its Heyting Algebra of truth values" ([Fre72, section 1.4]), also called a "J-operator" in [FS79, definition 2.11] and [PH2] — is a bijection, and shows how to visualize this. *I haven't been able to find mentions of this bijection in the literature... if you know any, please let me know!*

Contents

Status of these notes: this is a work in progress. The sections that still need changes before I upload a version of this to Arxiv are marked with "(TODO)". The compilation date of this PDF is at the footer. For the most recent version see:
`http://angg.twu.net/math-b.html#clops-and-tops`
My e-mail is eduardoochs@gmail.com.

1 Subobjects and inclusions

The *subobjects* of an object D of a topos $\mathbf{E}$ are the monics with codomain D *modulo isomorphism*. Here is an example in **Set**:



Here the monics $f : A \rightarrowtail D$, $g : B \rightarrowtail D$, $h : C \rightarrowtail D$ are all equivalent; in some texts they are "the same subobject". Let's make that precise. For us the elements of $\mathrm{Sub}(D)$ are the monics with codomain D. If $(f : A \rightarrowtail D)$, $(g : B \rightarrowtail D)$ are elements of $\mathrm{Sub}(D)$ then they are *equivalent* (notation: $f \equiv g$) iff there is an iso $A \leftrightarrow B$ making the obvious triangle commute. We write $[f]$ for the equivalence class made of an $f \in \mathrm{Sub}(D)$ and all other monics in $\mathrm{Sub}(D)$ equivalent to f, and we write $\overline{\mathrm{Sub}}(D)$ for $\mathrm{Sub}(D)$ modulo equivalence: so $[f] \in \overline{\mathrm{Sub}}(D)$.

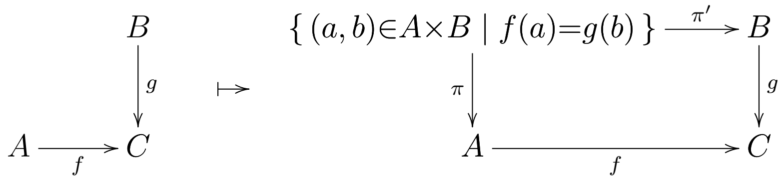
A monic $g : B \rightarrowtail D$ in **Set** is an *inclusion* if it obeys:

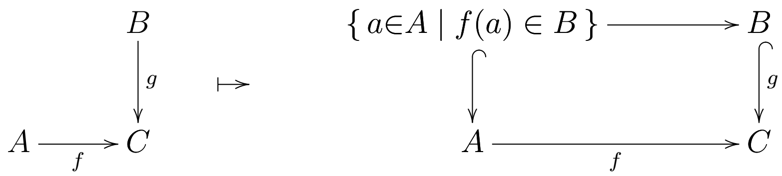
$$\forall b \in \mathrm{dom}(g).\, g(b) = b.$$

The usual way to formalize inclusions in toposes is via canonical subobjects. A topos $\mathbf{E}$ *has canonical subobjects* is it comes equiped with a class $\mathsf{CanSub}(\mathbf{E})$ of monics that obey a certain list of properties — see [LS86, p.200 onwards] — that are also obeyed by the inclusions in **Set**. Here we will do something similar but with a different list of properties, and in section 3 we will see how to translate our proofs, done in toposes with inclusions, to proofs in arbitrary toposes.

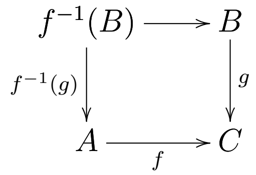
When $f : A \rightarrowtail C$ and $g : B \rightarrowtail C$ are subobjects of C we say that f *is contained in* g (notation: $f \subseteq g$) when there is a monic $m : A \rightarrowtail B$ making the obvious triangle commute. We call m the "mediating map".

In **Set** we have two different operations that take two maps f, g with a common codomain and produce pullbacks:



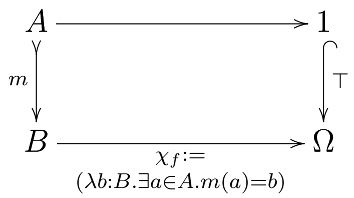


The second one only works when the right wall is an inclusion, but it produces pullbacks whose left walls are inclusions. In both cases we will write the left wall as $f^{-1}(g) : f^{-1}(B) \to A$,



and there will be no default name for the top wall. When the right wall is marked as an inclusion we will use the second pullback operation, otherwise the first one.

In **Set** the classifying map of a monic $m : A \rightarrowtail B$ is defined as:



and the "true" map $\top : 1 \hookrightarrow \Omega$ is the inclusion $\{1\} \hookrightarrow \{0, 1\}$.

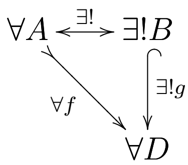
The *inclusion classified by a map* $f : B \to \Omega$ is the map $f^{-1}(\top)$; we will sometimes write it as $\sigma(f)$. Note that for any monic $m : A \rightarrowtail B$ we have $m \equiv \sigma(\chi_m)$, and we have $m = \sigma(\chi_m)$ if m is an inclusion; and for any $f : B \to \Omega$ we have $\chi_{\sigma(f)} = f$.

1.1 Inclusions, precisely

A *topos with inclusions* is a topos $\mathbf{E}$ endowed with a class of monics $\mathsf{Incs}(\mathbf{E})$, called the *inclusions*, and two pullback operations, as in the previous section, obeying the properties below:

Inc1) For any two object C and D of $\mathbf{E}$ there is at most one inclusion from C to D. When that inclusion map exists we write it as $C \hookrightarrow D$ — we don't need to name it — and we say that C *is a subset of* D (notation: $C \subseteq D$).

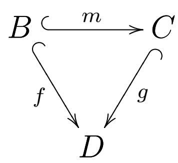
Inc2) Each $[f] \in \overline{\mathrm{Sub}}(D)$ contains exactly one inclusion map. This can be expressed as



in the variant of Freyd's diagrammatic language defined in [FavC, section 4.1]. We will say that this g is *the inclusion associated* (or: *equivalent*) *to* f, and write this as $\mathsf{can}(f) = g$.

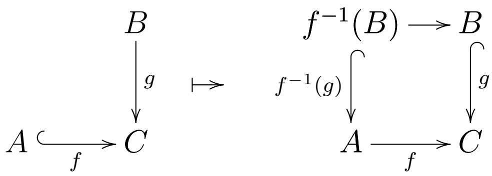
Inc3) The composite of two inclusions is an inclusion. Or, in the language of Inc1: if $B \subseteq C$ and $C \subseteq D$ then $B \subseteq D$, with $B \hookrightarrow D = B \hookrightarrow C \hookrightarrow D$.

Inc4) If $f : B \hookrightarrow D$ and $g : C \hookrightarrow D$ are inclusions with $f \subseteq g$ then the mediating map $m : B \rightarrowtail C$ is an inclusion. In the language of Inc1: $f \subseteq g$ implies $B \subseteq C$. We can visualize this as:



Inc5) The "true" map $\top : 1 \hookrightarrow \Omega$ is an inclusion.

Inc6) The second operation that produces pullbacks in $\mathbf{E}$ receives maps $f : A \to C$ and $g : B \hookrightarrow C$ and returns pullbacks whose left walls are inclusions. In a diagram:



Inc7) The *intersection* of two inclusions $B \hookrightarrow D$ and $C \hookrightarrow D$ is defined as their pullback:

$$\begin{array}{ccc} B{\cap}C & \hookrightarrow & C \\ \downarrow & & \downarrow \\ B & \hookrightarrow & D \end{array}$$

Note that its upper wall is the mediating map from the composite $B{\cap}C \hookrightarrow C \hookrightarrow D$ to $C \hookrightarrow D$, so it is an inclusion.

Using Inc2 we can see that $B{\cap}C$ and $C{\cap}B$ are the *same* subset of D, not just isomorphic subobjects.

We write $\mathsf{Incs}(D)$ for the class of inclusions with codomain D and $\mathsf{Subsets}(D)$ for the class of subsets of D. In a topos with inclusions we have:

$$\mathsf{Subsets}(D) \cong \mathsf{Incs}(D) \cong \overline{\mathrm{Sub}}(D) \cong \mathrm{Sub}(D),$$

where the first two '$\cong$'s are isomorphisms and the last one is just an "equivalence of categories": if we start with a monic f in $\mathrm{Sub}(D)$, take it to its equivalence class $[f]$ in $\overline{\mathrm{Sub}}(D)$, and then go back to $\mathrm{Sub}(D)$, what we get is $\mathsf{can}(f)$, and we have $f \equiv \mathsf{can}(f)$ but not necessarily $f = \mathsf{can}(f)$.

1.2 'And' and 'implies'

In section 2.2 we will need the "internal conjunction map", $(\land) : \Omega \times \Omega \to \Omega$, whose internal view is $(P, Q) \mapsto P \land Q$, and the "internal implication map", $(\to) : \Omega \times \Omega \to \Omega$, that works as $(P, Q) \mapsto (P{\to}Q)$. They are well explained in sections 13.3 and 13.4 of [McL92], but only in their forms "for adults", that work in arbitrary toposes. In this section I will just complement [McL92] by showing briefly how those definitions that hold in any topos are translations of definitions that make sense in **Set**.

The arrow $(\land)$ is built as the classifying map of the inclusion $\sigma(\land)$ in this diagram,

$$\begin{array}{ccc} \{\,(P,Q) \in \Omega \times \Omega \mid P \land Q\,\} & \longrightarrow & 1 \\ {\scriptstyle \sigma(\land)} \downarrow & & \downarrow \\ \Omega \times \Omega & \xrightarrow[\land := \chi_{\sigma(\land)}]{} & \Omega \end{array}$$

and the inclusion $\sigma(\wedge)$ is built as an equalizer. We have:

$$
\begin{array}{ll}
\multicolumn{2}{l}{\{\,(P,Q) \in \Omega \times \Omega \mid P \wedge Q\,\}} \\
= & \{\,(P,Q) \in \Omega \times \Omega \mid P = \top \wedge Q = \top\,\} \\
= & \{\,(P,Q) \in \Omega \times \Omega \mid \mathrm{id}_\Omega(P) = \top_\Omega(P) \wedge \mathrm{id}_\Omega(Q) = \top_\Omega(Q)\,\} \\
= & \{\,(P,Q) \in \Omega \times \Omega \mid (\mathrm{id}_\Omega \times \mathrm{id}_\Omega)(P,Q) = (\top_\Omega \times \top_\Omega)(P,Q)\,\} \\
= & \mathsf{Eq}((\mathrm{id}_\Omega \times \mathrm{id}_\Omega), (\top_\Omega \times \top_\Omega))
\end{array}
$$

$$
\{\,(P,Q) \in \Omega \times \Omega \mid P \wedge Q\,\} \xhookrightarrow{\sigma(\wedge)} \Omega \times \Omega \underset{(P,Q)\mapsto(\top,\top)}{\overset{(P,Q)\mapsto(P,Q)}{\rightrightarrows}} \Omega
$$

$$
\mathsf{Eq}((\mathrm{id}_\Omega \times \mathrm{id}_\Omega), (\top_\Omega \times \top_\Omega)) \xhookrightarrow{\mathsf{eq}((\mathrm{id}_\Omega\times\mathrm{id}_\Omega),(\top_\Omega\times\top_\Omega))} \Omega \times \Omega \underset{\top_\Omega\times\top_\Omega}{\overset{\mathrm{id}_\Omega\times\mathrm{id}_\Omega}{\rightrightarrows}} \Omega
$$

Where the map $\top_\Omega$ is defined as:

$$
A \xrightarrow{!_A} 1 \xrightarrow{\top} \Omega, \quad A \xrightarrow[\top_A := \top \circ !_A]{} \Omega \qquad\qquad \Omega \xrightarrow{!_\Omega} 1 \xrightarrow{\top} \Omega, \quad \Omega \xrightarrow[\top_\Omega := \top \circ !_\Omega]{} \Omega
$$

The arrow $(\to)$ is the classifier of the inclusion $\sigma(\to)$, that is built as another equalizer:

$$
\begin{array}{ccc}
\{\,(P,Q) \in \Omega \times \Omega \mid P \to Q\,\} & \longrightarrow & 1 \\
{\scriptstyle \sigma(\to)}\big\downarrow & & \big\downarrow \\
\Omega \times \Omega & \xrightarrow[(\to) := \chi_{\sigma(\to)}]{} & \Omega
\end{array}
$$

$$
\begin{array}{ll}
\multicolumn{2}{l}{\{\,(P,Q) \in \Omega \times \Omega \mid P \to Q\,\}} \\
= & \{\,(P,Q) \in \Omega \times \Omega \mid \top \le P \to Q\,\} \\
= & \{\,(P,Q) \in \Omega \times \Omega \mid \top \wedge P \le Q\,\} \\
= & \{\,(P,Q) \in \Omega \times \Omega \mid P \le Q\,\} \\
= & \{\,(P,Q) \in \Omega \times \Omega \mid P = P \wedge Q\,\} \\
= & \{\,(P,Q) \in \Omega \times \Omega \mid \pi(P,Q) = (\wedge)(P,Q)\,\} \\
= & \mathsf{Eq}(\pi, \wedge)
\end{array}
$$

$$
\{\,(P,Q) \in \Omega \times \Omega \mid P \to Q\,\} \xhookrightarrow{\sigma(\to)} \Omega \times \Omega \underset{(P,Q)\mapsto(P\wedge Q)}{\overset{(P,Q)\mapsto P}{\rightrightarrows}} \Omega
$$

$$
\mathsf{Eq}(\pi, \wedge) \xhookrightarrow{\mathsf{eq}(\pi,\wedge)} \Omega \times \Omega \underset{\wedge}{\overset{\pi}{\rightrightarrows}} \Omega
$$

Note that in **Set** we have:

$$
\begin{array}{rcl}
\{\,(P,Q) \in \Omega \times \Omega \mid P \wedge Q\,\} & = & \{(1,1)\}, \\
\{\,(P,Q) \in \Omega \times \Omega \mid P \to Q\,\} & = & \{(0,0),(0,1),(1,1)\}.
\end{array}
$$

2 Closure operators

A *closure operator* $\overline{(\cdot)}$ on a topos with inclusions $\mathbf{E}$ is a family of operations like this,

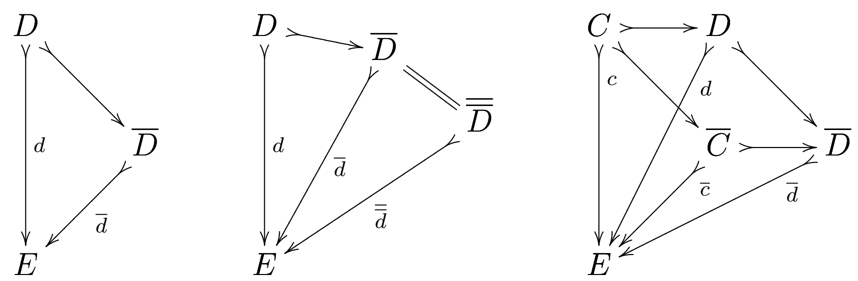
$$\begin{array}{rrcl} \overline{(\cdot)}_E : & \mathsf{Incs}(E) & \to & \mathsf{Incs}(E) \\ & (d : D \hookrightarrow E) & \mapsto & (\overline{d} : \overline{D} \hookrightarrow E), \end{array}$$

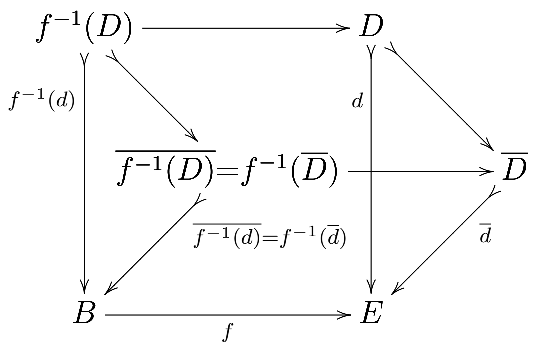
where we have one $\overline{(\cdot)}_E$ for each object E of the topos, and these '$\overline{(\cdot)}_E$'s obey:

C1) $d \subseteq \overline{d}$,
C2) $\overline{d} = \overline{\overline{d}}$,
C3) $c \subseteq d$ implies $\overline{c} \subseteq \overline{d}$,
C4) $\overline{c \cap d} = \overline{c} \cap \overline{d}$,
C5) $\overline{f^{-1}(d)} = f^{-1}(\overline{d})$,

for all inclusions $c : C \hookrightarrow E$ and $d : D \hookrightarrow E$ and for all maps $f : B \to E$.

We will draw the properties C1, C2, C3, C5 as:





Where all the '$\rightarrowtail$'s in the diagrams are inclusions.

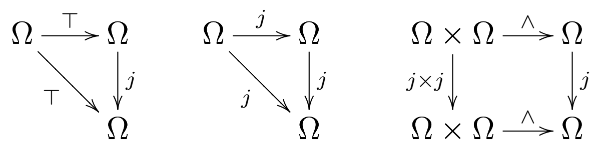
Important: in all diagrams from this section to section to 2.6 all the '$\rightarrowtail$'s will stand for inclusions. This is for typographical reasons, to make the diagrams a bit lighter. The distinction between '$\rightarrowtail$'s and '$\hookrightarrow$'s will reappear in section 3.

2.1 Topologies

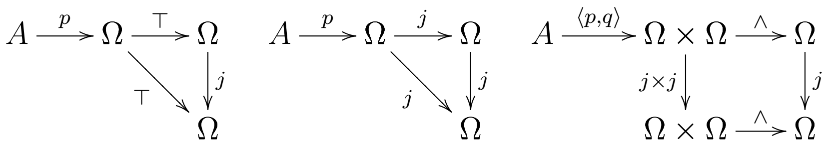
A *(Lawvere-Tierney) Topology* on a topos $\mathbf{E}$ is a map $j : \Omega \to \Omega$ obeying:

LT1) $j \circ j = j$,
LT2) $j \circ \top = \top$,
LT3) $j \circ \wedge = \wedge \circ (j \times j)$.

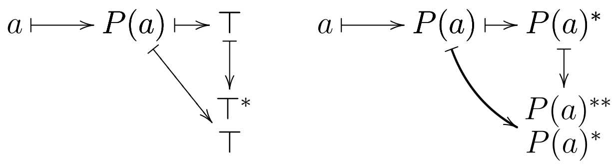
We draw LT1, LT2, and LT3 as:

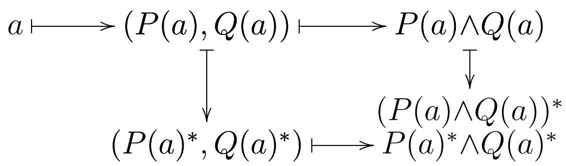


One way to grasp the intuitive meaning of LT1, LT2, and LT3 is to look at their internal views. If we have maps $p, q : A \to \Omega$, the internal views of



are:

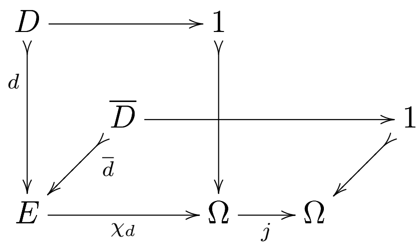




We are writing $j(P(a))$ as $P(a)^*$ to suggest a connection between topologies and the J-operators of [PH2]; we will develop this idea in section .... .

2.2 Topologies induce closure operators

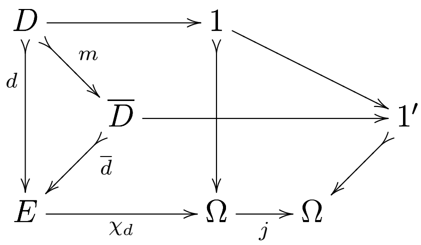
Theorem 2.2.1. Let **E** be a topos with inclusions, and let j be a topology on it. For each inclusion $d : D \hookrightarrow E$ let $\overline{d} : \overline{D} \hookrightarrow E$ be the inclusion that is classified by $j \circ \chi_d$, as in the diagram below:



Then this operation $d \mapsto \overline{d}$ is a closure operator — i.e., it obeys C1, C2, C3, C4, C5.

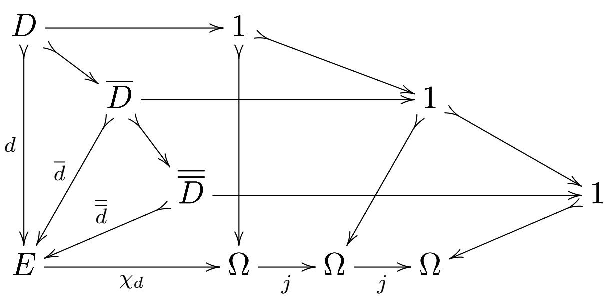
Proof.

For C1, rename the second 1 to $1'$ in the diagram above and draw the identity map $1 \to 1'$. The slanted rectangle with $\overline{D}$ in its upper left corner is a pullback. We can factor the maps $d : D \rightarrowtail E$ and $! : D \to 1'$ through it,



and this gives us a mediating map $m : D \rightarrowtail \overline{D}$. It is easy to check that this m is a monic and an inclusion.[1]

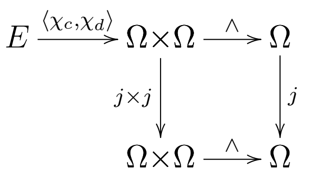
For C2, draw the diagram below:



[1] I thank David Michael Roberts for helping me with this.

The inclusion $\overline{d}$ is classified by $j \circ \chi_d$ and $\overline{\overline{d}}$ is classified by $j \circ j \circ \chi_d$. By LT1 we have $j \circ j = j$, and so $j \circ \chi_d = j \circ j \circ \chi_d$. This means that $\overline{d}$ and $\overline{\overline{d}}$ are two inclusions classified by the same map — so $\overline{d} = \overline{\overline{d}}$, and the inclusion $\overline{D} \hookrightarrow \overline{\overline{D}}$ is the identity.

To prove C4 we use the diagram below and the series of equalities at the right of it:



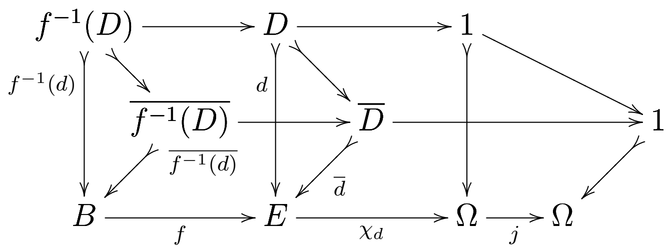
$$\begin{aligned} \chi_{(\overline{c \cap d})} &= j \circ \chi_{c \cap d} \\ &= j \circ \wedge \circ \langle \chi_c, \chi_d \rangle \\ &= \wedge \circ (j \times j) \circ \langle \chi_c, \chi_d \rangle \\ &= \wedge \circ \langle j \circ \chi_c, j \circ \chi_d \rangle \\ &= \wedge \circ \langle \chi_{\overline{c}}, \chi_{\overline{d}} \rangle \\ &= \chi_{(\overline{c} \cap \overline{d})} \end{aligned}$$

The inclusions $\overline{c \cap d}$ and $\overline{c} \cap \overline{d}$ are classified by the same map, so they are equal.

The proof of C3 is this series of inferences:

$$\dfrac{\dfrac{\dfrac{\dfrac{c \subseteq d}{c = c \wedge d}}{\overline{c} = \overline{c \wedge d}} \qquad \overline{c \wedge d} = \overline{c} \wedge \overline{d}}{\overline{c} = \overline{c} \wedge \overline{d}}}{\overline{c} \subseteq \overline{d}}$$

The proof of C5 is this diagram



plus these equalities:

$$\begin{aligned} \chi_{(\overline{f^{-1}(d)})} &= j \circ \chi_{f^{-1}(d)} \\ &= j \circ \chi_d \circ f \\ &= \chi_{\overline{d}} \circ f \\ &= \chi_{f^{-1}(\overline{d})} \end{aligned}$$

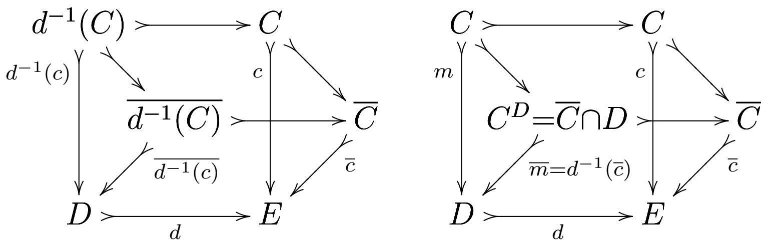
The inclusions $\overline{f^{-1}(d)}$ and $f^{-1}(\overline{d})$ are classified by the same map, so they are the same inclusion.

2.3 Restricting a $\overline{(\cdot)}_E$

In this section we will see how a closure operation $\overline{(\cdot)}_E$ can be "restricted" to a subset $D \subseteq E$.

Theorem 2.3.1. Let $\mathbf{E}$ be a topos with inclusions, with a closure operator $\overline{(\cdot)}$. If $C \subseteq D \subseteq E$ in it, then the closure of $m : C \hookrightarrow D$ can be calculated from the closures of $c : C \hookrightarrow D$ and $d : D \hookrightarrow E$ — and we have $\overline{m} = d^{-1}(\overline{c})$ and $\mathrm{dom}(m) = \overline{C} \cap D$.

Proof. draw the diagram at the left below, that is the diagram for C5 with some things renamed. The pullback of $c : C \hookrightarrow E$ and $d : D \hookrightarrow E$ is $C \cap D$, which is C; so $\overline{d}^{-1}(c) = m$, and we have the diagram at the right.



Our notation for the domain of the closure of an $m : C \hookrightarrow D$ when the name $\overline{C}$ is taken will be C^D, for "the closure of C in D"; the operation '$\cdot^D$' will generalize the '$\cdot^*$' of [PH2]. As C^D is the pullback of $\overline{c} : \overline{C} \hookrightarrow E$ and $d : D \hookrightarrow E$ we have $C^D = \overline{C} \cap D = C^E \cap D$.

Theorem 2.3.2. Let $\mathbf{E}$ be a topos with inclusions with closure operator $\overline{(\cdot)}$. If $D \subseteq E$ in $\mathbf{E}$, then $\overline{(\cdot)}_D$ can be obtained from $\overline{(\cdot)}_E$ in the following way:

$$\begin{array}{rrcl} \overline{(\cdot)}_D : & \mathsf{Incs}(D) & \to & \mathsf{Incs}(D) \\ & (m : C \hookrightarrow D) & \to & (\overline{m} : C^D \to D) \\ & & & := (d^{-1}(\overline{c}) : \overline{C} \cap D \hookrightarrow D) \end{array}$$

where c is $c : C \hookrightarrow E$ and $\overline{c}$ is its closure, $\overline{c} : \overline{C} \hookrightarrow E$.

Proof. This is an easy corollary of Theorem 2.3.

2.4 Dense and closed

For the next theorems we need some definitions:

An inclusion $c : C \to D$ is *dense* iff $\overline{c} = \mathrm{id}_D$.

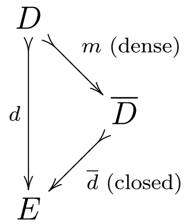
An inclusion $d : D \to E$ is *closed* iff $\overline{d} = d$.

Theorem 2.4.1. If an inclusion $a : A \hookrightarrow B$ is dense and closed then it is the identity.

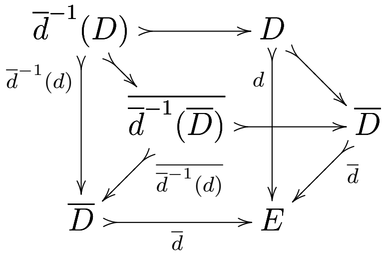
Proof:

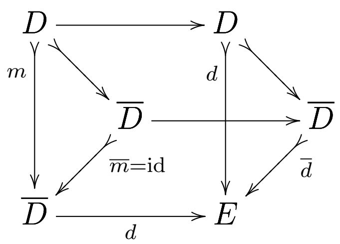
$$\dfrac{\dfrac{a \text{ dense}}{\overline{a} = \mathrm{id}_B} \qquad \dfrac{a \text{ closed}}{\overline{a} = a}}{a = \mathrm{id}_B}$$

Theorem 2.4.2. In a topos with inclusions $\mathbf{E}$ with closure operator $\overline{(\cdot)}$, for any inclusion $d : D \hookrightarrow E$ we have:



Proof. $\overline{\overline{d}} = \overline{d}$, so $\overline{d}$ is closed. To see that $m : D \hookrightarrow \overline{D}$ is dense, we build the diagram at the left below:

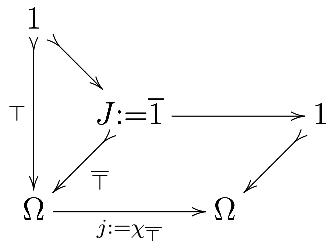




we have $\overline{d}^{-1}(D) = \overline{D} \cap D = D$ and $\overline{d}^{-1}(\overline{D}) = \overline{D} \cap \overline{D} = \overline{D}$, so we can rewrite it as the diagram at the right above, and we get that $\overline{m} = \mathrm{id}$.

2.5 Closure operators induce topologies

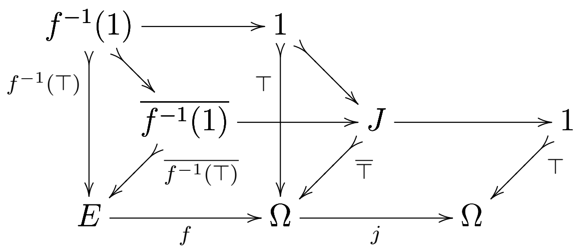
Let **E** be a topos with inclusions, and $\overline{(\cdot)}$ a closure operator on it. Build this diagram on it:



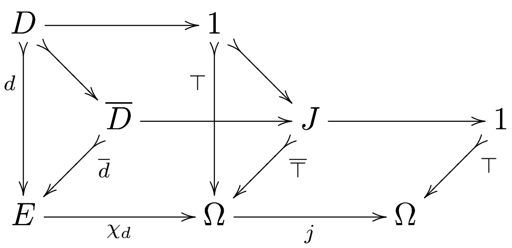
Here the closure of $\top : 1 \to \Omega$ is $\overline{\top} : \overline{1} \to \Omega$, and J is an alternate name for this $\overline{1}$; and $j := \chi_{\overline{\top}}$ is the map that classifies $\overline{\top}$.

Theorem 2.5.1. For every inclusion $d : D \hookrightarrow E$ we have $\chi_{\overline{d}} = j \circ \chi_d$, where j is the map above.

Proof. Take a map $f : E \to \Omega$, and add to the diagram above the diagram for $\overline{f^{-1}(\top)} = f^{-1}(\overline{\top})$. We get this:



This map f is the classifying map for some inclusion; let's call it $d : D \hookrightarrow E$, and rewrite f as χ_d. We get:

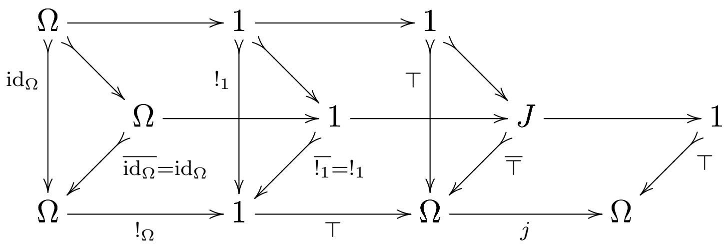


We have $\overline{d} = \overline{f^{-1}(\top)} = f^{-1}(\overline{\top}) = {\chi_d}^{-1}(\overline{\top}) = {\chi_d}^{-1}(j^{-1}(\top)) = (j \circ \chi_d)^{-1}(\top)$, and so $\chi_{\overline{d}} = \chi_{((j \circ \chi_d)^{-1}(\top))} = j \circ \chi_d$.

Theorem 2.5.2. The map j defined above is a topology.

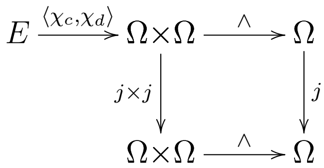
Proof. To prove LT1 we have to see that $j = j \circ j$. We have $\overline{d} = \overline{\overline{d}}$ for all inclusions d; so $\chi_{\overline{d}} = \chi_{\overline{\overline{d}}}$ always. We have $\chi_{\overline{d}} = j \circ \chi_d$ and $\chi_{\overline{\overline{d}}} = j \circ j \circ \chi_d$, so $j \circ \chi_d = j \circ j \circ \chi_d$ always holds. There is a way to make $\chi_d = \mathrm{id}$ here — which is when $d : D \hookrightarrow E$ is $\top : 1 \hookrightarrow \Omega$ — and so a particular case of $j \circ \chi_d = j \circ j \circ \chi_d$ is $j \circ \mathrm{id} = j \circ j \circ \mathrm{id}$, which gives us $j = j \circ j$.

To prove LT2 we have to see that $\top_\Omega = j \circ \top_\Omega$, i.e., that $\top \circ !_\Omega = j \circ \top \circ !_\Omega$. To do this we draw this diagram,



and check that its two upright squares and its three lower slanted squared are pullbacks. With this we get that both $\top \circ !_\Omega$ and $j \circ \top \circ !_\Omega$ classify id_Ω, so $\top \circ !_\Omega = j \circ \top \circ !_\Omega$.

To prove LT3 we start by choosing any two inclusions with the same codomain, $c : C \hookrightarrow E$ and $d : D \hookrightarrow E$. From the maps $\chi_c, \chi_d : E \to \Omega$ we build a map $\langle \chi_c, \chi_d \rangle : \Omega \to \Omega \times \Omega$, and we plug it on the diagram for LT3. We get:

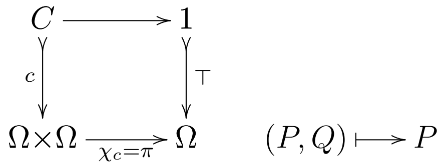


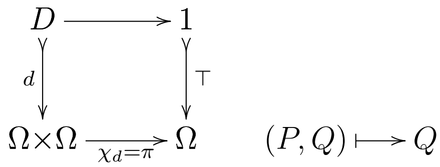
We have

$$\begin{aligned} \wedge \circ (j \times j) \circ \langle \chi_c, \chi_d \rangle &= \chi_{\overline{c} \cap \overline{d}} \\ j \circ \wedge \circ \langle \chi_c, \chi_d \rangle &= \chi_{\overline{c \cap d}} \end{aligned}$$

and C4 tells us that $\overline{c} \cap \overline{d} = \overline{c \cap d}$; so it is always true that $\wedge \circ (j \times j) \circ \langle \chi_c, \chi_d \rangle = j \circ \wedge \circ \langle \chi_c, \chi_d \rangle$. We can make $\langle \chi_c, \chi_d \rangle$ be the identity map if we take $E := \Omega \times \Omega$,

$\langle \chi_c, \chi_d \rangle = \mathrm{id}_{\Omega\times\Omega} = \langle \pi, \pi' \rangle$. The internal views of χ_c and χ_d are:

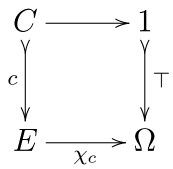


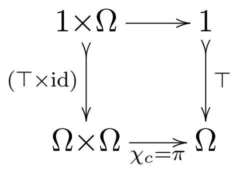


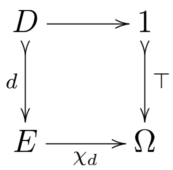
In **Set** we can construct the subsets C and D as:

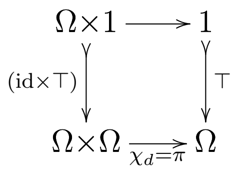
$$\begin{array}{rcl} C & = & \{\, (P,Q) \in \Omega \times \Omega \mid P = \top \,\} \\ & = & \{\top\} \times \Omega \\ D & = & \{\, (P,Q) \in \Omega \times \Omega \mid Q = \top \,\} \\ & = & \Omega \times \{\top\} \end{array}$$

This *suggests* that we can generalize that construction to any topos as:





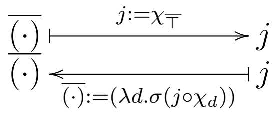




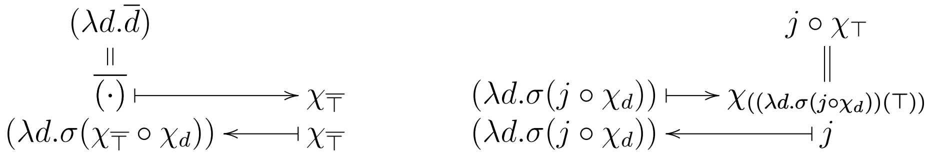
These constructions do work, but I will skip the details of the proof. So: with $c = (\top \times \mathrm{id})$ and $d = (\mathrm{id} \times \top)$ we have $\langle \chi_c, \chi_d \rangle = \mathrm{id}_{\Omega\times\Omega}$, and in this particular case our equality $\land \circ (j \times j) \circ \langle \chi_c, \chi_d \rangle = j \circ \land \circ \langle \chi_c, \chi_d \rangle$ reduces to $\land \circ (j \times j) = j \circ \land$ — and this proves LT3.

2.6 A bijection

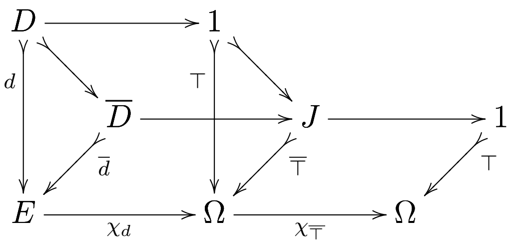
We saw that a closure operator induces a topology and that a topology induces a closure operator. Now we need to check that these two operations, that we can abbreviate as $\overline{(\cdot)} \mapsto j$ and $j \mapsto \overline{(\cdot)}$, as below,



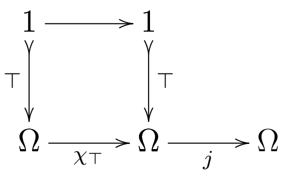
are inverses to one another — i.e., that the composites $\overline{(\cdot)} \mapsto j \mapsto \overline{(\cdot)}$ and $j \mapsto \overline{(\cdot)} \mapsto j$ are identity maps. We will organize all this visually as:



To prove that $\overline{(\cdot)} \mapsto j \mapsto \overline{(\cdot)}$ is the identity we need to check that in any topos with inclusions with a closure operator $\overline{(\cdot)}$ we have that $\overline{(\cdot)}$, i.e., $(\lambda d.\overline{d})$, is equal to $(\lambda d.\sigma(\chi_{\overline{\top}} \circ \chi_d))$. It is enough that check that we have $\overline{d} = \sigma(\chi_{\overline{\top}} \circ \chi_d)$ for any inclusion d. Look at the diagram below...



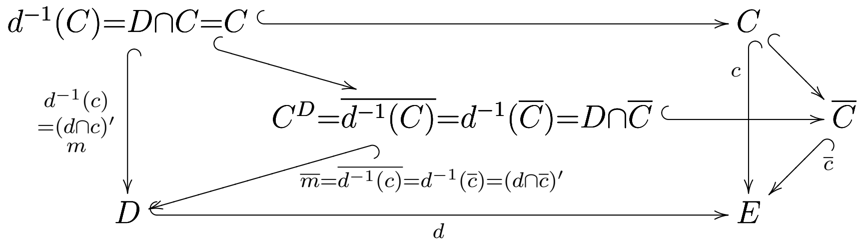
To prove that $j \mapsto \overline{(\cdot)} \mapsto j$ is the identity we need to check that in any topos with inclusions with a topology j we have $j = j \circ \chi_T$. Look at the diagram below:



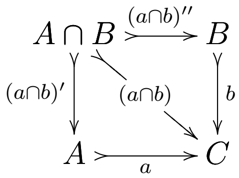
We have $\chi_\top = \mathrm{id}_\Omega$, and so $j \circ \chi_T = j \circ \mathrm{id} = j$.

3 Translating all this to toposes without inclusions

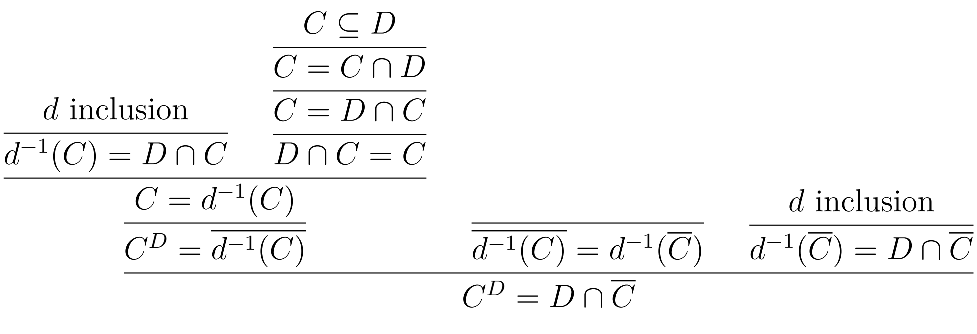
Let's start by an example – we will translate Theorem 2.3. This diagram condenses the two diagrams of the original proof into a single one:



where $(d \cap c)'$ and $(d \cap \overline{c})'$ are maps in intersection pullbacks. The convention is that if $a : A \rightarrowtail C$ and $b : B \rightarrowtail C$ are monics then the components of the diagram for $A \cap B$ are named like this:



This is how I would start to structure the proof above to implement it in a proof assistant. Most nodes in this tree



state that two inclusions with different constructions are isomorphic, and so they are the same morphism. For example, "$C = d^{-1}(C)$" is an abbreviation for this:

$$(m : C \hookrightarrow D) = (d^{-1}(c) : D^{-1}(C) \hookrightarrow D)$$

The properties of inclusions let us omit the codomains and the names of the arrows in many cases, and write only their domains.

We can regard the tree above as a *proof* of this *equality* of *inclusions* that appears at the root node:

$$(\overline{m} : C^D \hookrightarrow D) = ((d \cap \overline{c})' : D \cap \overline{C} \hookrightarrow D)$$

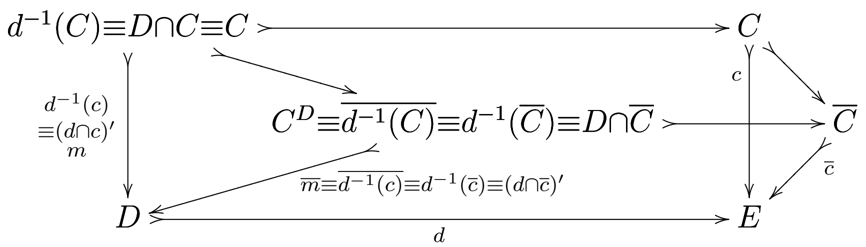
We can translate it to a *construction* of this *isomorphism* of *monics*:

$$(\overline{m} : C^D \rightarrowtail D) \equiv ((d \cap \overline{c})' : D \cap \overline{C} \rightarrowtail D)$$

Now the names of the morphisms are primary and the names of the objects secondary. I prefer write both, otherwise I feel that the translated tree becomes unreadable. Here is the translation of the upper left part of the previous tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{d \text{ monic}}{(d^{-1}(c) : d^{-1}(C) \rightarrowtail D) \equiv (\mathrm{id} \cap m : D \cap C \rightarrowtail D)} \quad \dfrac{\dfrac{\dfrac{(m : C \rightarrowtail D) \subseteq (\mathrm{id} : D \rightarrowtail D)}{(m : C \rightarrowtail D) \equiv (m \cap \mathrm{id} : C \cap D \rightarrowtail D)}}{(m : C \rightarrowtail D) \equiv (\mathrm{id} \cap m : D \cap C \rightarrowtail D)}}{(\mathrm{id} \cap m : D \cap C \rightarrowtail D) \equiv (m : C \rightarrowtail D)}}{(d^{-1}(c) : d^{-1}(C) \rightarrowtail D) \equiv (m : C \rightarrowtail D)}}{(m : C \rightarrowtail D) \equiv (d^{-1}(c) : d^{-1}(C) \rightarrowtail D)}}{(\overline{m} : C^D \rightarrowtail D) \equiv (\overline{d^{-1}(c)} : \overline{d^{-1}(C)} \rightarrowtail D)}$$

I tried to draw a diagram with all the morphisms in the tree above following my usual conventions, and I found the result too messy. But if we translate the original diagram to this,



and we define in the right way how to interpret the '≡'s in it, then everything works. In

$$\begin{array}{c} A_1\equiv A_2\equiv A_3 \\ \Big\downarrow {\scriptstyle f_1\equiv f_2\equiv f_3\equiv f_4\equiv f_5} \\ B_1\equiv B_2\equiv B_3\equiv B_4 \end{array}$$

the "object" $A_1{\equiv}A_2{\equiv}A_3$ means that we have three objects with known, but unnamed, isos between each one and the next, like this: $A_1 \leftrightarrow A_2 \leftrightarrow A_3$, and the "arrow" $f_1{\equiv}f_2{\equiv}f_3{\equiv}f_4{\equiv}f_5$ is in fact five "isomorphic" arrows, and each

f_i goes from some A_j to some B_k, but the diagram does not say what are these 'j's and 'k's; in this context "the 'f_i's are isomorphic" means that the diagram made by $A_1 \leftrightarrow A_2 \leftrightarrow A_3$, $B_1 \leftrightarrow B_2 \leftrightarrow B_3 \leftrightarrow B_4$, and all the '$f_i$'s commutes.

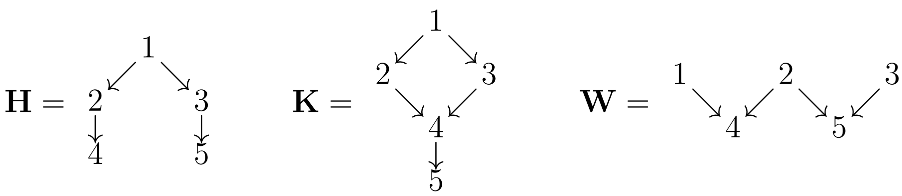
The translation sketched above works for all constructions and proofs in sections 2–2.6. It may be possible to characterize the class of constructions and proofs on which it works, but this is far beyond the scope of these notes.

4 Toposes of the form $\mathbf{Set}^{\mathbf{C}}$ and $\mathbf{Set}^{\mathbf{D}}$

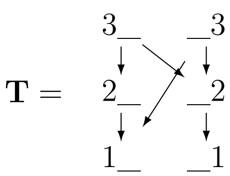
From here onwards we will reserve the symbol $\mathbf{C}$ for small categories and $\mathbf{D}$ for "DAG categories", that we define as follows: a category $\mathbf{D}$ is a *DAG category* iff there is a *finite* DAG (P, A) such that $\mathbf{D}$ is the transitive-reflexive closure (P, A^*) of (P, A), regarded as a category.

We are especially interested in two kinds of DAG categories:

1. The ones coming from the ZDAGs of [PH1], sections 1–2, with their objects labeled in the "reading order", as the categories $\mathbf{H}$ ("house"), $\mathbf{K}$ ("kite"), and $\mathbf{W}$ below:

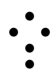


2. "2-column graphs", like the one below:

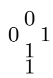


They will be discussed in the next section.

The paper [PH1] defines a language in which certain diagrams have precise meanings as mathematical objects, but these meanings depends on the context, and new meanings can be added. This bullet diagram



can be interpreted as a subset of $\mathbb{Z}^2$ (section 1) and as a DAG with arrows going downward (section 2); a diagram with '0's and '1's like

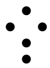
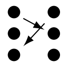


can be interpreted as:

1. a function, a characteristic function, or a subset (section 1),

2. a set or an open set in an arbitrary topology (section 11),
3. a stable subset (section 12),
4. an open set in the default topology (section 12),
5. an open set of an order topology (section 12),
6. a point in a certain partial order (section 13),
7. a point in a certain planar Heyting Algebra (in a "ZHA"; sections 4, 13, 16)

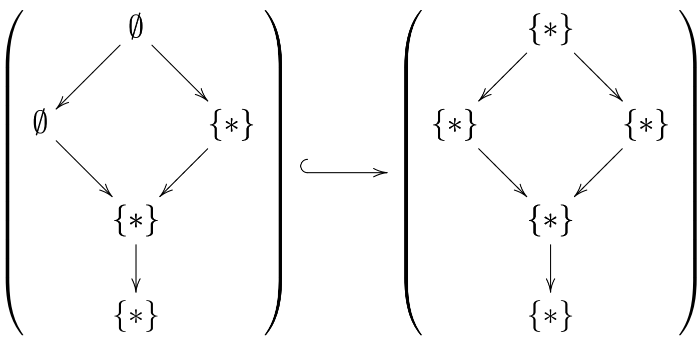
Here we will add some new meanings to these lists. When we say

K =  or **T** = 

these will be shorter versions of the definitions of the DAG categories **K** and **T** in the beginning of the section, and this

$$\begin{smallmatrix} & 0 & \\ 0 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix} \hookrightarrow \begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}$$

will be an inclusion in the category $\mathbf{Set}^{\mathbf{K}}$, that we can expand as:



Suppose that $A = \begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \\ & 0 & \end{smallmatrix}$ is an object of $\mathbf{Set}^{\mathbf{K}}$. This A has an arrow from a 1 to 0: the image of the morphism $2 \to 4$ in K by A is a morphism from $\{*\}$ to $\emptyset$ in **Set**, but $\mathrm{Hom}(\{*\}, \emptyset)$ is empty, so this is absurd. A diagram like $\begin{smallmatrix} & 0 & \\ 1 & & 0 \\ & 0 & \\ & 0 & \end{smallmatrix}$, "with an arrow $1 \to 0$", denotes a non-stable subset of a ZSet in [PH1, section 12], and a non-open subset in [PH1, section 15]; here it will denote something that is not an object of the DAG category that we are dealing with.

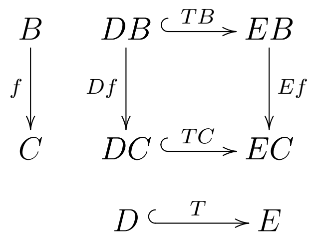
4.1 Inclusions in '$\mathbf{Set}^{\mathbf{C}}$'s and '$\mathbf{Set}^{\mathbf{D}}$'s

Suppose that $\mathbf{C}$ is a small category and that $\mathbf{E}$ is the topos $\mathbf{C}$. We can define a class of inclusions on this $\mathbf{Set}^{\mathbf{C}}$ as follows. A morphism $T : D \to E$ is said to be an *inclusion* if these two conditions hold:

SCInc1) For every object B of $\mathbf{C}$ the map $TB : DB \to EB$ is an inclusion in $\mathbf{Set}$,

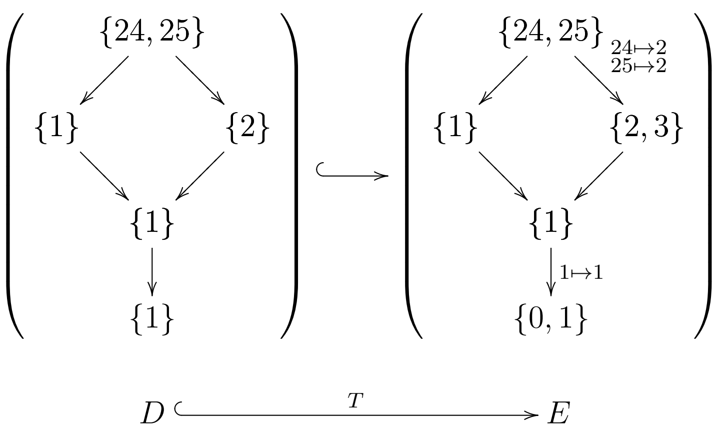
SCInc2) For every map $f : B \to C$ in $\mathbf{C}$ the map $Df : DB \to DC$ is the restriction of the map $Ef : EB \to EC$ to DB.

A morphism $T : D \to E$ in $\mathbf{Set}^{\mathbf{C}}$ is a natural transformation between two functors $D, E : \mathbf{C} \to \mathbf{Set}$, and using the conventions in section 5.3 of [PH2] we can draw the conditions SCInc1 and SCInc2 for it as:



with $\forall d \in DB. (Df)(d) = (Ef \circ TB)(d)$.

Here is an example of an inclusion in $\mathbf{Set}^{\mathbf{K}}$, drawn using the conventions from [FavC], sections 7.12–7.13::



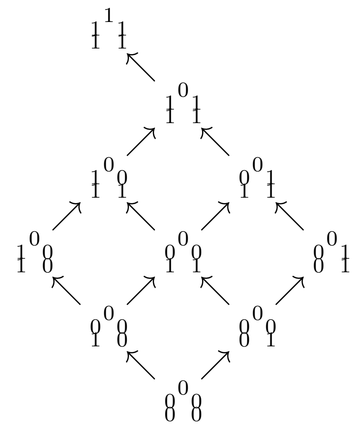
It is relatively easy to check that this class of inclusions in a $\mathbf{Set}^{\mathbf{C}}$ obeys the conditions Inc1, ..., Inc7 from section 1.1. We will leave the details to the reader, inclusing the two ways of defining pullbacks. The classifier will be discussed in section 4.6.

4.2 The logic of a $\mathbf{Set^D}$

Let $\mathbf{E}$ be a topos with inclusions. The *truth-values* of $\mathbf{E}$ are the subsets of the terminal $1_\mathbf{E}$, and the *logic* of $\mathbf{E}$, $\mathsf{Logic}(\mathbf{E})$, is the class, or set, of truth-values of $\mathbf{E}$. For some '$\mathbf{Set^D}$'s $\mathsf{Logic}(\mathbf{Set^D})$ is easy to calculate and to draw. For example:

$$\mathsf{Logic}(\mathbf{Set^K}) = \left\{ \begin{smallmatrix} &0& \\ 0&&0 \\ &0& \\ &0& \end{smallmatrix},\ \begin{smallmatrix} &0& \\ 0&&0 \\ &0& \\ &1& \end{smallmatrix},\ \begin{smallmatrix} &0& \\ 0&&0 \\ &1& \\ &1& \end{smallmatrix},\ \begin{smallmatrix} &0& \\ 0&&1 \\ &1& \\ &1& \end{smallmatrix},\ \begin{smallmatrix} &0& \\ 1&&0 \\ &1& \\ &1& \end{smallmatrix},\ \begin{smallmatrix} &0& \\ 1&&1 \\ &1& \\ &1& \end{smallmatrix},\ \begin{smallmatrix} &1& \\ 1&&1 \\ &1& \\ &1& \end{smallmatrix} \right\},$$

$\mathsf{Logic}(\mathbf{Set^H}) =$



In a DAG topos $\mathbf{E} = \mathbf{Set^D}$ where $\mathbf{D}$ is the DAG (P, A) regarded as a category we have:

$$\mathcal{O}_A(P) = \mathsf{Logic}(\mathbf{Set^D}) \cong \mathsf{Subsets}(1_\mathbf{E}) \cong \mathsf{Incs}(1_\mathbf{E}) \cong \overline{\mathrm{Sub}}(1_\mathbf{E}) \cong \mathrm{Sub}(1_\mathbf{E}),$$

The order topology $\mathcal{O}_A(P)$ is a Heyting Algebra. The section 16 of [PH1] shows, very succintly and with nice diagrams, how to interpret $\top, \bot, \land, \lor, \to$, and $\neg$ on an $\mathcal{O}_A(P)$ — but for the gory details of the translation between $\mathcal{O}_A(P)$ and $\mathrm{Sub}(1_\mathbf{E})$ it points to [Awo06, section 6.3].

4.3 Inner points and χ_{01}-diagrams

Let's now borrow some terminology from Kripke Semantics. The objects of $\mathbf{C}$ will be called *worlds*, or *stages*. An *inner point* of an object $E \in \mathbf{Set^C}$ is a pair (B, b) where $B \in \mathbf{C}$ and $b \in EB$ — here we will not follow the convention of calling b a "point of E at stage B". We denote the set of inner points of E as $\mathsf{Innerpoints}(E)$.

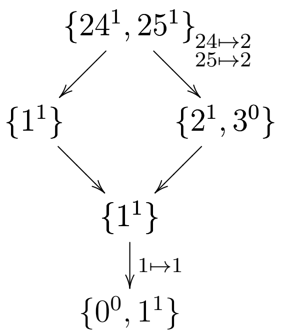
When (B, b) and (C, c) are inner points of $E \in \mathbf{Set^C}$ and there is a map $f : B \to C$ in $\mathbf{C}$ such that $Ef(b) = c$ we will say that (B, b) *sees* (C, c), or that (C, c) *is ahead of* (B, b). The *set of futures* of a point $(B, b) \in E \in \mathbf{Set^C}$, denoted $\mathsf{Futures}_E((B, b))$, is the set of inner points ahead of (B, b) in E.

In '$\mathbf{Set}^{\mathbf{D}}$'s we can use positional notations based on the ones of [PH1, section 1] to draw both inner points and sets of futures. In the example of an inclusion in $\mathbf{Set}^{\mathbf{K}}$ given in section 4.1 we have:

$$\begin{gathered}(3,2) \in E \in \mathbf{Set}^{\mathbf{K}}, \\ \mathsf{Futures}_E((3,2)) = \{(3,2),\ (4,1),\ (5,1)\}, \\ \mathsf{Futures}_E\left(\begin{smallmatrix} & & \cdot & \\ & \cdot & & 2 \\ & & \cdot & \\ & & \cdot & \end{smallmatrix}\right) = \left(\begin{smallmatrix} & & \cdot & \\ & \cdot & & 2 \\ & & 1 & \\ & & 1 & \end{smallmatrix}\right).\end{gathered}$$

If $m : D \hookrightarrow E$ is an inclusion in a $\mathbf{Set}^{\mathbf{C}}$ we define its *01-characteristic map* as the function χ_{01m} that takes each inner point $(B,b) \in E$ to 1 if $(B,b) \in D$, and to 0 otherwise. The χ_{01}*-diagram* of an inclusion $m : D \hookrightarrow E$ in a $\mathbf{Set}^{\mathbf{D}}$, $\chi_{01}(m)$, is obtained by drawing above each inner point (B,b) of E its image by χ_{01m}. The χ_{01}-diagram of the inclusion $T : D \hookrightarrow E$ of section 4.1 is:

$\chi_{01}(T) \quad =$



To obtain D from that diagram just drop the inner points with superscript 0, and then erase the remaining superscripts.

Take any inclusion $m : D \hookrightarrow E$ in a $\mathbf{Set}^{\mathbf{D}}$. Then its χ_{01}-diagram is *non-decreasing*, in the following sense: if a $(B,b) \in E$ sees a $(C,c) \in E$, then $\chi_{01m}((B,b)) \leq \chi_{01m}((C,c))$. Also, the functions from $\mathsf{Innerpoints}(E)$ to $\{0,1\}$ that are 0/1-characteristic maps of inclusions are exactly the ones that are non-decreasing. This gives us a simple way to build $\mathsf{Incs}(E)$ for a given $E \in \mathbf{Set}^{\mathbf{D}}$: get all the non-decreasing functions from the inner points of E to $\{0,1\}$, then convert each one to an inclusion.

We can mix χ_{01}-diagrams with futures, but we will draw the result in a very compact way. The χ_{01}-diagram of the futures in $m : D \hookrightarrow D$ of an inner point $(B,b) \in E$ is drawn like $\mathsf{Futures}_E((B,b))$, but for each (C,c) that is ahead of (B,b) we draw $\chi_{01m}((C,c))$ in the place that we would draw (C,c). For example, in the $T : D \hookrightarrow E$ of 4.1 we have:

$$\chi_{01}\mathsf{Futures}_T((3,2)) = \left(\begin{smallmatrix} & & \cdot & \\ & \cdot & & 0 \\ & & 1 & \\ & & 1 & \end{smallmatrix}\right).$$

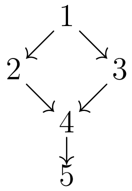
It is easy to see that if (C,c) is ahead of (B,b) then the $\chi_{01}\mathsf{Futures}_m((C,c))$ is $\chi_{01}\mathsf{Futures}_m((B,b))$ restricted to $\mathsf{Futures}_E((C,c))$, where E is the codomain of m. In the example,

$$\chi_{01}\mathsf{Futures}_T((3,2)) = \left(\substack{\cdot \\ \cdot\ \ 0 \\ 1 \\ 1}\right)$$
$$\chi_{01}\mathsf{Futures}_T((4,1)) = \left(\substack{\cdot \\ \cdot\ \ \cdot \\ 1 \\ 1}\right)$$

4.4 The classifier in a $\mathbf{Set}^{\mathbf{D}}$

Let $\mathbf{D}$ be the DAG category induced by a DAG (P,A), and let $\mathbf{E} = \mathbf{Set}^{\mathbf{D}}$. In this situation we will use '$\downarrow$' with two different meanings: if $B \in P$ then $\downarrow B$ will be the set of elements of P "ahead" of B, and if $Q \in \mathsf{Logic}(\mathbf{E})$ then $\downarrow Q$ is the set of elements of $\mathsf{Logic}(\mathbf{E})$ "below" Q. For example, when $\mathbf{D} = \mathbf{K}$,

$\mathbf{K} =$

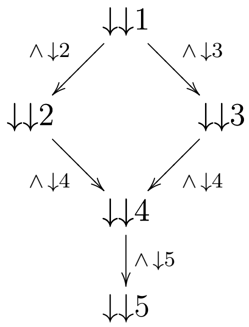


$$, \qquad \downarrow 3 = \downarrow \left(\substack{\cdot \\ \cdot\ \ \bullet \\ \cdot \\ \cdot}\right) = \downarrow \left(\substack{0 \\ 0\ \ 1 \\ 1 \\ 1}\right),$$

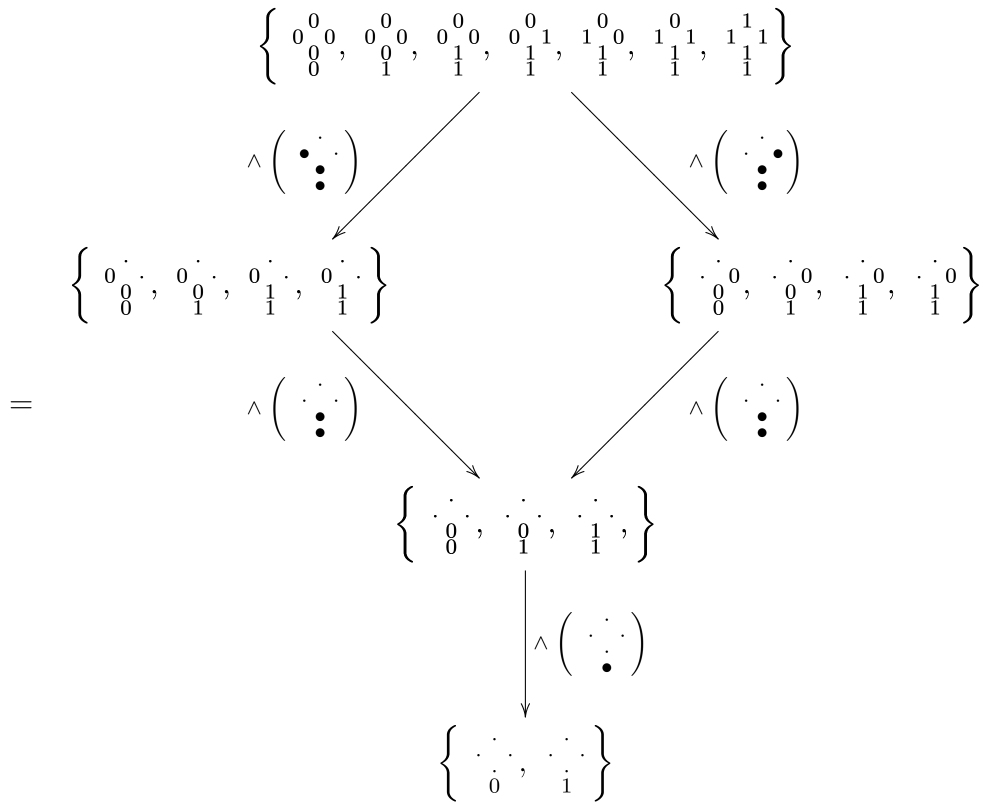
$$\downarrow \left(\substack{0 \\ 1\ \ 1 \\ 1 \\ 1}\right) = \left\{ \substack{\cdot \\ 0\ \ 0 \\ 0 \\ 0},\ \substack{\cdot \\ 0\ \ 0 \\ 0 \\ 1},\ \substack{\cdot \\ 0\ \ 0 \\ 1 \\ 1},\ \substack{\cdot \\ 0\ \ 1 \\ 1 \\ 1},\ \substack{\cdot \\ 1\ \ 0 \\ 1 \\ 1},\ \substack{\cdot \\ 1\ \ 1 \\ 1 \\ 1} \right\}$$

In a category $\mathbf{Set}^{\mathbf{D}}$ the classifier object Ω is the functor $\Omega : \mathbf{D} \to \mathbf{Set}$ that takes each object $B \in \mathbf{D}$ to $\downarrow\downarrow B$ and each morphism $B \to C$ in $\mathbf{D}$ to the function $(\wedge\downarrow C) : \downarrow\downarrow B \to \downarrow\downarrow C$. The classifier of $\mathbf{Set}^{\mathbf{K}}$ is this,

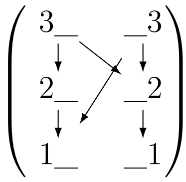
$\Omega \ =$



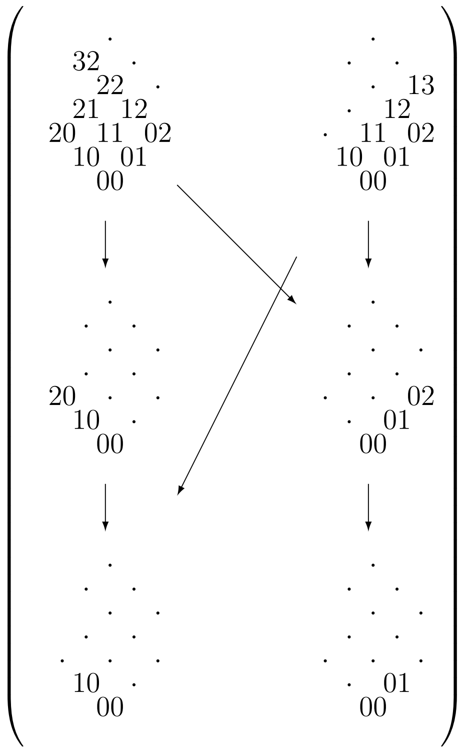
Here is the inner view of this $\Omega \in \mathbf{Set}^{\mathbf{K}}$:



When the DAG (P, A) is a 2-column graph we can draw each $\downarrow\!\!\downarrow B$ as a subset of the logic of the topos. For example, when $\mathbf{D} = \mathbf{T}$,

$\mathbf{T} =$ 

$$\mathsf{Logic}(\mathbf{Set}^{\mathbf{T}}) = \begin{array}{c} 33 \\ 32\quad 23 \\ 22\quad 13 \\ 21\quad 12 \\ 20\quad 11\quad 02 \\ 10\quad 01 \\ 00 \end{array}$$

$\Omega =$ 

4.5 Classifying maps in a $\mathbf{Set^D}$ (TODO)

4.6 The classifier in a $\mathbf{Set^D}$ (via Yoneda)

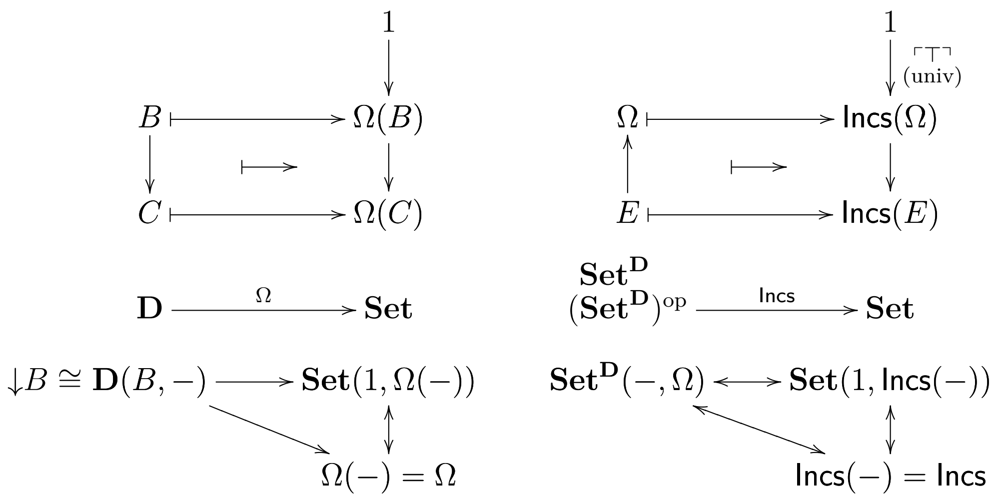
Most texts about basic topos theory define the classifier of a $\mathbf{Set^C}$ directly, as we did in sections 4.4–4.5: they define an object Ω whose action on objects takes each $B \in \mathbf{C}$ to the set of subfunctors of $\mathrm{Hom}(B,-)$ and whose action on morphisms is the "obvious" one, and then they show that this Ω obeys the properties that we expect from the classifier.

The book [LM92] does that but it also shows, in its pages 36–39, that we can use Yoneda to prove that $\Omega B \cong \mathrm{Sub}(\mathrm{Hom}(B,-))$. I struggled a lot to understand their proof, but I found that when I specialize it to '$\mathbf{Set^D}$'s it becomes easy to visualize. Let me show how.

Let (P, A) be a 2-column graph, let $\mathbf{D}$ be (P, A) regarded as a category, and let $\mathbf{E}$ be the topos $\mathbf{Set^D}$. Let Ω be any classifier object in $\mathbf{E}$. This Ω is a functor from $\mathbf{D}$ to $\mathbf{Set}$, and:

Theorem 4.6.1. The action on objects of Ω takes each $B \in \mathbf{D}$ to a set isomorphic to $\downarrow\!\downarrow B$.

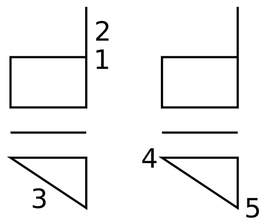
The proof needs the two diagrams below. The one on the left is the "standard Yoneda Lemma": it is the diagram Y1 of [FavC, section 7.2] with its functor $R : \mathbf{B} \to \mathbf{Set}$ replaced by $\Omega : \mathbf{D} \to \mathbf{Set}$. The diagram on the right is the "Yoneda Lemma for representable functors" — the second diagram from [FavC, section 7.6] with its representable functor $R : \mathbf{B} \to \mathbf{Set}$ replaced by $\mathsf{Incs} : (\mathbf{Set^D})^{\mathrm{op}} \to \mathbf{Set}$; its universal arrow is $\ulcorner\top\urcorner$, that selects the element $(\top : 1 \rightarrowtail \Omega) \in \mathsf{Incs}(\Omega)$.



Now we have this sequence of isomorphisms:

$$\begin{array}{rcl} \Omega(B) & \cong & \mathbf{Set}(1, \Omega(B)) \\ & \cong & \mathbf{Set}^{\mathbf{D}}(\downarrow B, \Omega) \\ & \cong & \mathsf{Incs}(\downarrow B) \end{array}$$

where the movement is:



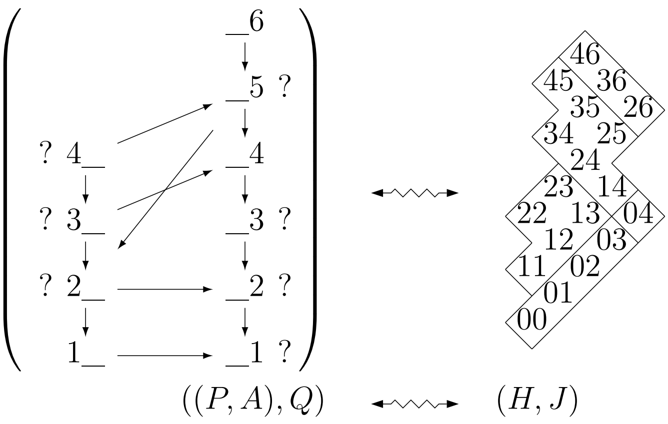
Let's look at an example. If **D** is the category **T** and $B = \_3$ then:

$$\mathsf{Logic}(\mathbf{Set}^{\mathbf{T}}) = \begin{array}{ccccc} & & 33 & & \\ & 32 & & 23 & \\ & & 22 & & 13 \\ & 21 & & 12 & \\ 20 & & 11 & & 02 \\ & 10 & & 01 & \\ & & 00 & & \end{array} \qquad \begin{array}{rcl} \Omega(\_3) & \cong & \mathbf{Set}(1, \Omega(\_3)) \\ & \cong & \mathbf{Set}^{\mathbf{D}}(\downarrow(\_3), \Omega) \\ & \cong & \mathsf{Incs}(\downarrow(\_3)) \\ & \cong & \mathsf{Incs}(13) \\ & \cong & \mathsf{Subsets}(13) \\ & \cong & \{00, 01, 02, 10, 11, 12, 13\} \end{array}$$

5 J-operators, slashings, and question marks

A *set of question marks* on 2-column graph (P, A) is a subset Q of P; a *slashing* S on a Planar Heyting Algebra H is a set of diagonal cuts on H that "do not stop midway". We say that two open subsets R_1 and R_2 of the order topology $\mathcal{O}_A(P)$ are *Q-equivalent* if they only differ in a subset of Q; we say that two elements R_1 and R_2 of H are *S-equivalent* if they belong to the same region, i.e., it is possible to go from R_1 to R_2 without crossing any one of the diagonal cuts.

The first theme treated in [PH2] is that there is a correspondence between sets of question marks and slashings. For example, here,



the Q-equivalence classes in $\mathcal{O}_A(P)$ coincide with the S-equivalence classes in H; see [PH2, sections 1–1.4].

Let H be a ZHA and S be a slashing on it. Every S-equivalence class of H has a top element; let's write $S(R)$ for the top element in the S-equivalence class of R.

A *J-operator* on a ZHA H is a function $J : H \to H$, that we usually write as $(\cdot)^*$, that obeys these three axioms: for any $R, R_1, R_2 \in H$,

J1) $R \leq R^*$,
J2) $R^* = R^{**}$,
J3) $(R_1 \wedge R_2)^* = R_1^* \wedge R_2^*$.

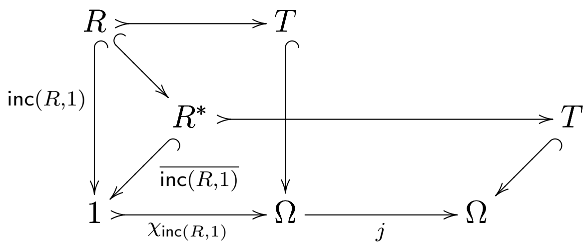
A J-operator induces an equivalence relation on H, in which $R_1 \sim_J R_2$ iff $R_1^* = R_2^*$. It is easy to prove that each R^* is the top element of its J-equivalence class. It is also possible to prove that every slashing S "is" a J-operator if we define $J(R) := S(R)$, and that the lines separating the equivalence classes of a J-operator always form a slashing. This is done in [PH2, sections 2–4].

5.1 Closure operators induce J-operators (TODO)

Let (P, A) be a 2-column graph. Let $\mathbf{E} = \mathbf{Set}^{\mathbf{D}}$ be the DAG topos generated by (P, A), and let $H := \mathsf{Logic}(\mathbf{E})$. Let $\overline{(\cdot)}$ be a closure operator on $\mathbf{E}$. Define an operation $(\cdot)^* : H \to H$ by defining R^* as $\mathrm{dom}(\overline{\mathsf{inc}(R, 1)})$. More formally,

$$\begin{array}{rrcl} (\cdot)^* : & \mathsf{Logic}(\mathbf{E}) & \to & \mathsf{Logic}(\mathbf{E}) \\ & R & \mapsto & \mathrm{dom}(\overline{\mathsf{inc}(R, 1)}) \end{array}$$

And as a diagram:

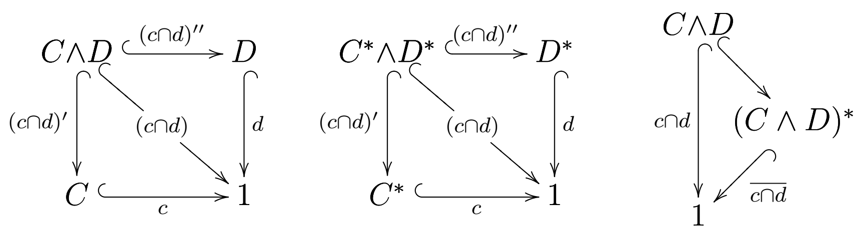


Note that we also have $R^* = \mathrm{dom}(\sigma(j \circ \chi_{\mathsf{inc}(R,1)}))$.

Theorem 5.1.1. The operation $(\cdot)^*$ defined above is a J-operator.

Proof. The proofs of J1 and J2, i.e., of $R \leq R^*$ and $R^* = R^{**}$, are just the parts C1 and C2 of Theorem 2.2 with a slightly different notation: just rewrite the E as 1, D as R, $\overline{D}$ as R^*, and $\overline{\overline{D}}$ as R^{**}.

The proof of J3 is better done in two steps. First we establish the notation, i.e., how the truth-values and their arrows to 1 will be named, using the notation for intersection pullbacks that we defined in Theorem 2.3:



And here is the second step:

$$\begin{array}{rcl} (C \wedge D)^* & = & \mathrm{dom}(\overline{c \cap d}) \\ & = & \mathrm{dom}(\overline{c} \cap \overline{d}) \\ & = & \mathrm{dom}(\overline{c}) \cap \mathrm{dom}(\overline{d}) \\ & = & C^* \cap D^* \end{array} \qquad \begin{array}{rcl} (C \wedge D)^* & = & \mathrm{dom}(\overline{\mathsf{inc}(C, 1) \cap \mathsf{inc}(D, 1)}) \\ & = & \mathrm{dom}(\overline{\mathsf{inc}(C, 1)} \cap \overline{\mathsf{inc}(D, 1)}) \\ & = & \mathrm{dom}(\overline{\mathsf{inc}(C, 1)}) \cap \mathrm{dom}(\overline{\mathsf{inc}(D, 1)}) \\ & = & C^* \cap D^* \end{array}$$

5.2 J-operators induce topologies (TODO)

Let's state this in its general form first, and then see an example that clarifies what it means and how it works.

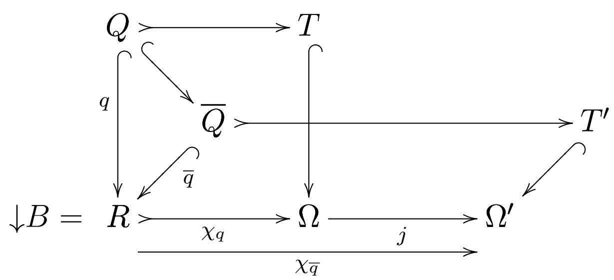
Theorem 5.2.1.

Suppose that: $\mathbf{D}$ is a DAG category,
$\mathbf{E}$ is $\mathbf{Set}^{\mathbf{D}}$,
j is a topology in $\mathbf{E}$,
$\overline{(\cdot)}$ is the closure operator associated to j,
B is an object of D,
R is ${\downarrow}B$ (so $R \in \mathsf{Logic}(E)$),
$(q : Q \hookrightarrow R) \in \mathsf{Incs}(R)$ (so $Q \leq R$),
$(\overline{q} : \overline{Q} \hookrightarrow R)$ is the closure of q (so $Q \leq \overline{Q} \leq R$).

Then: $RB = \{*\}$,
$\Omega B = {\downarrow}{\downarrow}B = {\downarrow}R$,
$(\chi_q B)(*) = Q$,
$(\chi_{\overline{q}} B)(*) = \overline{Q}$,

and so:

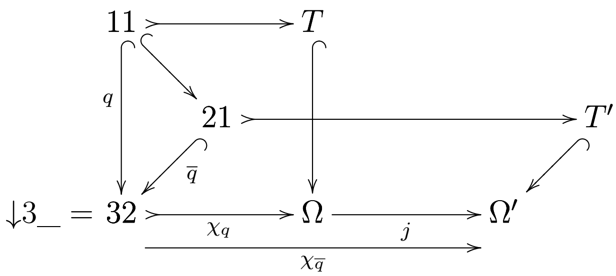
$$\begin{array}{rcl} jB(Q) & = & jB(\chi_q B(*)) \\ & = & (jB \circ \chi_q B)(*) \\ & = & (j \circ \chi_q)(B)(*) \\ & = & \chi_{\overline{q}}(B)(*) \\ & = & \overline{Q}. \end{array}$$



$$RB \xrightarrow{\chi_q B} \Omega B \xrightarrow{jB} \Omega B, \qquad RB \xrightarrow{\chi_{\overline{q}} B} \Omega B$$

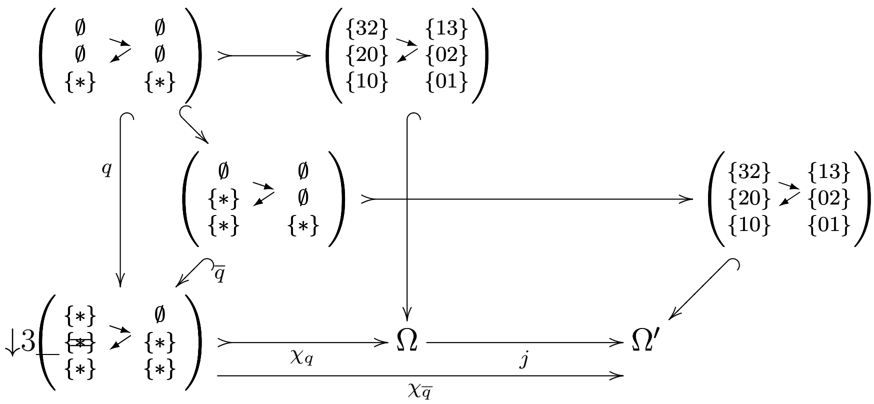
$$\begin{array}{rcl} * & \mapsto & Q \\ * & \mapsto & \overline{Q} \\ Q & \mapsto & \overline{Q} \end{array}$$

Example 5.2.2. If we make $\mathbf{D} = \mathbf{T}$, $B = 3\_$, $Q = 11$, and $\overline{Q} = 21$ in the previous diagram it becomes this:



$$\begin{array}{rcl} (3\_,*) & \longmapsto & (3\_,11) \\ (3\_,*) & \longmapsto & (3\_,21) \\ (3\_,11) & \longmapsto & (3\_,21) \end{array}$$

Here is its internal view:



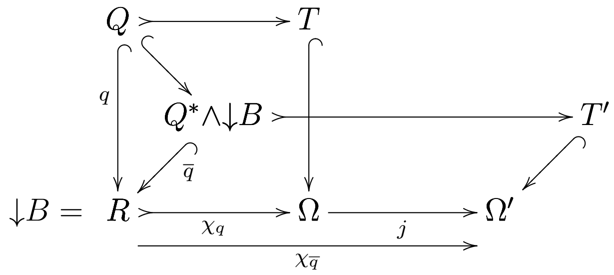
$$\begin{array}{rcl} (3\_,*) & \longmapsto & (3\_,11) \\ (3\_,*) & \longmapsto & (3\_,21) \\ (3\_,11) & \longmapsto & (3\_,21) \end{array}$$

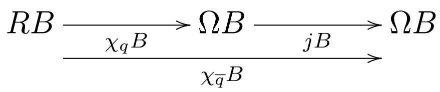
I did not expand the 'Ω's because their internal views would be too big – see the last diagram in section 4.4. Note that the images of R by the monics χ_q and $\chi_{\overline{q}}$ (that are not inclusions!) are, respectively:

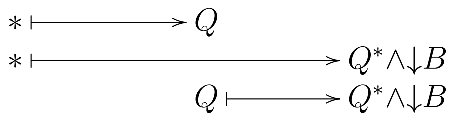
$$\begin{pmatrix} \{11\} & \searrow & \emptyset \\ \{10\} & \swarrow & \{01\} \\ \{10\} & & \{01\} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \{21\} & \searrow & \emptyset \\ \{20\} & \swarrow & \{01\} \\ \{10\} & & \{01\} \end{pmatrix}.$$

Theorem 5.2.3.

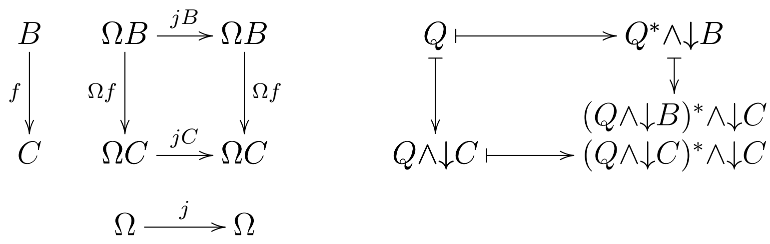
Suppose that:	(P, A) is a 2-column graph,
	$\mathbf{D}$ is (P, A) as a DAG category,
	$\mathbf{E}$ is $\mathbf{Set^D}$,
	H is $\mathsf{Logic}(\mathbf{E})$,
	$(\cdot)^*$ is a J-operator on H.
Then the operation	$j = \lambda B{:}P.\lambda Q{:}{\downarrow}Q.Q^* \wedge {\downarrow}B$
is a morphism	$j : \Omega \to \Omega$
that obeys	$j \circ j = j$,
	$j \circ \top_\Omega = \top_\Omega$,
	$\wedge \circ j = (j \times j) \circ \wedge$,
i.e.,	j is a topology on $\mathbf{E}$.



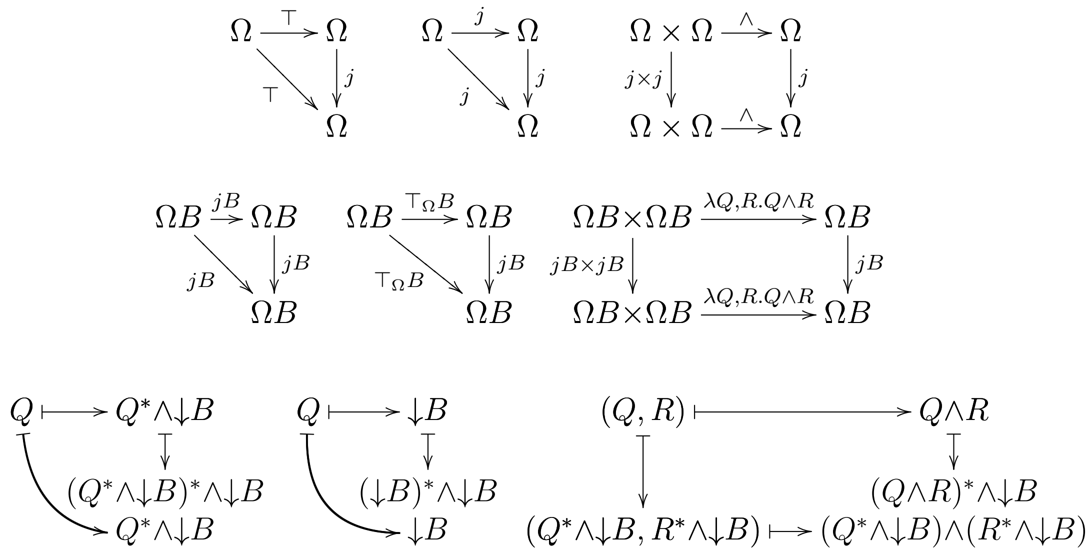




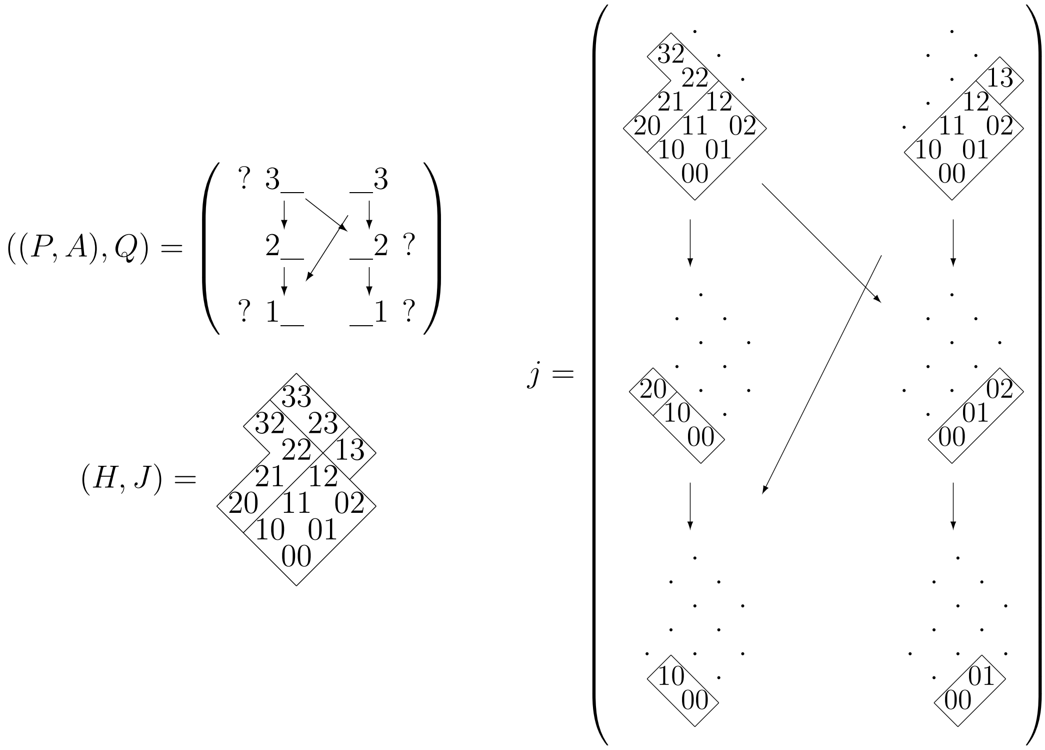
To prove that this $j : \Omega \to \Omega$ is a natural transformation in $\mathbf{Set^D}$ we need to check that the square at the right here commutes:



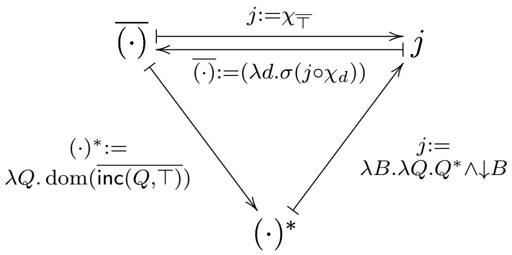
To prove that j obejs J1, J2, J3 we need to check that the lower three diagrams here commute:



Example 5.2.4. We can build a j by brute force from a 2CG with question marks and see what patterns emerge. TODO: explain the notation with slashings on sub-ZHAs of H.

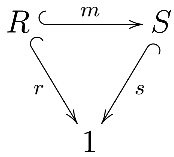


5.3 A bijection (TODO)

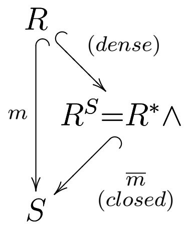


5.4 Dense and closed maps in a $\mathbf{Set}^{\mathbf{D}}$ (TODO)

Let $\mathbf{D}$ be a DAG topos with a J-operator. If R and S are two truth-values in it with $R \leq S$ we can name the inclusions between R, S, and 1 like this:

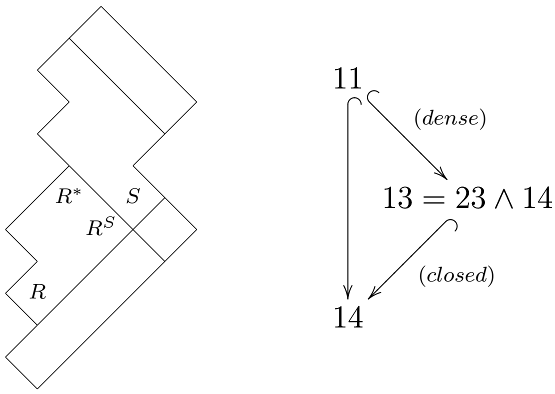


and can use the Theorem 2.3 to calculate $\overline{m}$ using just the J-operator. We have this:



where we call the mediating map d, and we know that d is dense and $\overline{m}$ is closed by Theorem 2.4.

This gives us lots of examples of dense and closed maps in a topos with a J-operator. For example, if $R = 11$ and $S = 14$ in the figure below, then $R^* = 23$ and $R^S = R^* \wedge S = 13$:



By trying many examples of this factorization I got two conjectures:

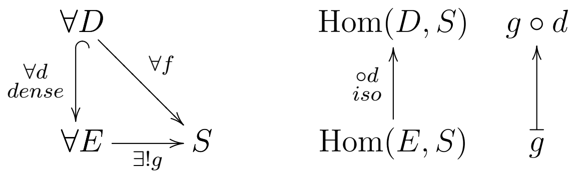
Conjecture 5.4.1. an inclusion $R \hookrightarrow R'$ in the logic is dense iff both R and R' belong to the same region of the slashing,

Conjecture 5.4.2. an inclusion $S' \hookrightarrow S$ in the logic is dense iff S' cannot be moved upwards toward S without crossing a line of the slashing.

They are both easy to prove. (...)

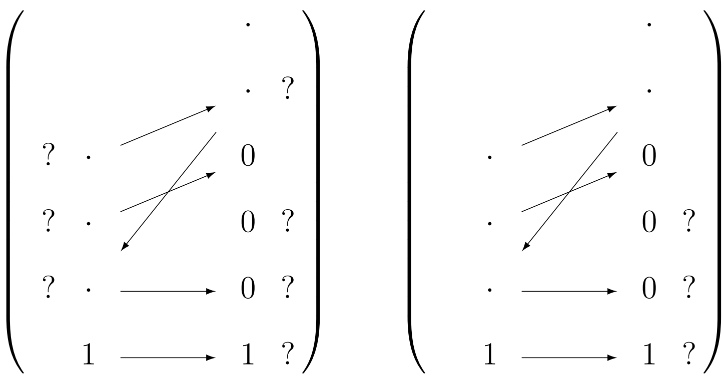
5.5 Sheaves in a $\mathbf{Set}^{\mathbf{D}}$ (TODO)

The usual definition of a sheaf in a topos $\mathbf{E}$ with a topology j is that an object $S \in \mathbf{E}$ is a (j-)sheaf iff for every dense monic $d : D \rightarrowtail E$ in $\mathbf{E}$ every map $f : D \to S$ factors uniquely through $d : D \rightarrowtail E$. I prefer to write this backwards: S is a sheaf iff for every dense monic $d : D \rightarrowtail E$ the map of hom-sets $(\circ d) : \mathrm{Hom}(E, S) \to \mathrm{Hom}(D, S)$ is a bijection. In a diagram:



In a $\mathbf{Set}^{\mathbf{D}}$ we can get lots of properties that sheaves must obey by understanding what the condition above means on a very small family of dense maps — the “basic dense maps”. Suppose that we have a DAG with question marks $((P, A), Q)$ and that our topos $\mathbf{E}$ with topology j is generated by that $((P, A), Q)$. For every point B of the set of question marks Q let $bd(B) : D \hookrightarrow E$ be the inclusion in which E is $\downarrow B$ and D is E minus the point B. We will call these ‘$bd(B)$’s the *basic dense maps* of our topos.

5.6 From question marks to a closure operator (TODO)



5.7 Some topological lemmas (TODO)

Let $(X, \mathcal{O}(X))$ be a topological space.

Lemma 5.7.1. If $A, B \subseteq X$ then $\mathsf{int}(A \cap B) = \mathsf{int}(A) \cap \mathsf{int}(B)$.

Proof. Suppose that $c \in \mathsf{int}(A) \cap \mathsf{int}(B)$. Then we have open sets $W, V \in \mathcal{O}(X)$ that make all the inferences in the tree at the left below true, and so $c \in \mathsf{int}(A \cap B)$. Now suppose that $d \in \mathsf{int}(A \cap B)$. We have an open set $U \in \mathcal{O}(X)$ that makes all the inferences in the tree at the right below true, and so $d \in \mathsf{int}(A) \cap \mathsf{int}(B)$. This means that $\mathsf{int}(A) \cap \mathsf{int}(B) \subseteq \mathsf{int}(A \cap B)$ and $\mathsf{int}(A \cap B) \subseteq \mathsf{int}(A) \cap \mathsf{int}(B)$, and so $\mathsf{int}(A \cap B) = \mathsf{int}(A) \cap \mathsf{int}(B)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{c \in \mathsf{int}(A) \cap \mathsf{int}(B)}{c \in \mathsf{int}(A)}}{c \in V \subseteq \mathsf{int}(A)}}{c \in V \subseteq A}}{c \in V \cap W \subseteq A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{c \in \mathsf{int}(A) \cap \mathsf{int}(B)}{c \in \mathsf{int}(B)}}{c \in V \subseteq \mathsf{int}(B)}}{c \in W \subseteq B}}{c \in V \cap W \subseteq B}}{\dfrac{c \in V \cap W \subseteq A \cap B}{c \in \mathsf{int}(A \cap B)}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{d \in \mathsf{int}(A \cap B)}{d \in U \subseteq \mathsf{int}(A \cap B)}}{d \in U \subseteq A \cap B}}{d \in U \subseteq A}}{d \in \mathsf{int}(A)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{d \in \mathsf{int}(A \cap B)}{d \in U \subseteq \mathsf{int}(A \cap B)}}{d \in U \subseteq A \cap B}}{d \in U \subseteq B}}{d \in \mathsf{int}(B)}}{d \in \mathsf{int}(A) \cap \mathsf{int}(B)}$$

Lemma 5.7.2. If $W, V \in \mathcal{O}(X)$, $W \subseteq V$, and $A \subseteq X$, then $\mathsf{int}(W \cup A) \cap V = \mathsf{int}(W \cup (A \cap V))$.
Proof:

$$\begin{aligned} \mathsf{int}(W \cup A) \cap V &= \mathsf{int}(W \cup A) \cap \mathsf{int}(V) \\ &= \mathsf{int}((W \cup A) \cap V) \\ &= \mathsf{int}((W \cap V) \cup (A \cap V)) \\ &= \mathsf{int}(W \cup (A \cap V)). \end{aligned}$$

5.8 Planar Heyting Algebras and 2CGs (TODO)

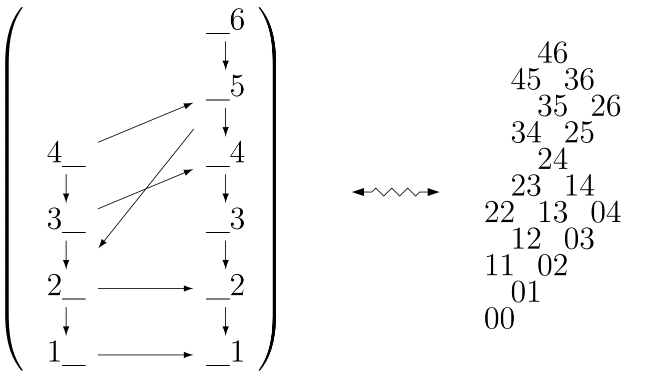
[This section is obsolete (?)]

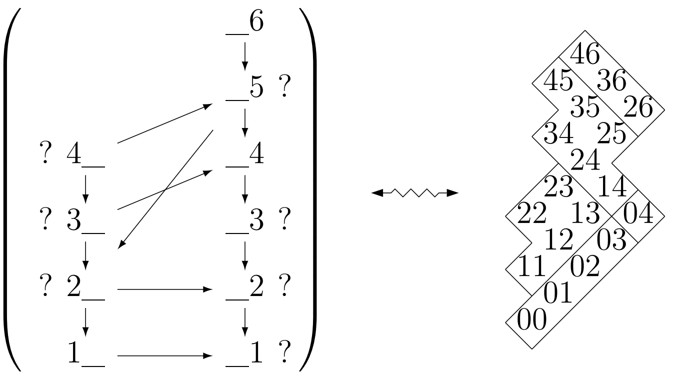
The preprints [PH1] and [PH2] explain how to use 2-column graphs ("2CGs") to develop visual intuition about intuitionistic logic (the first one) and J-operators (the second one). [PH2] stops just short from showing the connections between J-operators and sheaves; we will do that in the next sections.

The central construction in [PH1] can be stated as: every 2CG is associated to a Planar Heyting Algebra (a "ZHA") and vice-versa, and the central construction in [PH2] is: every 2CG with question marks is associated to a ZHA with a J-operator and vice-versa. This can be represented in the general case as:

$$\begin{array}{rcl} (P,A) & \leftrightsquigarrow & H \\ ((P,A),Q) & \leftrightsquigarrow & (H,J) \end{array}$$

where the '$\leftrightsquigarrow$' is pronounced "is associated to". Here is a nice particular case:





For the details see [PH2], sections 1–3.

For a quick explanation of how to interpret the logical connectives in $\mathcal{O}_A(P)$ and $\mathsf{Incs}(1_{\mathbf{E}})$, see section 16 of [PH1]; for a way to visualize these Heyting Algebras, see [PH1], section 13.

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