Sieves on a category $\mathbf{C}$:
(See [LM92, pages 38, 109])

$$
\begin{array}{rcl}
t_C & := & \{\, f \mid \mathrm{cod}(f) = C \,\} \\
\mathsf{Sieves\_on}(C) & := & \{\, S \subseteq t_C \mid \forall \begin{pmatrix} B \xrightarrow{f} C \\ {\scriptstyle g}\uparrow \\ A \end{pmatrix} . \begin{pmatrix} f \in S \\ \downarrow \\ f \circ g \in S \end{pmatrix} \,\}
\end{array}
$$

$$
\begin{array}{ccc}
C & \longmapsto & \mathsf{Sieves\_on}(C) \\
{\scriptstyle h}\uparrow & \longmapsto & \downarrow{\scriptstyle h^*} \\
D & \longmapsto & \mathsf{Sieves\_on}(D)
\end{array}
\qquad
\begin{array}{c}
S \\
\downarrow \\
h^*(S) := \\
\{\, m \mid \mathrm{cod}(m) = D,\ h \circ m \in S \,\}
\end{array}
\qquad
\begin{array}{ccc}
 & & C \\
 & & \uparrow{\scriptstyle h} \\
B & \xrightarrow{m} & D
\end{array}
$$

$$
\begin{array}{l}
\mathbf{C} \\
\mathbf{C}^{\mathrm{op}} \xrightarrow{\mathsf{Sieves\_on}} \mathbf{Set}
\end{array}
$$

Sieves on a topological space $\mathcal{O}(X)$:
(See [LM92, page 70])

$$
\begin{array}{rcl}
t_U & := & \{\, V \subseteq \mathcal{O}(X) \mid V \subseteq U \,\} \\
 & = & \mathcal{O}(U) \\
\mathsf{Sieves\_on}(U) & := & \{\, \mathcal{S} \subseteq t_U \mid \forall \begin{pmatrix} V \subseteq U \\ \cup\!\mid \\ W \end{pmatrix} . \begin{pmatrix} V \in \mathcal{S} \\ \downarrow \\ W \in \mathcal{S} \end{pmatrix} \,\} \\
 & = & \mathcal{D}(t_U) \\
 & = & \mathcal{D}(\mathcal{O}(U))
\end{array}
$$

$$
\begin{array}{ccc}
U & \longmapsto & \mathsf{Sieves\_on}(U) \\
{\scriptstyle V \subseteq U}\uparrow & \longmapsto & \downarrow{\scriptstyle (V \subseteq U)^*} \\
V & \longmapsto & \mathsf{Sieves\_on}(V)
\end{array}
\qquad
\begin{array}{c}
\mathcal{S} \\
\downarrow \\
(V \subseteq U)^*(\mathcal{S}) := \\
\{\, W \in \mathcal{O}(V) \mid W \in \mathcal{S} \,\} \\
= \mathcal{S} \cap \mathcal{O}(V)
\end{array}
\qquad
\begin{array}{ccc}
 & & U \\
 & & \uparrow{\scriptstyle V \subseteq U} \\
W & \xrightarrow{W \subseteq V} & V
\end{array}
$$

$$
\begin{array}{l}
\mathcal{O}(X) \\
\mathcal{O}(X)^{\mathrm{op}} \xrightarrow{\mathsf{Sieves\_on}} \mathbf{Set}
\end{array}
$$

Motivating case:
Start with the case $\mathsf{Sieves\_on} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$.
A "sieve on $U$" is an element $\mathcal{S} \in \mathsf{Sieves\_on}(U)$.
We say that a sieve $\mathcal{U}$ on $U$ "covers $U$" iff $\bigcup\mathcal{U} = U$.
Let $J_{\mathrm{can}}(U) := \{\, \mathcal{U} \in \mathsf{Sieves\_on}(U) \mid \bigcup\mathcal{U} = U \,\}$.
This $J_{\mathrm{can}}$ is a subfunctor of $\mathsf{Sieves\_on} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$...
Its action on morphisms is a restriction of the one in $\mathsf{Sieves\_on}$;
we can define it as $J_{\mathrm{can}}(V \subseteq U)(\mathcal{U}) := \mathcal{U} \cap \mathcal{O}(V)$.
This $J_{\mathrm{can}}$ obeys these three properties:

$$\begin{array}{rcl}
\mathsf{hasmax}_{J_{\mathrm{can}}} & := & (\forall U \in \mathcal{O}(X).\mathcal{O}(U) \in J_{\mathrm{can}}(U)) \\
\mathsf{trans}_{J_{\mathrm{can}}} & := & (\forall V \subseteq U.\forall \mathcal{U} \in J_{\mathrm{can}}(U).(V \subseteq U)^*(\mathcal{U}) \in J_{\mathrm{can}}(V)) \\
\mathsf{stab}_{J_{\mathrm{can}}} & := & \left(\begin{array}{c} \forall U.\forall \mathcal{U} \in J_{\mathrm{can}}(U).\forall \mathcal{S} \in \mathsf{Sieves_on}(U). \\ (\forall V \in \mathcal{U}.((V \subseteq U)^*(\mathcal{S}) \in J_{\mathrm{can}}(V))) \to (\mathcal{S} \in J_{\mathrm{can}}(U)) \end{array}\right)
\end{array}$$

We will define "Grotopness" as their conjunction:

$$\mathsf{Gro\_top}_{J_{\mathrm{can}}} := \mathsf{hasmax}_{J_{\mathrm{can}}} \land \mathsf{trans}_{J_{\mathrm{can}}} \land \mathsf{stab}_{J_{\mathrm{can}}}$$

And we will draw $\mathsf{Gro\_top}_{J_{\mathrm{can}}}$ in this way:

$$\begin{array}{lcc}
U & & (\mathcal{O}(U) \in J_{\mathrm{can}}(U)) \\
\\
U & & (\mathcal{U} \in J_{\mathrm{can}}(U)) \\
\uparrow & & \downarrow \\
V & & (\mathcal{U} \cap \mathcal{O}(V) \in J_{\mathrm{can}}(V)) \\
\\
U \quad \mathcal{U} & \mathcal{S} & (\mathcal{S} \in J_{\mathrm{can}}(U)) \\
\uparrow \;\; \in & \downarrow & \uparrow \\
\forall V & \mathcal{S} \cap \mathcal{O}(V) & (\forall V \in \mathcal{U}.\, \mathcal{S} \cap \mathcal{O}(V) \in J_{\mathrm{can}}(V))
\end{array}$$

It turns out that $\mathsf{Gro\_top}_{J_{\mathrm{can}}}$ is true.

We will generalize this to:
A *Grothendieck topology on* $\mathcal{O}(X)$ is any operation $J$
that takes each $U \in \mathcal{O}(X)$ to a subset $J(U) \subseteq \mathsf{Sieves\_on}(U)$
that obeys $\mathsf{Gro\_top}_J$.
The property $\mathsf{trans}_J$ will guarantee that this $J$
is a subfunctor of $\mathsf{Sieves\_on} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$.

First generalization...
Start with the case $\mathsf{Sieves\_on} : \mathcal{O}(X)^{\text{op}} \to \mathbf{Set}$.
Let $J$ be an operation that takes
each $U \in \mathcal{O}(X)$ to a subset $J(U) \subseteq \mathsf{Sieves\_on}(U)$.
Define $\mathsf{hasmax}_J$, $\mathsf{trans}_J$, $\mathsf{stab}_J$ as:

$$\begin{array}{rcl}
\mathsf{hasmax}_J & := & (\forall U \in \mathcal{O}(X). \mathcal{O}(U) \in J(U)) \\
\mathsf{trans}_J & := & (\forall V \subseteq U. \forall \mathcal{U} \in J(U). (V \subseteq U)^*(\mathcal{U}) \in J(V)) \\
\mathsf{stab}_J & := & \left( \begin{array}{c} \forall U. \forall \mathcal{U} \in J(U). \forall \mathcal{S} \in \mathsf{Sieves_on}(U). \\ (\forall V \in \mathcal{U}. ((V \subseteq U)^*(\mathcal{S}) \in J(V))) \to (\mathcal{S} \in J(U)) \end{array} \right)
\end{array}$$

We will define $\mathsf{Gro\_top}_J$ as their conjunction:

$$\mathsf{Gro\_top}_J := \mathsf{hasmax}_J \land \mathsf{trans}_J \land \mathsf{stab}_J$$

and we will say that this $J$ is a Grothendieck topology on $\mathcal{O}(X)$
iff $\mathsf{Gro\_top}_J$ is true.

We will draw $\mathsf{Gro\_top}_J$ in this way:

$$\begin{array}{cccc}
U & & & (\mathcal{O}(U) \in J(U)) \\
\\
U & & & (\mathcal{U} \in J(U)) \\
\uparrow & & & \downarrow \\
V & & & (\mathcal{U} \cap \mathcal{O}(V) \in J(V)) \\
\\
U \quad \mathcal{U} & & \mathcal{S} & (\mathcal{S} \in J(U)) \\
\uparrow \quad \rotatebox{45}{$\in$} & & \downarrow & \uparrow \\
\forall V & & \mathcal{S} \cap \mathcal{O}(V) & (\forall V \in \mathcal{U}. \mathcal{S} \cap \mathcal{O}(V) \in J(V))
\end{array}$$

Second generalization:
Start with the case $\mathsf{Sieves\_on} : \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$.
Let $J$ be an operation that takes
each $C \in \mathbf{C}$ to a subset $J(C) \subseteq \mathsf{Sieves\_on}(C)$.
Define $\mathsf{hasmax}_J$, $\mathsf{trans}_J$, $\mathsf{stab}_J$ as:

$$\begin{array}{rcl} \mathsf{hasmax}_J & := & (\forall C \in \mathbf{C}.\, t_C \in J(C)) \\ \mathsf{trans}_J & := & (\forall (h : D \to C).\, \forall S \in J(C).\, h^*(S) \in J(D)) \\ \mathsf{stab}_J & := & \left( \begin{array}{c} \forall C \in \mathbf{C}.\, \forall S \in J(C).\, \forall R \in \mathsf{Sieves\_on}(C). \\ (\forall (h : D \to C) \in S.(h^*(R) \in J(D))) \to (R \in J(C)) \end{array} \right) \end{array}$$

We will define $\mathsf{Gro\_top}_J$ as their conjunction:

$$\mathsf{Gro\_top}_J := \mathsf{hasmax}_J \wedge \mathsf{trans}_J \wedge \mathsf{stab}_J$$

and we will say that this $J$ is a Grothendieck topology on $\mathbf{C}$
iff $\mathsf{Gro\_top}_J$ is true.

We will draw $\mathsf{Gro\_top}_J$ in this way:

$$\begin{array}{ccc} C & & (t_C \in J(C)) \end{array}$$

$$\begin{array}{ccc} C & & (S \in J(C)) \\ {\scriptstyle h}\uparrow & & \downarrow \\ D & & (h^*(S) \in J(D)) \end{array}$$

$$\begin{array}{cccc} C \;\in\; S & R & & (R \in J(C)) \\ {\scriptstyle \forall h}\uparrow & \mapsto & & \uparrow \\ \forall D & h^*(R) & & (\forall (h : D \to C) \in S.\, h^*(R) \in J(D)) \end{array}$$

This is exactly the definition in [LM92, page 109] —
I have just reorganized it in a more visual way.

Let $\mathbf{P} \equiv (P, \leq)$ be a partial order.
"$D$ is a down-set of $P$" means: $D \subseteq P$ and
$\forall p_1, p_2 \in P.(p_1 \leq p_2) \to (p_2 \in D \to p_1 \in D)$.
$\mathcal{D}(\mathbf{P}) := \{\, D \subseteq P \mid D \text{ is a down-set of } \mathbf{P} \,\}$.
Note that $\mathcal{D}(\mathbf{P}) \subseteq \mathcal{P}(P)$.

Let $\mathbf{Q} \equiv (Q, \leq, \top, \wedge)$ be a partial order
with maximal element $\top$ and binary meet operation $\wedge$.
"$U$ is an up-set of $\mathbf{Q}$" means $U \subseteq Q$ and
$\forall q_1, q_2 \in Q.(q_1 \leq q_2) \to (q_1 \in U \to q_2 \in U)$.
"$U$ is closed by binary meets" means
$\forall q_1, q_2 \in Q.(q_1 \in U \wedge q_2 \in U) \to (q_1 \wedge q_2 \in U)$.
"$U$ is a filter in $\mathbf{Q}$" means:
$U$ is an up-set of $\mathbf{Q}$ closed by binary meets, with $\top \in U$.
$\mathcal{U}(\mathbf{Q}) := \{\, U \subseteq Q \mid U \text{ is a up-set of } \mathbf{Q} \,\}$.
$\mathcal{F}(\mathbf{Q}) := \{\, U \subseteq Q \mid U \text{ is a filter on } \mathbf{Q} \,\}$.
Note that $\mathcal{U}(\mathbf{Q}), \mathcal{F}(\mathbf{Q}) \subseteq \mathcal{P}(Q)$.

Archetypal case:
$(X, \mathcal{O}(X))$ is a topological space.
$\mathcal{O}(X) \equiv (\mathcal{O}(X), \subseteq)$ is a partial order.
$\mathcal{O}(U) \equiv (\mathcal{O}(U), \subseteq)$ is a partial order for any $U \in \mathcal{O}(X)$.
"$\mathcal{S}$ is a sieve on $U$" means $\mathcal{S} \in \mathcal{D}(\mathcal{O}(U))$.
"$\mathcal{U}$ is a covering sieve on $U$" means $\mathcal{U} \in \mathcal{D}(\mathcal{O}(U))$ and $\bigcup \mathcal{U} = U$.
$\mathsf{Sieves\_on}(U) := \{\, \mathcal{S} \mid \mathcal{S} \text{ is a sieve on } U \,\}$
$\mathsf{Sieves\_on}(V \subseteq U)(\mathcal{S}) := (V \subseteq U)^*(\mathcal{S}) := \mathcal{S} \cap \mathcal{O}(V)$
$\mathsf{Covsieves\_on}(U) := \{\mathcal{U} \mid \mathcal{U} \text{ is a covering sieve on } U\,\}$
$\mathsf{Covsieves\_on}(V \subseteq U)(\mathcal{U}) := (V \subseteq U)^*(\mathcal{U}) := \mathcal{U} \cap \mathcal{O}(V)$
$J_{\text{can}}(U) := \mathsf{Covsieves\_on}(U)$
$J_{\text{can}}(V \subseteq U)(\mathcal{U}) := (V \subseteq U)^*(\mathcal{U}) := \mathcal{U} \cap \mathcal{O}(V)$
$\mathsf{hasmax}_{J_{\text{can}}} := (\forall U \in \mathcal{O}(X).\mathcal{O}(U) \in J_{\text{can}}(U))$
$\mathsf{trans}_{J_{\text{can}}} := (\forall V \subseteq U.\forall \mathcal{U} \in J_{\text{can}}(U).(V \subseteq U)^*(\mathcal{U}) \in J_{\text{can}}(V))$
$$\mathsf{stab}_{J_{\text{can}}} := \begin{pmatrix} \forall U.\forall \mathcal{U} \in J_{\text{can}}(U).\forall \mathcal{S} \in \mathsf{Sieves\_on}(U). \\ (\forall V \in \mathcal{U}.((V \subseteq U)^*(\mathcal{S}) \in J_{\text{can}}(V))) \to (\mathcal{S} \in J_{\text{can}}(U)) \end{pmatrix}$$
$\mathsf{Gro\_top}_{J_{\text{can}}} := \mathsf{hasmax}_{J_{\text{can}}} \wedge \mathsf{trans}_{J_{\text{can}}} \wedge \mathsf{stab}_{J_{\text{can}}}$
$\mathsf{Gro\_top}_{J_{\text{can}}}$ is true.

Grotopness (on topological spaces):
$(X, \mathcal{O}(X))$ is a topological space.
$\mathcal{O}(X) \equiv (\mathcal{O}(X), \subseteq)$ is a partial order.
$\mathcal{O}(U) \equiv (\mathcal{O}(U), \subseteq)$ is a partial order for any $U \in \mathcal{O}(X)$.
"$\mathcal{S}$ is a sieve on $U$" means $\mathcal{S} \in \mathcal{D}(\mathcal{O}(U))$.
"$\mathcal{U}$ is a covering sieve on $U$" means $\mathcal{U} \in \mathcal{D}(\mathcal{O}(U))$ and $\bigcup \mathcal{U} = U$.
$\mathsf{Sieves\_on}(U) := \{\, \mathcal{S} \mid \mathcal{S} \text{ is a sieve on } U \,\}$
$\mathsf{Sieves\_on}(V \subseteq U)(\mathcal{S}) := (V \subseteq U)^*(\mathcal{S}) := \mathcal{S} \cap \mathcal{O}(V)$
$\mathsf{Covsieves\_on}(U) := \{\, \mathcal{U} \mid \mathcal{U} \text{ is a covering sieve on } U \,\}$
$\mathsf{Covsieves\_on}(V \subseteq U)(\mathcal{U}) := (V \subseteq U)^*(\mathcal{U}) := \mathcal{U} \cap \mathcal{O}(V)$
$J(U) \subseteq \mathsf{Sieves\_on}(U)$
$J(V \subseteq U)(\mathcal{U}) := (V \subseteq U)^*(\mathcal{U}) := \mathcal{U} \cap \mathcal{O}(V)$
$\mathsf{hasmax}_J := (\forall U \in \mathcal{O}(X).\mathcal{O}(U) \in J(U))$
$\mathsf{trans}_J := (\forall V \subseteq U.\forall \mathcal{U} \in J(U).(V \subseteq U)^*(\mathcal{U}) \in J(V))$
$$\mathsf{stab}_J := \left( \begin{matrix} \forall U.\forall \mathcal{U} \in J(U).\forall \mathcal{S} \in \mathsf{Sieves\_on}(U). \\ (\forall V \in \mathcal{U}.((V \subseteq U)^*(\mathcal{S}) \in J(V))) \to (\mathcal{S} \in J(U)) \end{matrix} \right)$$
$\mathsf{Gro\_top}_J := \mathsf{hasmax}_J \wedge \mathsf{trans}_J \wedge \mathsf{stab}_J$

A Grothendieck topology $J$ on $\mathcal{O}(X)$
is a subfunctor $J : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$
of $\mathsf{Sieves\_on} : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$
that obeys $\mathsf{Gro\_top}_J$.

From Lindenhovius, p.8., with a different notation:
From Mac Lane/Moerdijk, p.109:

From Mac Lane/Moerdijk, p.110:

Definition 1. A (Grothendieck) topology on a category $\mathbf{C}$ is a function $J$ which assigns to each object $C$ of $\mathbf{C}$ a collection $J(C)$ of sieves on $C$, in such a way that:

(i) the maximal sieve $t_C = \{\, f \mid \mathrm{cod}(f) = C \,\}$ is in $J(C)$;

(ii) (stability axiom) if $S \in J(C)$, then $h^*(S) \in J(D)$ for any arrow $h : D \to C$;

(iii) (transitivity axiom) if $S \in J(C)$ and $R$ is any sieve on $C$ such that $h^*(R) \in J(D)$ for all $h : D \to C$ in $S$, then $R \in J(C)$.

$$\begin{array}{rcl}
J(C) & \subseteq & \mathsf{Sieves_on}(C) \\
J(C) & \in & \mathcal{P}(\mathsf{Sieves_on}(C)) \\
J & : & (C : \mathbf{C}) \to \mathcal{P}(\mathsf{Sieves_on}(C)) \\
\mathsf{hasmax}_J & := & \forall C.\ t_C \in J(C) \\
\mathsf{trans}_J & := & \forall (h : D \to C).\ \forall S \in J(C).\ h^*(S) \in J(D) \\
\mathsf{stab}_J & := & \forall C. \forall S \in J(C).\ \forall R \in \mathsf{Sieves_on}(X). \\
& & (\forall (D \stackrel{h}{\to} C).\ h^*(R) \in J(D)) \to (R \in J(C))
\end{array}$$

We draw $(J, \mathsf{hasmaxs}_J, \mathsf{trans}_J, \mathsf{stab}_J)$ as:

$$\begin{array}{ccc}
C & & (t_C \in J(C)) \\
\\
C & & (S \in J(C)) \\
{\scriptstyle h}\uparrow & & \downarrow \\
D & & (h^*(S) \in J(D)) \\
\\
C \in S & R & (R \in J(C)) \\
{\scriptstyle \forall h}\uparrow & \downarrow & \uparrow \\
\forall D & h^*(R) & (\forall h \in S.\ h^*(R) \in J(D))
\end{array}$$

# References

[LM92] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic: a first introduction to topos theory.* Springer, 1992.