

Notes on [Lambek86], a.k.a.:

“Cartesian Closed Categories and Typed λ -Calculi”, available at:

https://link.springer.com/chapter/10.1007%2F3-540-17184-3_44

or as pages 136–175 of LNCS 242:

<https://link.springer.com/book/10.1007/3-540-17184-3>

These notes are at:

<http://angg.twu.net/LATEX/2020lambek86.pdf>

The introduction says:

While the material in the first part has been published before [L1974, 1980, LS1984], an attempt is made to look at some of it from a different point of view and to clarify some difficult points. At any rate, it is hoped that it will serve as an introduction to the forthcoming book “Introduction to higher order categorical logic”, written in collaboration with Phil Scott.

2. Cartesian Categories

Page 141:

Although the conjunction calculus contains no symbol for implication, it admits the following form of the *deduction theorem*:

Proposition 2.1: if $\varphi(x) : B \rightarrow C$ is a proof from the assumption $x : T \rightarrow A$, there is a proof $\kappa_{x \in A} \varphi(x) : A \wedge B \rightarrow C$ in \mathcal{L} not depending on the assumption x .

(...)

Proof: there are four cases in the proof of the deduction theorem:

- (1) $\varphi(x) = k : B \rightarrow C$, a proof in \mathcal{L} ;
- (2) $\varphi(x) = x : T \rightarrow A$, where $B = T$ and $C = A$;
- (3) $\varphi(x) = \chi(x)\psi(x)$, where $\psi(x) : B \rightarrow D$ and $\chi(x) : D \rightarrow C$;
- (4) $\varphi(x) = \langle \psi(x), \chi(x) \rangle$, where $\psi(x) : B \rightarrow D$, $\chi(x) : B \rightarrow E$ and $C = D \wedge E$.

We define $\kappa_{x \in A} \varphi(x)$ by induction on the “length” of $\varphi(x)$:

- (1) $\kappa_{x \in A} k = k\pi'_{A,B}$;
- (2) $\kappa_{x \in A} x = \pi_{A,T}$;
- (3) $\kappa_{x \in A} (\chi(x)\psi(x)) = \kappa_{x \in A} \chi(x) \langle \pi_{A,B}, \kappa_{x \in A} \psi(x) \rangle$;
- (4) $\kappa_{x \in A} \langle \psi(x), \chi(x) \rangle = \langle \kappa_{x \in A} \psi(x), \kappa_{x \in A} \chi(x) \rangle$;

$$\underbrace{\kappa_{x \in A}(\underbrace{k}_{:B \rightarrow C})}_{:A \times B \rightarrow C} = \underbrace{\underbrace{k}_{:B \rightarrow C} \underbrace{\pi'_{A,B}}_{:A \times B \rightarrow B}}_{:A \times B \rightarrow C}$$

$$\underbrace{\kappa_{x \in A}(\underbrace{x}_{:T \rightarrow A})}_{:A \times T \rightarrow A} = \underbrace{\pi_{A,T}}_{:A \times T \rightarrow A}$$

$$\underbrace{\kappa_{x \in A}(\underbrace{\chi(x)}_{:D \rightarrow C} \underbrace{\psi(x)}_{:B \rightarrow D})}_{:A \times B \rightarrow C} = \underbrace{\kappa_{x \in A} \underbrace{\chi(x)}_{:D \rightarrow C} \langle \underbrace{\pi_{A,B}}_{:A \times B \rightarrow A}, \underbrace{\kappa_{x \in A} \underbrace{\psi(x)}_{:B \rightarrow D}}_{:A \times B \rightarrow D} \rangle}_{:A \times B \rightarrow C}$$

$$\underbrace{\kappa_{x \in A} \langle \underbrace{\psi(x)}_{:B \rightarrow E}, \underbrace{\chi(x)}_{:B \rightarrow D} \rangle}_{:A \times B \rightarrow D \times E} = \langle \underbrace{\kappa_{x \in A} \underbrace{\psi(x)}_{:B \rightarrow E}}_{:A \times B \rightarrow E}, \underbrace{\kappa_{x \in A} \underbrace{\chi(x)}_{:B \rightarrow D}}_{:A \times B \rightarrow D} \rangle_{:A \times B \rightarrow D \times E}$$