On a way to visualize (some) sheaves

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Abstract

This is an attempt to connect the way in which sheaves are presented in [LM92] with the approach "for children" from [PH1], [PH2], and [FavC] — but these notes are a work in progress that is still in a very preliminary form.

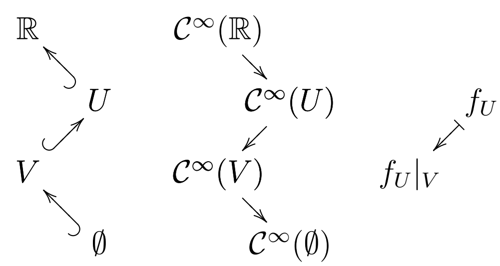
1 Sheaves

The archetypal example of a sheaf is the operation

$$\begin{array}{rrcl} \mathcal{C}^\infty : & \mathcal{O}(\mathbb{R})^{\text{op}} & \to & \mathbf{Set} \\ & U & \mapsto & \{\, f : U \to \mathbb{R} \mid f \text{ is } \mathcal{C}^\infty \,\} \end{array}$$

that expects open sets of $\mathbb{R}$ and returns sets of functions; more precisely, for each subset $U \subseteq \mathbb{R}$ it returns $\mathcal{C}^\infty(U)$, the set of infinitely differentiable functions from U to $\mathbb{R}$.

This $\mathcal{C}^\infty$ is a *contravariant functor*. The topology $\mathcal{O}(\mathbb{R})$ is a preorder category whose morphisms are the inclusions, and the image by $\mathcal{C}^\infty$ of each inclusion map $V \hookrightarrow U$ is a *restriction map* $\mathcal{C}^\infty(U) \to \mathcal{C}^\infty(V)$. In a diagram:



$$\mathcal{O}(\mathbb{R})$$
$$\mathcal{O}(\mathbb{R})^{\mathrm{op}} \xrightarrow{\mathcal{C}^\infty} \mathbf{Set}$$

This diagram follows the conventions from [FavC, section 2]. In brief:

(CAI) the diagram $\emptyset \hookrightarrow V \hookrightarrow U \hookrightarrow \mathbb{R}$ is drawn above $\mathcal{O}(\mathbb{R})$ to indicate that it is inside the category $\mathcal{O}(\mathbb{R})$,

(CFSh) the image of $\emptyset \hookrightarrow V \hookrightarrow U \hookrightarrow \mathbb{R}$ by the functor $\mathcal{C}^\infty$ is drawn as a diagram with the same shape as the original; in particular, the unnamed arrow $\mathcal{C}^\infty(V) \leftarrow \mathcal{C}^\infty(U)$ is the image by $\mathcal{C}^\infty$ of the (unnamed) inclusion map $V \hookrightarrow U$,

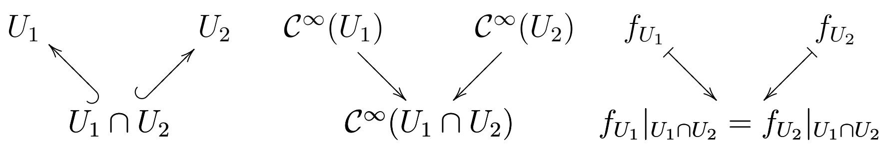
(C$\mapsto$) the arrow $f_U \mapsto f_U|_V$ is the internal view of the unnamed arrow $\mathcal{C}^\infty(U) \to \mathcal{C}^\infty(V)$, and $f_U|_V := \mathcal{C}^\infty(U \hookleftarrow V)(f_U)$.

The rationale for having an '$\mathcal{O}(\mathbb{R})$' above the '$\mathcal{O}(\mathbb{R})^{\mathrm{op}}$' is explained in [FavC, section 7.4], and our reasons for drawing topological spaces with the "everything" on top and the "nothing" at the bottom are explained in [PH1]; in short, that's because we will at some point treat $\mathcal{O}(\mathbb{R})$ as a logic, in which $\mathbb{R}$ is 'true', 'top', and '$\top$'.

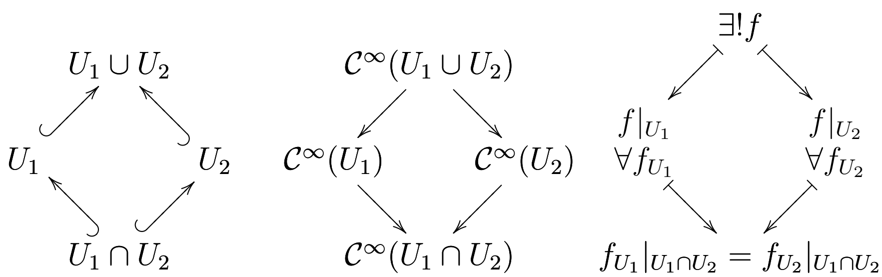
1.1 The unique glueing property

$\mathcal{C}^\infty$ has the "unique glueing property". The UGP can be formalized in several different, and slightly incompatible, ways.

Take any two open sets $U_1, U_2 \in \mathcal{O}(\mathbb{R})$ and choose functions $f_{U_1} \in \mathcal{C}^\infty(U_1)$ and $f_{U_2} \in \mathcal{C}^\infty(U_2)$. We say that f_{U_1} and f_{U_2} are *compatible* when their restrictions to $\mathcal{C}^\infty(U_1 \cap U_2)$ are the same — i.e., when $f_{U_1}|_{U_1\cap U_2} = f_{U_2}|_{U_1\cap U_2}$. In a diagram:



Our first version of the unique glueing property is this. For all $U_1, U_2 \in \mathcal{O}(\mathbb{R})$, for every *compatible* pair $(f_{U_1}, f_{U_2}) \in \mathcal{O}(U_1) \times \mathcal{O}(U_2)$ has a *unique glueing*: there is a unique $f \in \mathcal{C}^\infty(U_1 \cup U_2)$ such that this f restricts to f_{U_1} and f_{U_2}, i.e., $f|_{U_1} = f_{U_1}$ and $f|_{U_2} = f_{U_2}$. In a diagram:



Our second version of the UGP deals with compatible *families* of functions. Take an index set I, a family of open sets $(U_i)_{i\in I}$ such that each $U_i \in \mathcal{O}(\mathbb{R})$, and a family of functions $(f_i)_{i\in I}$ such that each $f_i \in U_i$. We say that this family of functions is *pairwise compatible* if

$$\forall i, j \in I. f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j},$$

and we say that this family *has a unique glueing* if there is a unique $f \in \mathcal{C}^\infty(U)$, where $U := \bigcup_{i\in I} U_i$, such that

$$\forall i \in I. f_i = f|_{U_i}.$$

A standard reference for this is [LM92], section II.1, but it uses some notational tricks that took me years (really!!!) to understand... they are easy to decypher when we know some notations from dependent types, though.

Let $A = \{2, 3\}$, $B = \{4, 5\}$, $C_2 = \{20, 21\}$, $C_3 = \{32, 33\}$. The notations for the set of all pairs (a, b) with $a \in A$ and $b \in B$ and for all function that take each $a \in A$ to a $b \in B$ are well-known: $A \times B$ and B^A, but we will sometimes write B^A as $A{\to}B$. The standard notations for the set of all pairs (a, c) with $a \in A$ and $c \in C_a$ and for the set of all functions that take each $a \in A$ to a $c \in C_a$ are less familiar (see [HOTT, sections 1.4 and 1.6]): $\Sigma_{a \in A}.C_a$ and $\Pi_{a \in A}.C_a$. Some programming languages with support for dependent types, such as Agda, implement a notation that looks like an extension of '$\times$' and '$\to$':

$$\begin{array}{rcl} (a:A) \times C_a & := & \Sigma_{a \in A}.C_a \\ (a:A) \to C_a & := & \Pi_{a \in A}.C_a \end{array}$$

So we have:

$$\begin{array}{rcl} (a:A) \times C_a & := & \{(2,20),(2,21),(3,32),(3,33)\}, \\ (a:A) \to C_a & := & \{\{(2,20),(3,32)\}, \\ & & \ \{(2,20),(3,33)\}, \\ & & \ \{(2,21),(3,32)\}, \\ & & \ \{(2,21),(3,33)\}\}. \end{array}$$

A family of pairwise functions on $\mathcal{C}^\infty$ is a triple

$$(I, \mathcal{U}, \mathcal{F}) \quad : \quad (I : \mathsf{Sets}) \times (\mathcal{U} : I \to \mathcal{O}(\mathbb{R})) \times (\mathcal{F} : (i : I) \to \mathcal{C}^\infty(\mathcal{U}i))$$

obeying:

$$\forall i, j \in I.(\mathcal{F}i)|_{(\mathcal{U}i \cap \mathcal{U}j)} = (\mathcal{F}j)|_{(\mathcal{U}i \cap \mathcal{U}j)}.$$

Our third version of the UGP needs "downward-closedness". For a subset $\mathcal{V} \subseteq \mathcal{O}(\mathbb{R})$ we define

$$\downarrow\mathcal{V} := \{\, W \in \mathcal{O}(\mathbb{R}) \mid \exists V \in \mathcal{V}.W \subseteq V \,\},$$

and we say that a $\mathcal{V} \subseteq \mathcal{O}(\mathbb{R})$ is *sieve* when $\mathcal{V} = \downarrow\mathcal{V}$. A *compatible family* $\mathcal{G}$ *on a sieve* $\mathcal{V}$ is a family $\mathcal{G} : (V : \mathcal{V}) \to \mathcal{C}^\infty(V)$ obeying:

$$\forall V, W \in \mathcal{V}.\ W \subseteq V \to \mathcal{G}W = \mathcal{G}V|_W.$$

A compatible family $\mathcal{G}$ on a sieve $\mathcal{V}$ has a unique glueing iff there is a unique $f \in \mathcal{C}^\infty(U)$, where $U = \bigcup \mathcal{V}$, such that

$$\forall V \in \mathcal{V},\ f|_V = \mathcal{G}V.$$

A good way to understand how these ideas can be generalized is to work on cases where everything is finite and everything can be drawn explicitly.

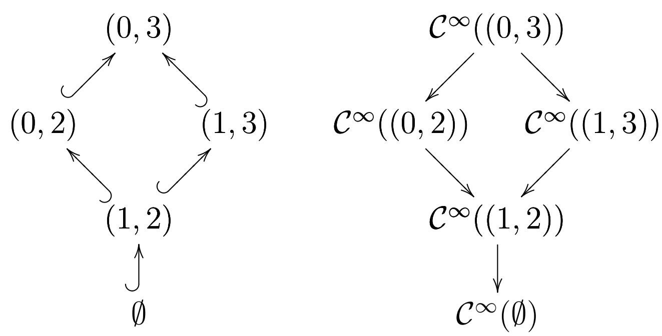
1.2 Presheaves (on some finite topologies)

In the previous sections we worked with a fixed topology, $\mathcal{O}(\mathbb{R})$, and with a fixed contravariant functor, $\mathcal{C}^\infty : \mathcal{O}(\mathbb{R})^{\text{op}} \to \mathbf{Set}$. We saw that $\mathcal{C}^\infty$ obeys three different "unique glueing properties" that were easy to understand; we will now generalize this, in two steps.

Let X be the open interval $(0,3) \subset \mathbb{R}$, and let

$$\mathcal{O}(X) := \{\emptyset,\ (1,2),\ (0,2),\ (1,3),\ (0,3)\},$$

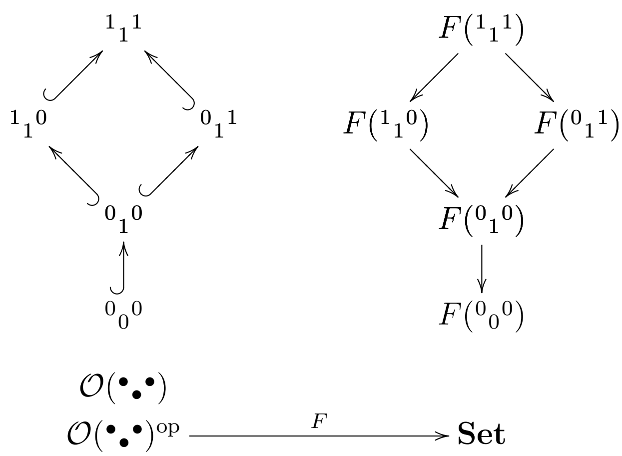
which is a subtopology of the usual topology on $(0,3)$. As $\mathcal{O}(X)$ is finite we can draw it, and its image by $\mathcal{C}^\infty$, explicitly:



$$\mathcal{O}(X)$$
$$\mathcal{O}(X)^{\text{op}} \xrightarrow{\ \mathcal{C}^\infty\ } \mathbf{Set}$$

In [PH1, sections 1, 2, 12, and 13], we saw how to interpret diagrams like $^\bullet{}_\bullet{}^\bullet$ as directed acyclical graphs (DAGs), how to define an order topology on a DAG, and how to draw these topologies. If we replace X by $^\bullet{}_\bullet{}^\bullet$ and $\mathcal{C}^\infty$

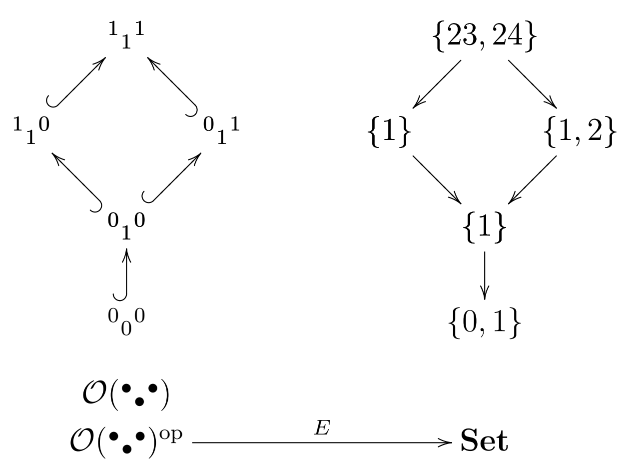
by an arbitrary contravariant functor $F : \mathcal{O}({}^{\bullet}{}_{\bullet}{}^{\bullet})^{\mathrm{op}} \to \mathbf{Set}$ we get this,



that by (an intentional) coincidence has the same shape as the previous diagram. The trick to "pronounce" things like ${}^{0}{}_{1}{}^{1}$ is explained in [PH1, section 1]: if we read aloud the digits of ${}^{0}{}_{1}{}^{1}$ in "reading order", i.e., for top to bottom and in each line from left to right, then it becomes "zero-one-one".

Now that our topology has a definite shape we can use that shape, with '0's and '1's at the right positions, to talk of subsets of it. For example, $\{{}^{1}{}_{1}{}^{0}, {}^{0}{}_{1}{}^{1}\} = \begin{smallmatrix}&0&\\1&&1\\&0&\\&0&\end{smallmatrix}$, and this is not a sieve, because $\downarrow \begin{smallmatrix}&0&\\1&&1\\&0&\\&0&\end{smallmatrix} = \begin{smallmatrix}&0&\\1&&1\\&1&\\&1&\end{smallmatrix}$; also, $\bigcup \begin{smallmatrix}&0&\\1&&1\\&0&\\&0&\end{smallmatrix} = \bigcup\{{}^{1}{}_{1}{}^{0}, {}^{0}{}_{1}{}^{1}\} = {}^{1}{}_{1}{}^{1}$, and $\downarrow\{\bigcup \begin{smallmatrix}&0&\\1&&1\\&0&\\&0&\end{smallmatrix}\} = \begin{smallmatrix}&1&\\1&&1\\&1&\\&1&\end{smallmatrix}$.

We can also choose other presheaves and test if they obey the unique glueing properties. Let E be this functor:



For the first version of the UGP, let $U_1 = {}^1{}_1{}^0$, $U_2 = {}^0{}_1{}^1$, $f_{U_1} = 1$, $f_{U_2} = 2$. This is a compatible pair, because $U_1 \cap U_2 = {}^0{}_1{}^0$ and $f_{U_1}|_{U_1\cap U_2} = f_{U_2}|_{U_1\cap U_2} = 1$ — but there is no $f \in E(U_1 \cup U_2)$ that restricts to both f_{U_1} and f_{U_2}; to check this we just need to test the two candidates, 23 and 24.

The second version of the UGP needs an index set. Take $I = \{42, 200\}$ — 42 and 200 are two of my favorite numbers —, and $\mathcal{U} = \{(42, {}^1{}_1{}^0), (200, {}^0{}_1{}^1)\}$, $\mathcal{F} = \{(42, 1), (200, 2)\}$. This is a pairwise compatible family of functions, but it has two possible glueings. We have $U := \bigcup_{i\in I} U_i = {}^1{}_1{}^1$, and both 23 and 24 are glueings, i.e., are possible values for the $f \in E(\bigcup_{i\in I} U_i)$ that obeys $\forall i \in I.f_i = f|_{U_i}$.

The third version of the UGP is the nicest to draw. A pair $(\mathcal{V} \subseteq \mathcal{O}({}^1{}_1{}^1), \mathcal{G} : (V : \mathcal{V}) \to E(V))$ can be drawn using the conventions for drawing partial functions from [PH1, section 1]: we call $\mathcal{V}$ the *support* of the family $(\mathcal{V}, \mathcal{G})$, and for each $V \in \mathcal{O}({}^\bullet{}_\bullet{}^\bullet)$ we draw $\mathcal{G}V$ on the position of V if $V \in \mathcal{V}$; when $V \notin \mathcal{V}$ we draw a '$\cdot$'. For example, let

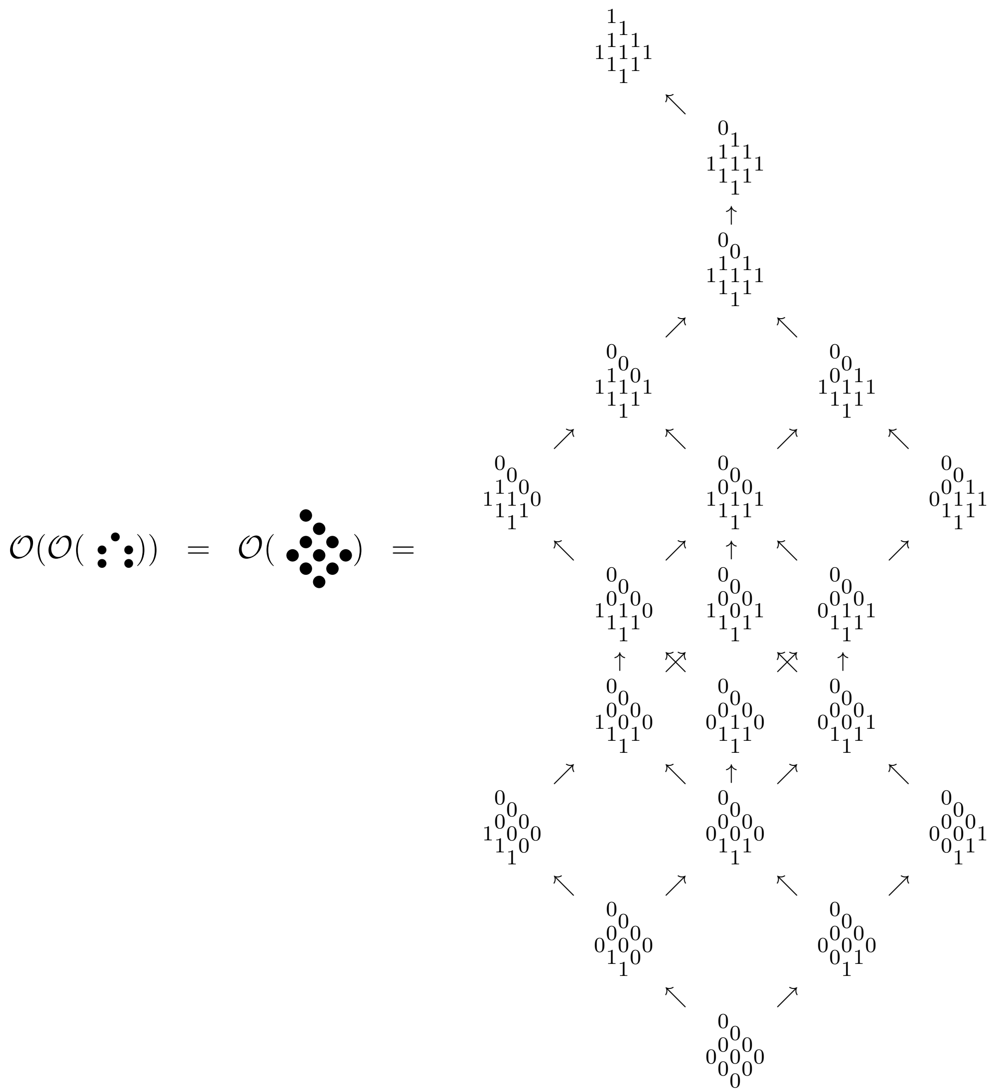
$$\begin{aligned}
\mathcal{V} &= \{{}^1{}_1{}^0, {}^0{}_1{}^1, {}^0{}_0{}^0\} = \begin{smallmatrix} & \cdot & \\ \bullet & & \bullet \\ & \cdot & \\ & \bullet & \end{smallmatrix} = \begin{smallmatrix} & 0 & \\ 1 & & 1 \\ & 0 & \\ & 1 & \end{smallmatrix}, \\
\mathcal{G} &= \{({}^1{}_1{}^0, 1), ({}^0{}_1{}^1, 2), ({}^0{}_0{}^0, 0)\} = \begin{smallmatrix} & \cdot & \\ 1 & & 2 \\ & \cdot & \\ & 0 & \end{smallmatrix}, \\
(\mathcal{V}, \mathcal{G}) &= \left(\begin{smallmatrix} & \cdot & \\ \bullet & & \bullet \\ & \cdot & \\ & \bullet & \end{smallmatrix}, \begin{smallmatrix} & \cdot & \\ 1 & & 2 \\ & \cdot & \\ & 0 & \end{smallmatrix} \right).
\end{aligned}$$

This $\mathcal{V} = \begin{smallmatrix} & \cdot & \\ \bullet & & \bullet \\ & \cdot & \\ & \bullet & \end{smallmatrix}$ isn't a sieve, and this $\mathcal{G} = \begin{smallmatrix} & \cdot & \\ 1 & & 2 \\ & \cdot & \\ & 0 & \end{smallmatrix}$ isn't compatible downwards. We want to restrict our attention to sieves, and it turns out the set of all sieves on $\mathcal{O}({}^\bullet{}_\bullet{}^\bullet)$ is exactly $\mathcal{O}(\mathcal{O}({}^\bullet{}_\bullet{}^\bullet))$, the order topology on the partial order $\begin{smallmatrix} & \bullet & \\ \bullet & & \bullet \\ & \bullet & \\ & \bullet & \end{smallmatrix}$. A very good place to learn about this is the section about down-sets and up-sets of [DP02, page 20 onwards]; for an ordered set P they define $\mathcal{O}(P)$ as the "ordered set of down-sets of P" in a way that is very easy to iterate

$$
\begin{array}{ccc}
\left(\begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}\right) & \mapsto & \left\{ \begin{smallmatrix} & 23 & \\ 1 & & 2 \\ & 1 & \\ & 1 & \end{smallmatrix},\ \begin{smallmatrix} & 24 & \\ 1 & & 2 \\ & 1 & \\ & 1 & \end{smallmatrix} \right\} \\
\uparrow & & \downarrow \\
\left(\begin{smallmatrix} & 0 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}\right) & \mapsto & \left\{ \begin{smallmatrix} & \cdot & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix},\ \begin{smallmatrix} & \cdot & \\ 1 & & 2 \\ & 1 & \\ & 1 & \end{smallmatrix} \right\}
\end{array}
\qquad
\begin{array}{ccc}
\left(\begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 0 & \\ & 1 & \end{smallmatrix}\right) & \mapsto & \left\{ \begin{smallmatrix} & \cdot & \\ \cdot & & \cdot \\ & \cdot & \\ & 0 & \end{smallmatrix},\ \begin{smallmatrix} & \cdot & \\ \cdot & & \cdot \\ & \cdot & \\ & 1 & \end{smallmatrix} \right\} \\
\uparrow & & \downarrow \\
\left(\begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 0 & \\ & 0 & \end{smallmatrix}\right) & \longmapsto & \left\{ \begin{smallmatrix} & \cdot & \\ \cdot & & \cdot \\ & \cdot & \\ & \cdot & \end{smallmatrix} \right\}
\end{array}
$$

$$
\mathrm{Sub}\left(\begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}\right) \qquad\qquad \mathrm{Sub}\left(\begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}\right)
$$

$$
\mathrm{Sub}\left(\begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}\right)^{\mathrm{op}} \xrightarrow{E} \mathbf{Set} \qquad\qquad \mathrm{Sub}\left(\begin{smallmatrix} & 1 & \\ 1 & & 1 \\ & 1 & \\ & 1 & \end{smallmatrix}\right)^{\mathrm{op}} \xrightarrow{E} \mathbf{Set}
$$



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