Notes on Set Comprehensions

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WARNING: this is an attempt to join in a single PDF file many of my notes on set comprehensions, including some that were written in Portuguese, and that I need to translate to English... but this is currently a mess!

URL of the current/latest version:
https://anggtwu.net/LATEX/2026-set-comprehensions.pdf

These notes are mentioned here:
https://anggtwu.net/LATEX/2026bad-foundations.pdf#page=6

1 Introduction

TODO: expand this one-page explanation:
https://anggtwu.net/LATEX/2026bad-foundations.pdf#page=5

My favorite way of presenting variables to students with bad foundations is by starting from a certain set of exercises on set comprehensions in the next section, in which all variables are bound and vary over small finite sets, and all the resulting sets are easy to draw; this lets us avoid most of the difficulties in [EllermeijerHeck, p.6] – "The meaning of variable is variable in mathematics" – in a first moment; we start by "variables that vary", and then we build the other possible meanings for variables on top of that.

I started using that approach in [MPG], that was in Portuguese and for freshmen students, and reused it in [PH1] and other papers. This appendix contains the exercises from [MPG] with a better introduction, and then a discussion of the rationale.

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These two kinds of set comprehensions are written similarly but they correspond to different programs:

$$\{\, 10a \mid a \in \{2,3,4\} \,\} \quad = \quad \{20,30,40\}$$

```
for a=2,4 do
  print(10*a)
end
```

$$\{\, a \in \{2,3,4\} \mid a^2 < 10 \,\} \quad = \quad \{2,3\}$$

```
for a=2,4 do
  if a^2<10 then
    print(a)
  end
end
```

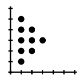
One way to unify them is to create a third notation, that uses a '`;`' instead of a '`|`', and in which all expressions at the left of the '`;`' are either "generators" or "filters", and after the '`;`' we have a "result expression". Generators, filters, and result expressions correspond to the '`for`'s, '`if`'s, and '`print`'s in the Lua programs above, and if we annotate the parts with underbraces to indicate the generators, the filters, and result expressions, we get this:

$$\{\, \underbrace{10a}_{\text{expr}} \mid \underbrace{a \in \{2,3,4\}}_{\text{gen}} \,\} \quad = \quad \{\, \underbrace{a \in \{2,3,4\}}_{\text{gen}} \,;\, \underbrace{10a}_{\text{expr}} \,\}$$

$$\{\, \underbrace{a \in \{2,3,4\}}_{\text{gen}} \mid \underbrace{a^2 < 10}_{\text{filt}} \,\} \quad = \quad \{\, \underbrace{a \in \{2,3,4\}}_{\text{gen}}, \underbrace{a^2 < 10}_{\text{filt}} \,;\, \underbrace{a}_{\text{expr}} \,\}$$

We can calculate the result of a set comprehension by using a tree. For example, in this case,

$$\{\, \underbrace{x \in \{1,\ldots,5\}}_{\text{gen}}, \underbrace{y \in \{x,\ldots,6-x\}}_{\text{gen}} \,;\, \underbrace{(x,y)}_{\text{expr}} \,\} \quad =$$



we can draw – or, rather, typeset – that tree in several ways. Here are two ways, that follow the same conventions but have different columns:

x	y	(x,y)
1	1	$(1,1)$
	2	$(1,2)$
	3	$(1,3)$
	4	$(1,4)$
	5	$(1,5)$
2	2	$(2,2)$
	3	$(2,3)$
	4	$(2,4)$
3	3	$(3,3)$
4 |		
5 |		

x	$6-x$	$\{x,\ldots,6-x\}$	y	(x,y)
1	5	$\{1,2,3,4,5\}$	1	$(1,1)$
			2	$(1,2)$
			3	$(1,3)$
			4	$(1,4)$
			5	$(1,5)$
2	4	$\{2,3,4\}$	2	$(2,2)$
			3	$(2,3)$
			4	$(2,4)$
3	3	$\{3\}$	3	$(3,3)$
4	2	$\{\}$	|	
5	1	$\{\}$	|	

Most students feel that the second tree/table is much easier to follow, but that they wouldn't be able to draw it fully by themselves... it is obvious how the columns of the first tree were chosen, but it in the second one it is not – *hint: because this needs the idea of subexpressions...*

Most students find the the second table easier to follow than the first one, but the columns in the first table were chosen in an obvious way, and it is not so easy to explain how we chose the columns of the second table. This is a nice example of how explanations can be expanded and contracted.

2 Exercises

Source:
https://anggtwu.net/LATEX/material-para-GA.pdf#page=10

SC1) Draw these sets:

$$\begin{array}{rcl}
A & := & \{(1,4),(2,4),(1,3)\} \\
B & := & \{(1,3),(1,4),(2,4)\} \\
C & := & \{(1,3),(1,4),(2,4),(2,4)\} \\
D & := & \{(1,3),(1,4),(2,3),(2,4)\} \\
E & := & \{(0,3),(1,2),(2,1),(3,0)\}
\end{array}$$

SC2) Draw these sets:

$$\begin{array}{rcl}
A & := & \{x \in \{1,2\};(x,3-x)\} \\
B & := & \{x \in \{1,2,3\};(x,3-x)\} \\
C & := & \{x \in \{0,1,2,3\};(x,3-x)\} \\
D & := & \{x \in \{0,0.5,1,\ldots,3\};(x,3-x)\} \\
E & := & \{x \in \{1,2,3\}, y \in \{3,4\};(x,y)\} \\
F & := & \{x \in \{3,4\}, y \in \{1,2,3\};(x,y)\} \\
G & := & \{x \in \{3,4\}, y \in \{1,2,3\};(y,x)\} \\
H & := & \{x \in \{3,4\}, y \in \{1,2,3\};(x,2)\} \\
I & := & \{x \in \{1,2,3\}, y \in \{3,4\}, x+y<6;(x,y)\} \\
J & := & \{x \in \{1,2,3\}, y \in \{3,4\}, x+y>4;(x,y)\} \\
K & := & \{x \in \{1,2,3,4\}, y \in \{1,2,3,4\};(x,y)\} \\
L & := & \{x,y \in \{0,1,2,3,4\};(x,y)\} \\
M & := & \{x,y \in \{0,1,2,3,4\}, y=3;(x,y)\} \\
N & := & \{x,y \in \{0,1,2,3,4\}, x=2;(x,y)\} \\
O & := & \{x,y \in \{0,1,2,3,4\}, x+y=3;(x,y)\} \\
P & := & \{x,y \in \{0,1,2,3,4\}, y=x;(x,y)\} \\
Q & := & \{x,y \in \{0,1,2,3,4\}, y=x+1;(x,y)\} \\
R & := & \{x,y \in \{0,1,2,3,4\}, y=2x;(x,y)\} \\
S & := & \{x,y \in \{0,1,2,3,4\}, y=2x+1;(x,y)\}
\end{array}$$

SC3) Draw these sets:

$$\begin{array}{rcl}
A & := & \{\,(x,0) \mid x \in \{0,1,2,3\}\,\} \\
B & := & \{\,(x,x/2) \mid x \in \{0,1,2,3\}\,\} \\
C & := & \{\,(x,x) \mid x \in \{0,1,2,3\}\,\} \\
D & := & \{\,(x,2x) \mid x \in \{0,1,2,3\}\,\} \\
E & := & \{\,(x,1) \mid x \in \{0,1,2,3\}\,\} \\
F & := & \{\,(x,1+x/2) \mid x \in \{0,1,2,3\}\,\} \\
G & := & \{\,(x,1+x) \mid x \in \{0,1,2,3\}\,\} \\
H & := & \{\,(x,1+2x) \mid x \in \{0,1,2,3\}\,\} \\
I & := & \{\,(x,2) \mid x \in \{0,1,2,3\}\,\} \\
J & := & \{\,(x,2+x/2) \mid x \in \{0,1,2,3\}\,\} \\
K & := & \{\,(x,2+x) \mid x \in \{0,1,2,3\}\,\} \\
L & := & \{\,(x,2+2x) \mid x \in \{0,1,2,3\}\,\} \\
M & := & \{\,(x,2) \mid x \in \{0,1,2,3\}\,\} \\
N & := & \{\,(x,2-x/2) \mid x \in \{0,1,2,3\}\,\} \\
O & := & \{\,(x,2-x) \mid x \in \{0,1,2,3\}\,\} \\
P & := & \{\,(x,2-2x) \mid x \in \{0,1,2,3\}\,\}
\end{array}$$

Now we can define the cartesian product.
These definitions are standard:

$$\begin{array}{rcl}
A \times B & := & \{a \in A, b \in B; (a,b)\} \\
A^2 & := & A \times A
\end{array}$$

So, for example:

$$\begin{array}{rcl}
\{1,2\} \times \{3,4\} & = & \{(1,3),(1,4),(2,3),(2,4)\} \\
\{3,4\}^2 & = & \{3,4\} \times \{3,4\} \\
 & = & \{(3,3),(3,4),(4,3),(4,4)\}
\end{array}$$

SC4) Let:

$$\begin{array}{rcl}
A & = & \{1,2,4\} \\
B & = & \{2,3\} \\
C & = & \{2,3,4\}
\end{array}$$

Draw these sets:

a) $A \times A$	d) $B \times A$	g) $C \times A$
b) $A \times B$	e) $B \times B$	h) $C \times B$
c) $A \times C$	f) $B \times C$	i) $C \times C$

SC5) Draw these sets:

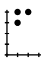
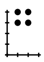
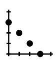
$$
\begin{aligned}
A &:= \{x, y \in \{0,1,2,3\}; (x,y)\} \\
B &:= \{x, y \in \{0,1,2,3\}, y = 2; (x,y)\} \\
C &:= \{x, y \in \{0,1,2,3\}, x = 1; (x,y)\} \\
D &:= \{x, y \in \{0,1,2,3\}, y = x; (x,y)\} \\
E &:= \{x, y \in \{0,1,2,3,4\}, y = 2x; (x,y)\} \\
F &:= \{(x,y) \in \{0,1,2,3,4\}^2, y = 2x; (x,y)\} \\
G &:= \{(x,y) \in \{0,1,2,3,4\}^2, y = x; (x,y)\} \\
H &:= \{(x,y) \in \{0,1,2,3,4\}^2, y = x/2; (x,y)\} \\
I &:= \{(x,y) \in \{0,1,2,3,4\}^2, y = x/2+1; (x,y)\} \\
J &:= \{\, (x,y) \in \{0,1,2,3,4\}^2 \mid y = 2x \,\} \\
K &:= \{\, (x,y) \in \{0,1,2,3,4\}^2 \mid y = x \,\} \\
L &:= \{\, (x,y) \in \{0,1,2,3,4\}^2 \mid y = x/2 \,\} \\
M &:= \{\, (x,y) \in \{0,1,2,3,4\}^2 \mid y = x/2+1 \,\} \\
N &:= \{\, (x,y) \in \{1,2,3\}^2 \mid 0x + 0y = 0 \,\} \\
O &:= \{\, (x,y) \in \{1,2,3\}^2 \mid 0x + 0y = 2 \,\} \\
P &:= \{\, (x,y) \in \{1,2,3\}^2 \mid x \ge y \,\}
\end{aligned}
$$

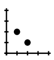
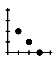
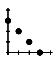
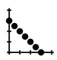
SC6) Draw these sets:

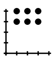
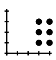
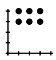
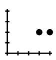
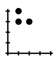
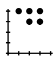
$$
\begin{aligned}
J' &:= \{\, (x,y) \in \mathbb{R}^2 \mid y = 2x \,\} \\
K' &:= \{\, (x,y) \in \mathbb{R}^2 \mid y = x \,\} \\
L' &:= \{\, (x,y) \in \mathbb{R}^2 \mid y = x/2 \,\} \\
M' &:= \{\, (x,y) \in \mathbb{R}^2 \mid y = x/2+1 \,\} \\
N' &:= \{\, (x,y) \in \mathbb{R}^2 \mid 0x + 0y = 0 \,\} \\
O' &:= \{\, (x,y) \in \mathbb{R}^2 \mid 0x + 0y = 2 \,\} \\
P' &:= \{\, (x,y) \in \mathbb{R}^2 \mid x \ge y \,\}
\end{aligned}
$$

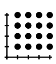
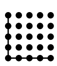
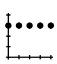
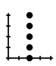
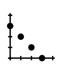
3 Answers

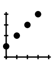
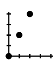
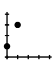
Source:
https://anggtwu.net/LATEX/material-para-GA.pdf#page=12

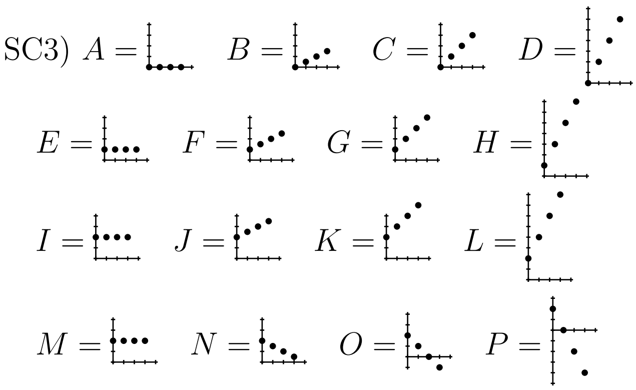
SC1) $A = B = C =$  $D =$  $E =$ 

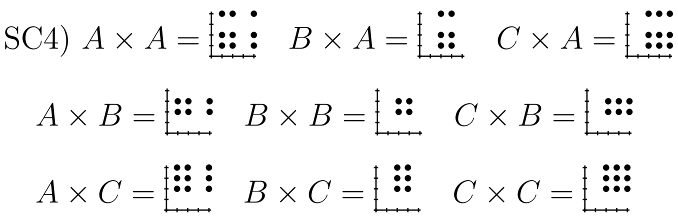
SC2) $A =$  $B =$  $C =$  $D =$ 

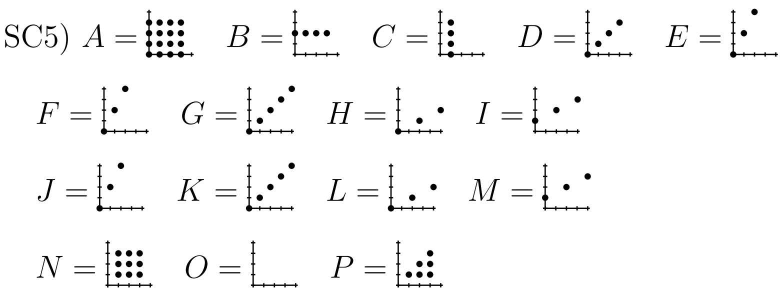
$E =$  $F =$  $G =$  $H =$  $I =$  $J =$ 

$K =$  $L =$  $M =$  $N =$  $O =$  $P =$ 

$Q =$  $R =$  $S =$ 







4 Maxima

TODO: explaind the DSL in:
https://anggtwu.net/eev-maxima.html#set-comprehensions
http://anggtwu.net/myqdraw/myqdraw3.mac.html#set-comprehensions

5 Haskell

TODO: explain this syntax, including the guards:
http://anggtwu.net/e/haskell.e.html#comprehension
http://anggtwu.net/HASKELL/Comprehensions1.hs.html

6 Lua

TODO: explaind the DSL in:
http://anggtwu.net/LUA/Comprehensions2.lua.html

7 Making and testing hypotheses

TODO: translate the collaborative game from:
http://anggtwu.net/LATEX/2026logica-para-pessoas.pdf#page=14
http://anggtwu.net/LATEX/2024-2-C2-somas-de-riemann.pdf#page=19

8 Venn Diagrams

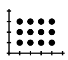
This is from [VanHiele, p.120]:

> In this guided learning process, leading from intuition to first abstraction, the concept of ‘set’ is of no use. ‘Set’ is the terminal point of an abstraction: It is the loosest link facilitating “counting.” It is easy to see that ‘set’ has no intuitive contents by consulting the textbooks that introduce sets. Some speak of ””ele- ments with a common property.” Indeed, the only real common property is that they all are elements of the same set. Other textbooks start with a set of stamps or a set of boulders. This also creates confusion, for ‘collecting” does not play a part in a mathematical set. The identity of the elements of such sets is often problematic, as well. A concept is likely to be element of a set if it is defined in such a way that it can be recognized. This condition is necessary and sufficient. But sets used as examples in textbooks almost always have elements that are very difficult to recognize.

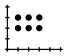
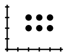
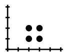
TODO: show how to translate between the Venn diagram in
[Epp, p.350, exercise 17] and the world with 12 villages from:
https://anggtwu.net/LATEX/2019ebl-mesa-slides.pdf#page=25

9 Propositions

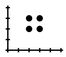
Here “adults” would prefer the terminology of Kripke frames, but students prefer a terminology inspired by videogames. Imagine that we are in a world that has these twelve villages,

$W =$ 

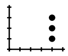
And these are the villages in which we can find ‘p’riests, ‘q’artographers, and leathe‘r’smiths:

$P =$  $Q =$  $R =$ 

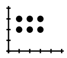
It’s easy to visualize the villages that have both priests and cartographers – it is $P \cap Q$:

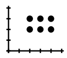
$P \cap Q =$ 

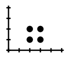
Until this point we were talking about sets that were easy to draw, and everything was very concrete. Now note that each village has coordinates. Saying simply ‘$x = 4$’ is confusing, but if we translate that to the set of villages in which $x = 4$ things become much easier to visualize:

$\{\, (x, y) \in W \mid x = 4 \,\} =$ 

Note that we have:

$P = \{\, (x, y) \in W \mid x \leq 3 \land y \geq 2 \,\} =$ 

$Q = \{\, (x, y) \in W \mid x \geq 2 \land y \geq 2 \,\} =$ 

$R = \{\, (x, y) \in W \mid 2 \leq x \leq 3 \land y \leq 2 \,\} =$ 

So, in a sense, P “is” $x \leq 3 \land y \geq 2$. How do the books handle this? This is a good opportunity to show that 1) some books explain their abuses of language, 2) other books are simply sloppy, 3) disambiguation and understanding abuses of language are usually tasks left to the reader, 4) sometimes we need to invent new notations ourselves to formalize abuses and sloppiness, 5) some things can be formalized *sort of* easily if we use dependent variables, 6) [but] I am not aware of any book that explains the rules for dependent variables really well, 7) we can look at computer languages to see how they implement ideas that the books leave underspecified,

9.1 Solving equations

TODO: discard?

Now we can discuss what means "solving a set of equations" – remember that students with bad foundations don't understand that well; see [Hewitt1, p.6]. Strang starts his book [Strang4] with the equations `eq1` and `eq2` below, and he derives consequences from them until he reaches $x = -1$, $y = 2$; these are two translations of his argument to Maxima:

```
(%i1) eq1 : 1*x + 2*y = 3;
(%o1)
        2 y + x = 3

(%i2) eq2 : 4*x + 5*y = 6;
(%o2)
        5 y + 4 x = 6

(%i3) eq3 :          4*eq1;
(%o3)
        4 (2 y + x) = 12

(%i4) eq4 :  expand(4*eq1);
(%o4)
        8 y + 4 x = 12

(%i5) eq5 : eq2 - eq4;
(%o5)
        - (3 y) = -6

(%i6) eq6 : eq5 / -3;
(%o6)
        y = 2

(%i7) eq7 : subst(eq6, eq1);
(%o7)
        x + 4 = 3

(%i8) eq8 : eq7 - 4;
(%o8)
        x = -1
```

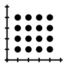
```
(%i1) eq1 : 1*x + 2*y = 3;
(%o1)
        2 y + x = 3

(%i2) eq2 : 4*x + 5*y = 6;
(%o2)
        5 y + 4 x = 6

(%i3) solve([eq1,eq2],[x,y]);
(%o3)
        [[x = -1, y = 2]]

(%i4)
```

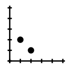
We can't spend too much class time on exercises for the students with the worst foundations, but we need to have a lot of material prepared for them. Here is an exercise that I find very nice. Now our world has the set of villages in the left below, and the second column shows how to solve in Maxima a problem that is simpler than the '$x + 2y = 3, 4x + 5y = 6$' from Strang – but it shows only the commands, not the log with the input and output lines. The exercise is: a) write the log by hand; b) complete the third column, that is translation of the calculations to how we usually write them in Maths;

$W\ =$ 

```
eq1: x + y = 3;
eq2: x - y = 1;
eq3: eq2+y;
eq4: subst(eq3,eq1);
eq5: eq4-1;
eq6: eq5/2;
eq7: subst(eq6,eq1);
eq8: eq7-1;
```

$$\begin{array}{rcl} x+y & \overset{(1)}{=} & 3 \\ x-y & \overset{(2)}{=} & 1 \\ \ldots & \overset{(3)}{=} & \ldots \\ \ldots & \overset{(4)}{=} & \ldots \\ \ldots & \ldots & \ldots \end{array}$$

c) draw the set of worlds that obeys each one of the equations `eq1` to '`eq8`', using a complete notation, like this:

$\{\,(x,y) \in W \mid x+y=3\,\}\ =$ 

d), e) do something similar for the two systems of equations below – they're harder to understand, so read the next pages of Strang's book:

```
eq1: x + y = 3;
eq2: x + y = 4;
```

(no solution)

```
eq1:  x +  y = 4;
eq2: 2x + 2y = 8;
```

(whole line)

9.2 Functions of (x, y)

TODO: discard?

Up to this point we only saw one way to draw sets like this one:

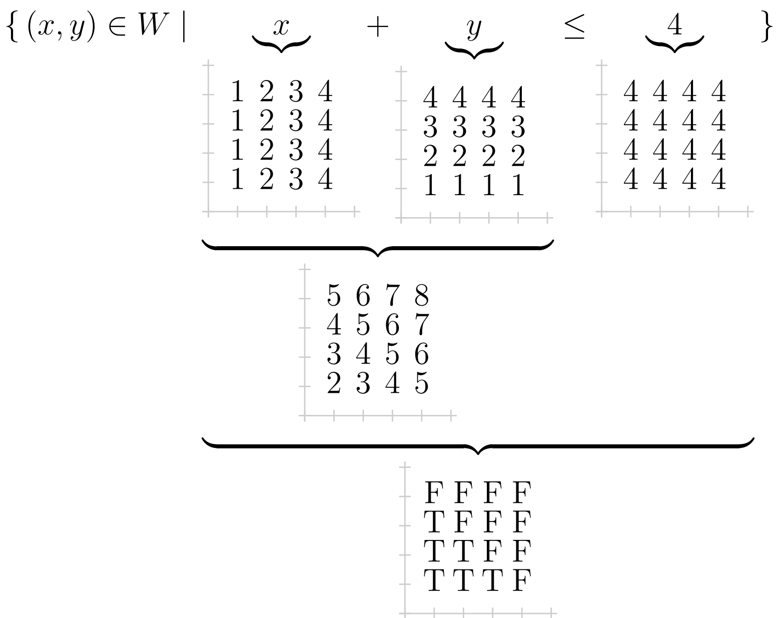
$$\{ (x, y) \in W \mid x + y \leq 4 \}$$

We calculated the result of the comprehension as a set of pairs, and we drew these pairs. There are other ways, of course – and one of these ways is: for each $(x, y) \in W$ we check if $x + y \leq 4$ is true there or not, and if that is true we draw a '•'; if not, we draw nothing.

We can adapt that way to functions of x and y whose results are not truth values. For example, if we draw the result of $x+y$ centered on each point $(x, y) \in W$ we get this,

$$\begin{array}{cccc} 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \end{array}$$

and this gives us a way to visualize (some) subexpressions, and how they are combined:



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