Logic and Categories, or: Heyting Algebras for Children
Eduardo Ochs - UniLog 2015, Istanbul
http://angg.twu.net/math-b.html#istanbul

1. A comparison between HAs and Cartesian Closed Categories:

$$
\begin{array}{rl}
(id) & \forall P.(P \leq P) \\
(comp) & \forall P,Q,R.(P \leq Q)\&(Q \leq R) \to (P \leq R) \\
\\
(\top) & \forall P.(P \leq \top) \\
(\bot) & \forall P.(\bot \leq P) \\
(\&) & \forall P,Q,R.(P \leq Q\&R) \leftrightarrow (P \leq Q)\&(P \leq R) \\
(\vee) & \forall P,Q,R.(P \vee Q \leq R) \leftrightarrow (P \leq R)\&(Q \leq R) \\
(\to) & \forall P,Q,R.(P \leq Q \to R) \leftrightarrow (P\&Q \leq R) \\
\\
(\neg) & \neg P := P \to \bot \\
(\leftrightarrow) & P \leftrightarrow Q := (P \to Q)\&(Q \to P)
\end{array}
$$

$$
\begin{array}{rl}
(id) & \mathrm{id}_A \in \mathrm{Hom}(A,A) \\
(comp) & (;)_{A,B,C} : \mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C) \\
\\
(\top) & \forall A.(\mathrm{Hom}(A,1) \simeq 1) \\
(\bot) & \forall A.(\mathrm{Hom}(0,A) \simeq 1) \\
(\&) & \forall A,B,C.(\mathrm{Hom}(A,B \times C) \simeq \mathrm{Hom}(A,B) \times \mathrm{Hom}(A,C)) \\
(\vee) & \forall A,B,C.(\mathrm{Hom}(A+B,C) \simeq \mathrm{Hom}(A,C) \times \mathrm{Hom}(B,C)) \\
(\to) & \forall A,B,C.(\mathrm{Hom}(A,C^B) \simeq \mathrm{Hom}(A \times B,C)
\end{array}
$$

$$
\dfrac{}{Q\&R \leq Q}\,\pi \qquad \dfrac{}{Q\&R \leq R}\,\pi' \qquad \dfrac{P\&Q \leq R}{P \leq Q{\to}R}\,\sharp
$$

$$
\dfrac{}{P \leq \top}\,! \qquad \dfrac{P \leq Q \quad P \leq R}{P \leq Q\&R}\,\langle,\rangle \qquad \dfrac{P \leq Q{\to}R}{P\&Q \leq R}\,\flat
$$

$$
\dfrac{}{P \leq P \vee Q}\,\iota \qquad \dfrac{}{Q \leq P \vee Q}\,\iota'
$$

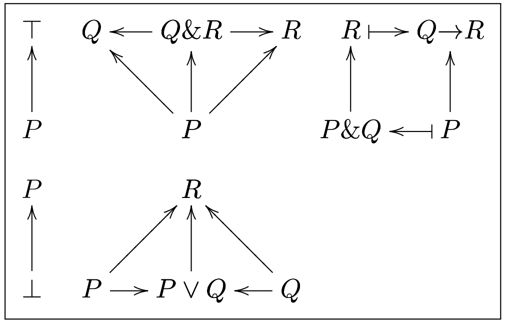
$$
\dfrac{}{\bot \leq P}\,¡ \qquad \dfrac{P \leq R \quad Q \leq R}{P \vee Q \leq R}\,[,]
$$

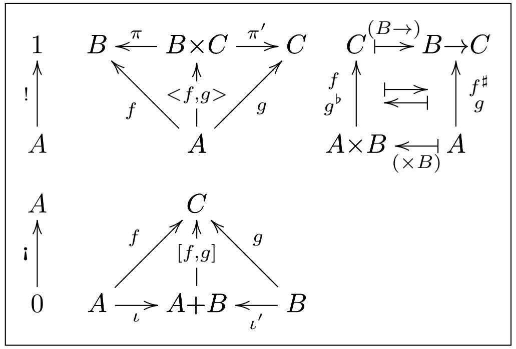
$$
\dfrac{}{\pi : B \times C \to B}\,\pi \qquad \dfrac{}{\pi' : B \times C \to C}\,\pi' \qquad \dfrac{f : A \times B \to C}{f^\sharp : A \to B{\to}C}\,\sharp
$$

$$
\dfrac{}{! : A \to 1}\,¡ \qquad \dfrac{f : A \to B \quad g : A \to C}{\langle f,g\rangle : A \to B \times C}\,\langle,\rangle \qquad \dfrac{g : A \to B{\to}C}{g^\flat : A \times B \to C}\,\flat
$$

$$
\dfrac{}{\iota : A \to A+B}\,\iota \qquad \dfrac{}{\iota' : B \to A+B}\,\iota'
$$

$$
\dfrac{}{¡ : 0 \to A}\,¡ \qquad \dfrac{f : A \to B \quad g : A \to C}{\langle f,g\rangle : A \to B \times C}\,\langle,\rangle
$$





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2. Calculating $21\&12$ and $21{\to}12$ by brute force: