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KAN EXTENSIONS

LET $\mathbf{A} \equiv \begin{smallmatrix}\bullet&\bullet&\\&\bullet&\bullet\end{smallmatrix}$,

$\mathbf{B} \equiv \begin{smallmatrix}\bullet&\\&\bullet\end{smallmatrix}$,

$$(f:\mathbf{A}\to\mathbf{B}) \equiv \left(\begin{smallmatrix}1&2&\\&3&4\end{smallmatrix} \to \begin{smallmatrix}5&\\&6\end{smallmatrix}\right)$$

$$\equiv \left(\begin{smallmatrix}5&5&\\&6&6\end{smallmatrix} \to \begin{smallmatrix}5&\\&6\end{smallmatrix}\right).$$

THEN:

$$\begin{tikzcd}[ampersand replacement=\&]
\begin{pmatrix}F_1&F_2&\\&F_3&F_4\end{pmatrix} \arrow[rr, mapsto] \arrow[dd] \& \& \begin{pmatrix}F_1+F_2&\\&F_3+_{F_2}F_4\end{pmatrix} \arrow[dd] \\
\& \leftrightarrow \& \\
\begin{pmatrix}G_5&G_5&\\&G_6&G_6\end{pmatrix} \arrow[dd] \& \& \begin{pmatrix}G_5&\\&G_6\end{pmatrix} \arrow[ll, mapsto, "f^*"'] \arrow[dd] \\
\& \leftrightarrow \& \\
\begin{pmatrix}H_1&H_2&\\&H_3&H_4\end{pmatrix} \arrow[rr, mapsto] \& \& \begin{pmatrix}H_1\times_{H_3}H_2&\\&H_3\times H_4\end{pmatrix}
\end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&]
\mathrm{Set}^{\begin{smallmatrix}\bullet&\bullet&\\&\bullet&\bullet\end{smallmatrix}} \arrow[r, shift left=2] \arrow[r, shift right=2] \& \mathrm{Set}^{\begin{smallmatrix}\bullet&\\&\bullet\end{smallmatrix}} \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&]
\begin{smallmatrix}\bullet&\bullet&\\&\bullet&\bullet\end{smallmatrix} \arrow[r, "f"] \& \begin{smallmatrix}\bullet&\\&\bullet\end{smallmatrix}
\end{tikzcd}$$

THE GENERALIZATION IS:

$$\begin{tikzcd}[ampersand replacement=\&]
(A\mapsto FA) \arrow[rr, mapsto] \arrow[d] \& \& (B\mapsto \mathrm{colim}(F\circ U_{(f\downarrow B)})) \arrow[d] \\
(A\mapsto GfA) \arrow[u, phantom, "\leftrightarrow" near end, shift left=40] \& \& (B\mapsto GB) \arrow[ll, mapsto, "f^*"'] \arrow[d] \\
(A\mapsto HA) \arrow[u, leftarrow] \arrow[rr, mapsto] \& \leftrightarrow \& (B\mapsto \lim(H\circ U_{(B\downarrow f)}))
\end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&]
\mathrm{Set}^{\mathbf{A}} \arrow[r, shift left=2] \arrow[r, shift right=2] \& \mathrm{Set}^{\mathbf{B}} \arrow[l]
\end{tikzcd} \quad \left(\text{OR } \mathbf{C}^{\mathbf{A}} \rightleftarrows \mathbf{C}^{\mathbf{B}}\right)$$

$$\begin{tikzcd}[ampersand replacement=\&]
\mathbf{A} \arrow[r, "f"] \& \mathbf{B}
\end{tikzcd}$$

WHICH IS SUPER-HARD TO UNDERSTAND...

LET $\mathbf{A} \equiv \begin{pmatrix}A_1&&A_2&\\&{}_{\alpha_{13}}\searrow\ \swarrow&&\searrow^{\alpha_{24}}\\&A_3&&A_4\end{pmatrix}$,

$\mathbf{B} \equiv \begin{pmatrix}B_5&&\\&\searrow^{\beta_{56}}&\\&&B_6\end{pmatrix}$.

THEN:

$$(f\downarrow B_5) \equiv \left(\begin{pmatrix}A_1\mapsto fA_1\\ \downarrow\beta_{55}\\ B_5\end{pmatrix} \quad \begin{pmatrix}A_2\mapsto fA_2\\ \downarrow\beta_{55}\\ B_5\end{pmatrix}\right) \equiv \left((A_1,\beta_{55}) \quad (A_2,\beta_{55})\right),$$

$$(f\downarrow B_6) \equiv \left(\begin{array}{cccc}
\begin{pmatrix}A_1\mapsto fA_1\\ \downarrow\beta_{56}\\ B_6\end{pmatrix} & & \begin{pmatrix}A_2\mapsto fA_2\\ \downarrow\beta_{56}\\ B_6\end{pmatrix} & \\
{}^{(\alpha_{13},id)}\searrow & \swarrow & \searrow^{(\alpha_{24},id)} & \\
& \begin{pmatrix}A_3\mapsto fA_3\\ \downarrow\beta_{66}\\ B_6\end{pmatrix} & & \begin{pmatrix}A_4\mapsto fA_4\\ \downarrow\beta_{66}\\ B_6\end{pmatrix}
\end{array}\right) \equiv \begin{pmatrix}(A_1,\beta_{56})&&(A_2,\beta_{56})&\\ {}^{(\alpha_{13},id)}\searrow&\swarrow&&\searrow^{(\alpha_{24},id)}\\&(A_3,\beta_{66})&&(A_4,\beta_{66})\end{pmatrix}$$

$$(B_5\downarrow f) \equiv \left(\begin{array}{cccc}
\begin{pmatrix}B_5\\ \downarrow\beta_{55}\\ A_1\mapsto fA_1\end{pmatrix} & & \begin{pmatrix}B_5\\ \downarrow\beta_{55}\\ A_2\mapsto fA_2\end{pmatrix} & \\
{}^{(id,\alpha_{13})}\searrow & \swarrow & \searrow^{(id,\alpha_{24})} & \\
& \begin{pmatrix}B_5\\ \downarrow\beta_{56}\\ A_3\mapsto fA_3\end{pmatrix} & & \begin{pmatrix}B_5\\ \downarrow\beta_{56}\\ A_4\mapsto fA_4\end{pmatrix}
\end{array}\right) \equiv \begin{pmatrix}(\beta_{55},A_1)&&(\beta_{55},A_2)&\\ {}^{(id,\alpha_{13})}\searrow&\swarrow&&\searrow^{(id,\alpha_{24})}\\&(\beta_{56},A_3)&&(\beta_{56},A_4)\end{pmatrix}$$

$$(B_6\downarrow f) \equiv \left(\begin{pmatrix}B_6\\ \downarrow\beta_{66}\\ A_3\mapsto fA_3\end{pmatrix} \quad \begin{pmatrix}B_6\\ \downarrow\beta_{66}\\ A_4\mapsto fA_4\end{pmatrix}\right) \equiv \left((\beta_{66},A_3) \quad (\beta_{66},A_4)\right)$$

AND THE INTERNAL DIAGRAM OF THE GENERALIZATION IS:

$$\begin{tikzcd}[ampersand replacement=\&]
\& \begin{pmatrix}FA_1&&FA_2&\\&\searrow\swarrow&&\searrow\\&FA_3&&FA_4\end{pmatrix} \arrow[rr, mapsto] \arrow[d] \& \& \begin{pmatrix}\mathrm{colim}(F\circ U_{(f\downarrow B_5)})\\ \searrow\\ \mathrm{colim}(F\circ U_{(f\downarrow B_6)})\end{pmatrix} = \begin{pmatrix}FA_1+FA_2&\\&\searrow\\&FA_3+_{FA_2}FA_4\end{pmatrix} \arrow[d] \\
\begin{pmatrix}GB_5&&GB_5&\\&\searrow\swarrow&&\searrow\\&GB_6&&GB_6\end{pmatrix} = \& \begin{pmatrix}GfA_1&&GfA_2&\\&\searrow\swarrow&&\searrow\\&GfA_3&&GfA_4\end{pmatrix} \arrow[d] \& \leftrightarrow \& \begin{pmatrix}GB_5&&\\&\searrow&\\&&GB_6\end{pmatrix} \arrow[ll, mapsto, "f^*"] \arrow[d] \\
\& \begin{pmatrix}HA_1&&HA_2&\\&\searrow\swarrow&&\searrow\\&HA_3&&HA_4\end{pmatrix} \arrow[rr, mapsto] \& \leftrightarrow \& \begin{pmatrix}\lim(H\circ U_{(B_5\downarrow f)})\\ \searrow\\ \lim(H\circ U_{(B_6\downarrow f)})\end{pmatrix} = \begin{pmatrix}HA_1\times_{HA_3}HA_2&\\&\searrow\\&HA_3\times HA_4\end{pmatrix}
\end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&]
\mathrm{Set}^{\mathbf{A}} \arrow[r, shift left=2] \arrow[r, shift right=2] \& \mathrm{Set}^{\mathbf{B}} \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&]
\mathbf{A} \arrow[r, "f"] \& \mathbf{B}
\end{tikzcd}$$

WHERE:

$$\begin{tikzcd}[ampersand replacement=\&]
(f\downarrow B_5) \arrow[r, "U_{(f\downarrow B_5)}"] \arrow[rr, bend right, "F\circ U_{(f\downarrow B_5)}"'] \& \mathbf{A} \arrow[r, "F"] \& \mathrm{Set},
\end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&]
(f\downarrow B_6) \arrow[r, "U_{(f\downarrow B_6)}"] \arrow[rr, bend right, "F\circ U_{(f\downarrow B_6)}"'] \& \mathbf{A} \arrow[r, "F"] \& \mathrm{Set},
\end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&]
(B_5\downarrow f) \arrow[r, "U_{(B_5\downarrow f)}"] \arrow[rr, bend right, "H\circ U_{(B_5\downarrow f)}"'] \& \mathbf{A} \arrow[r, "H"] \& \mathrm{Set},
\end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&]
(B_6\downarrow f) \arrow[r, "U_{(B_6\downarrow f)}"] \arrow[rr, bend right, "H\circ U_{(B_6\downarrow f)}"'] \& \mathbf{A} \arrow[r, "H"] \& \mathrm{Set},
\end{tikzcd}$$

AND PASSING TO INTERNAL DIAGRAMS WE GET:

$$\begin{tikzcd}[ampersand replacement=\&]
\left((A_1,\beta_{55}) \quad (A_2,\beta_{55})\right) \arrow[r] \arrow[rr, bend right, "F\circ U_{(f\downarrow B_5)}"'] \& (A_1 \quad A_2) \arrow[r] \& (FA_1 \quad FA_2)
\end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&]
\begin{pmatrix}(A_1,\beta_{56})&&(A_2,\beta_{56})&\\&\searrow\swarrow&&\searrow\\&(A_3,\beta_{66})&&(A_4,\beta_{66})\end{pmatrix} \arrow[r] \arrow[rr, bend right, "F\circ U_{(f\downarrow B_6)}"'] \& \begin{pmatrix}A_1&&A_2&\\&\searrow\swarrow&&\searrow\\&A_3&&A_4\end{pmatrix} \arrow[r] \& \begin{pmatrix}FA_1&&FA_2&\\&\searrow\swarrow&&\searrow\\&FA_3&&FA_4\end{pmatrix}
\end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&]
\begin{pmatrix}(\beta_{55},A_1)&&(\beta_{55},A_2)&\\&\searrow\swarrow&&\searrow\\&(\beta_{56},A_3)&&(\beta_{56},A_4)\end{pmatrix} \arrow[r] \arrow[rr, bend right, "H\circ U_{(B_5\downarrow f)}"'] \& \begin{pmatrix}A_1&&A_2&\\&\searrow\swarrow&&\searrow\\&A_3&&A_4\end{pmatrix} \arrow[r] \& \begin{pmatrix}HA_1&&HA_2&\\&\searrow\swarrow&&\searrow\\&HA_3&&HA_4\end{pmatrix}
\end{tikzcd}$$

$$\begin{tikzcd}[ampersand replacement=\&]
\left((\beta_{66},A_3) \quad (\beta_{66},A_4)\right) \arrow[r] \arrow[rr, bend right, "H\circ U_{(B_6\downarrow f)}"'] \& (A_3 \quad A_4) \arrow[r] \& (HA_3 \quad HA_4)
\end{tikzcd}$$

AND $\lim(H\circ U_{(B_5\downarrow f)}) = \lim\begin{pmatrix}HA_1&&HA_2&\\&\searrow\swarrow&&\searrow\\&HA_3&&HA_4\end{pmatrix}$

$= \lim\begin{pmatrix}HA_1&&HA_2\\&\searrow\swarrow&\\&HA_3&\end{pmatrix}$

$= HA_1\times_{HA_3}HA_2,$

A PULLBACK;

AND $\mathrm{colim}(F\circ U_{(f\downarrow B_6)} = \mathrm{colim}\begin{pmatrix}FA_1&&FA_2&\\&\searrow\swarrow&&\searrow\\&FA_3&&FA_4\end{pmatrix}$

$= \mathrm{colim}\begin{pmatrix}\cancel{FA_1}&&FA_2&\\&\swarrow&&\searrow\\&FA_3&&FA_4\end{pmatrix}$

$= FA_3+_{FA_2}FA_4,$

A PUSHOUT.