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THE INCOMPLETENESS OF THEORIES OF GAMES*

ABSTRACT. We first state a few previously obtained results that lead to general undecidability and incompleteness theorems in axiomatized theories that range from the theory of finite sets to classical elementary analysis. Out of those results we prove several incompleteness theorems for axiomatic versions of the theory of noncooperative games with Nash equilibria; in particular, we show the existence of *finite* games whose equilibria cannot be proven to be computable.

KEY WORDS: Chaitin, incompleteness, noncooperative games, Richardson's functor, undecidability

1. Introduction

The Santa Fe Institute has recently started a multidisciplinary program headed by J. Casti and J. Traub whose aim is to investigate the limits of scientific knowledge [2, 3]. Rather reasonably, concepts such as those that deal with the complexity and intractability of computations and with noncomputable functions [20] were taken as starting points for that investigation which is still under course, as our scientific endeavor is mainly concerned with the development of mathematical models for reality and their application. Since economics and other areas in the social sciences are now heavily dependent on sophisticated mathematical tools, the limitations intrinsically inherent in those tools have to be investigated and pondered in their applications. Examples of deep mathematical questions that arise in the biological and social sciences may be seen in the widely used nonlinear reaction–diffusion equations in ecology. Their computational intractability was already well-known; there are related systems whose chaotic behavior can be proved; and now the Gödel incompleteness phenomenon has been shown to be of import in those models [8].

More precisely, we may say that economists and social scientists have until now been concerned with positive (or negative) existential theorems that prove (or disprove) facts about particular mathematical models in their domains of activity. Archetypal examples are the Arrow–Debreu model

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and Arrow's Impossibility Theorem. Yet, more recently that concern with existential results has been extended to metamathematical aspects of economic models, and specifically with their computational aspects, such as the algorithmic resources required by the economic agents to carry out the effective implementation of those models. The existence of a journal solely dedicated to those questions, *Computational Economics*, bears witness to this growing concern.

Internal and External Epistemological Questions

The use of metamathematical tools has also spread to questions with a stronger philosophical flavor in economics. Epistemic logic and recursion theory have been used in the theory of games to discuss and assess the status of notions of solution, equilibria and rationality proposed by different authors. The special issue of *Theory and Decision* that deals with the relations between logic and game theory is very illuminating; see [1]. In such an approach we deal with the *internal epistemology and computability* of the theory of games. This means that we consider epistemological and computational questions that affect the *players* and, in general, the inner structure of the game – almost all papers published in economic journals adopt this point of view.

Another possibility, more in line with the problem on the limits of scientific knowledge is the one that may be called the *external epistemological and computability* of game theory. Here we deal with the relation between the scientist who uses the game theoretical model and the model itself. We have an economic model, or alternatively a class of models: and here the question is, what can we get out of these?

We consider the questions arising from this second point of view; i.e. we show that when we see game theory as a formal axiomatized theory, some unsuspected incompleteness phenomena appear that establish limits to the external epistemological power of the theory of games. It should be stressed that analogous results could be obtained for any mathematized microeconomic theory – here game theory is used for being an archetypal example in economic theory.

Previous Results

In this external epistemological approach the most striking result in the metamathematics of economic models was the recent proof by Lewis [16] that the (formal) theory of Walrasian models with a computable presentation is an undecidable theory. (A weaker but equally interesting result by the same author is the proof of the Gödel-incompleteness of the theory of Hamiltonian models [17].) In a similar vein Prasad [18] demonstrated the existence of a class of countably infinite games for which the problem of finding a Nash equilibrium is algorithmically unsolvable. When faced with the same question in the finite situation, Prasad remarks:

> For such games (i.e., with finite numbers of players and strategies) it is easy to describe a computational procedure for finding Nash equilibria. Since the problem is inherently finite, one approach would be to consider exhaustively all possible strategy combinations and to check if any player can gain from unilateral deviation.

On Descriptions of Finite Sets

Let's think a bit about that. Prasad argues that we can check for Nash equilibria in finite Nash games since the set of strategies is finite; out of its finiteness we can make a finite list of things that matter in the game and by brute force comparison we end up with the desired equilibria.

That approach certainly holds when we look at the problem in an extensional way, that is, when the sets of strategies and utilities are considered by themselves without the mediation of any formal language. Such is the case in all informal approaches. Yet when we go from the domain of informal mathematics into that of formalized or axiomatized theories, we cease to handle directly our intuitive, almost concrete initial objects. Formalized theories are about strings of symbols that purport to represent our intuitions about concrete objects. When we look at finite games within axiomatic theories, those naive intuitions about their solvability break down.

We can be more precise here. First, it has been known since R. Harrop's theorem that according to the description given to a finite set we may not be able to single out its elements [11, 15]. We can give an informal example of that phenomenon:

> *A description of a finite set with undecidable properties.* Let n_1 and n_2 be two different positive integers. Let $\theta(k)$ be an explicit expression for a noncomputable function defined over the positive integers which however is known to take values 0 or 1 (see Section 2 for several examples of that object). Then $n(k) = \{n_1\theta(k) + n_2(1 - \theta(k))\}$ is equal to $\{n_1\}$ or $\{n_2\}$ for each integer k.
> Yet for arbitrary integers k, there is no decision procedure to check whether $n(k) = \{n_1\}$ or $n(k) = \{n_2\}$.

Summary of the Paper

The core of this paper is based on techniques developed by two of the authors to solve some questions in dynamical systems theory and in other domains of (axiomatized) physics [5, 6, 7, 8, 10]. Section 2 in this paper reviews our construction of Richardson's Functor from Diophantine equations into elementary real analysis and lists several general undecidability and incompleteness results for the language of analysis. Section 3 states and proves our results on theories of noncooperative games with Nash equilibria. Finally Section 4 presents our conclusions.

Preliminary Concepts and Notation

Our formal languages are built out of a finite alphabet, and its *expressions* are defined as being the finite sequences of letters from the basic alphabet. We therefore reduce everything to finite sequences of letters.

To be precise, we suppose that our theories are formalized within a first-order classical language with equality and the description operator ι.

We follow the notation of [5] with a few changes that are explicitly indicated; in particular ω will denote the set of natural numbers, Z is the set of integers, and R are the real numbers. Let T be a first-order consistent axiomatic theory that contains formalized Peano's arithmetic N and such that T is strong enough to include the concept of set and classical elementary analysis. (We can simply take $T = \text{ZFC}$, where ZFC is Zermelo–Fraenkel set theory with the axiom of choice.) If L_T is the formal language of T, we suppose that we can form within T a recursive coding for L_T so that it becomes a set L_T of formal expressions in an adequate interpretation of T. Terms in L_T will be denoted by lower case italic letters, such as $x, y, z, f, g, a, \ldots$ Predicates or properties (i.e. open formulas) in L_T will be noted $P, Q, \ldots$ The use of Greek letters and more particular notation features (such as p, q for polynomial functional symbols) will be clear from the context. More generally, arbitrary expressions in L_T will be written $\xi, \zeta \ldots$

We emphasize that proofs in T are algorithmically defined ways of handling the objects of L_T; for the concept of algorithm see [5]. A review of concepts from computation theory and applications (algorithms, Turing machines, formal systems and the like) can be found in [5]. Ideas from logical number theory, such as the Matijasevich–Davis–Robinson–Jones theorem and universal polynomials can be found in [12, 14].

2. Undecidability and Incompleteness in T

We now review the above described previous results by two of the authors on the undecidability and incompleteness of classical analysis. Obviously our exposition is not rigorous and the reader is referred to the cited bibliography for complete details.

DEFINITION 2.1. T is *arithmetically consistent* if and only if the standard model $\mathbf{N}$ for N is a model for the first-order arithmetic theorems of T. We also say that T is arithmetically consistent in $\mathbf{M}$ if $\mathbf{M}$ is a model of T and T is arithmetically consistent.

REMARK 2.2. Arithmetic consistency is a stronger requirement than simple consistency, but it is (metamathematically) satisfied by ZFC. Nevertheless, in several of our results below arithmetic consistency may be replaced by simple consistency.

Richardson's Functor and the Incompleteness of Analysis

Let now $\mathcal{P}$ be the algebra of polynomial expressions on a finite number of unknowns over the integers Z; we identify $\mathcal{P}$ to the set of expressions for Diophantine polynomials in T. Let $\mathcal{E}$ be the set of expressions for real elementary functions on a finite number of unknowns, while $\mathcal{F}$ is the set of expressions for real-valued elementary functions on a single variable [5]. We assert:

PROPOSITION 2.3 (Richardson's Functor). *Let $p_m(x_1, x_2, \ldots, x_n) = 0$ be a family of expressions for Diophantine equations parametrized by the positive integer m in an arithmetically consistent theory T. Then there is an algorithmic procedure α: $\mathcal{P} \to \mathcal{E}$ such that out of $p_m \in \mathcal{P}$ we can explicitly obtain an expression*

$$f_m(x_1, x_2, \ldots, x_n) = \alpha p_m(x_1, x_2, \ldots, x_n),$$

$f_m \in \mathcal{E}$, such that $f_m = 0$ if and only if $f_m \leq 1$ if and only if there are positive integers $x_1, x_2, \ldots, x_n$ such that $p_m(x_1, \ldots, x_n) = 0$ (here we commit an abuse of language: $f_m = 0$ and $f_m \leq 1$ obviously mean that the function represented by expression f_m takes the stated values; in order to avoid rendering too clumsy the statements of our results, we will also adopt this same convention in the next propositions).

Moreover there are algorithmic procedures $\jmath'$, $\jmath''$: $\mathcal{P} \to \mathcal{F}$ such that we can obtain out of an expression p_m two other expressions for one-variable functions, $g_m(x) = \jmath' p_m(x_1, \ldots)$ and $h_m(x) = \jmath'' p_m(x_1, \ldots)$ such that there are positive integers $x_1, \ldots$ with $p_m(x_1, \ldots) = 0$ if and only if $g_m(x) = 0$ and $h_m(x) \leq 1$, for all real-valued x.

Proof. See [5]. □

PROPOSITION 2.4 (Incompleteness of Real Analysis). *If T is arithmetically consistent, and if we add the absolute value function $|x|$ to $\mathcal{F}$ and close it to obtain an extended set of expressions $\mathcal{F}^*$, we have:*

1. *We can algorithmically construct in T a denumerable family of expressions for real-valued, positive-definite functions $k_m(x) \geq 0$ so that there is no general algorithm to decide whether one has, for all real x, $k_m(x) = 0$.*
2. *For a model $\mathbf{M}$ in which T is arithmetically consistent, there is an expression for a real-valued function $k(x)$ such that $\mathbf{M} \models \forall x \in \mathrm{R} k(x) = 0$ while $T \nvdash \forall x \in \mathrm{R} k(x) = 0$ and $T \nvdash \exists x \in \mathrm{R} k(x) \neq 0$.*

Proof. See [5]. □

If k_m (as in Proposition 2.4) results out of p_m, we write $k_m = \lambda p_m$.

Equality is Undecidable in L_T

COROLLARY 2.5. *If T is arithmetically consistent in $\mathbf{M}$ then for a real-valued function f (defined also over the reals) there is an expression $\xi \in L_T$ such that $\mathbf{M} \models \xi = f$, while $T \nvdash \xi = f$ and $T \nvdash \neg(\xi = f)$.*

Proof. Put $\xi = f + k(x)$, for $k(x)$ as in Proposition 2.4. □

The Halting Function and Expressions for Complete Degrees in the Arithmetical Hierarchy

Let now $M_n(q)$ be the Turing machine of index n that acts upon the natural number q. Let $\theta(n, q)$ be the *halting function* for $M_n(q)$, that is, $\theta(n, q) = 1$ if and only if $M_n(q)$ stops over q, and $\theta(n, q) = 0$ if and only if $M_n(q)$ doesn't stop over q. Remember that the halting function $\theta(n, q)$ cannot be algorithmically implemented.

Let $p_{n,q}(x_1, x_2, \ldots, x_n)$ be a universal polynomial [14]. Since $\mathcal{F}^*$ has an expression for $|x|$ (informally one might have $|x| = +\sqrt{x^2}$), it has an expression for the sign function $\sigma(\pm x) = \pm 0$, $\sigma(0) = 0$. Therefore we can algorithmically build within the language of analysis (where we can express quotients and integrations) an expression for the halting function $\theta(n, q)$:

PROPOSITION 2.6 (The Halting Function). *If T is arithmetically consistent, then:*

$$\theta(n, q) = \sigma(G_{n,q}),$$

$$G_{n,q} = \int_{-\infty}^{+\infty} \frac{C_{n,q}(x) e^{-x^2}}{1 + C_{n,q}(x)} \, \mathrm{d}x,$$

$$C_{n,q}(x) = \lambda p_{n,q}(x_1, \ldots, x_r).$$

Proof. See [5]. □

REMARK 2.7. Other expressions are available for the halting function; see [8] for them. One of those expressions is 'quasi-arithmetical, that is, it can be constructed in N plus a functional symbol.

There are infinitely many expressions for $\theta(n, q)$ in L_T; however due to incompleteness some of them will never be proved to equal the halting function in T.

DEFINITION 2.8. A predicate P in L_T is *nontrivial* if there are terms x, y such that $T \vdash P(x)$ and $T \vdash \neg P(y)$.

If $\xi \in L_T$ is any expression in that language, we write $||\xi||$ for its complexity, as measured by the number of letters from T's alphabet in ξ. Also we define the *complexity of a proof* $C_T(\xi)$ of ξ in L_T to be the minimum length that a deduction of ξ from the axioms of T can have, as measured by the total number of letters in the expressions that belong to the proof. Let P be any nontrivial predicate, and let $\mathcal{B} \supset \mathcal{F}^*$.

Then:

PROPOSITION 2.9. *If T is arithmetically consistent, then:*

1. *There is an expression $\xi \in \mathcal{B}$ so that $T \nvdash \neg P(\xi)$ and $T \nvdash P(\xi)$, but $\mathbf{M} \models P(\xi)$, where T is arithmetically consistent in* **M**.
2. *There is a denumerable set of expressions for functions $\xi_m(x) \in \mathcal{B}$, $m \in \omega$, such that there is no general decision procedure to ascertain, for an arbitrary m, whether $P(\xi_m)$ or $\neg P(\xi_m)$ is provable in T.*
3. *Given the set $K = \{m : T \vdash P(\widehat{m})\}$, and given an arbitrary total recursive function $g \colon \omega \to \omega$, there is an infinite number of values for m so that $C_T(P(\widehat{m})) > g(||P(\widehat{m})||)$.*

Here $\widehat{m}$ recursively codes the set ξ_m of expressions in L_T. That result was the first general incompleteness theorem obtained by two of the authors [5]. It is (in a certain sense) parallel to Rice's Theorem in recursion theory [13].

These results form the core of our techniques for obtaining undecidable expressions in L_T. Using the same principles and standard concepts from recursion theory it is possible to extend our theorems to problems involving all degrees of the arithmetical hierarchy and even beyond it. For our present purposes we only state two of such generalizations that will be used in the following section (their proofs and all the technical details can be found in [8]):

DEFINITION 2.10. The sentences $\xi, \zeta \in L_T$ are *demonstrably equivalent* in T (or are T-equivalent) if and only if $T \vdash \xi \leftrightarrow \zeta$.

DEFINITION 2.11. The sentence $\xi \in L_T$ is *arithmetically expressible* in T if and only if there is an arithmetic sentence ζ such that $T \vdash \xi \leftrightarrow \zeta$. Similarly we can define when a term is said to be arithmetically expressible in T.

PROPOSITION 2.12. *If T satisfies some simple conditions* (*see* [8]), *then for any nontrivial predicate P in* N *there is a $\zeta \in L_T$ such that the assertion $P(\zeta)$ is T-demonstrably equivalent to a Π_{m+1} assertion, but not to any assertion with a lower rank in the arithmetic hierarchy.*

PROPOSITION 2.13 (Nonarithmetic incompleteness). *If T is arithmetically consistent then given any nontrivial property P:*

1. *There is a family of expressions $\zeta_m \in L_T$ such that there is no general algorithm to check, for every $m \in \omega$, whether or not $T \vdash P(\zeta_m)$.*
2. *There is an expression $\zeta \in L_T$ such that $\mathbf{M} \models P(\zeta)$ while $T \nvdash P(\zeta)$ and $T \nvdash \neg P(\zeta)$.*
3. *Neither ζ_m nor ζ are arithmetically expressible.*

REMARK 2.14. We have thus produced out of every nontrivial predicate in T intractable problems that cannot be reduced to arithmetic problems. It should also be noted that the general nonarithmetic undecidable statement $P(\zeta)$ has been obtained without the help of any kind of forcing construction.

Problems Equivalent to Some Specific Intractable Problem

REMARK 2.15. A related result is the equivalence between proving famous arithmetic conjectures and the provability of a given nontrivial property P within our formal system. The original example dealt with Fermat's Last Theorem [10], but we can consider here, the well-known $P?NP$ question, which da Costa and Doria have strong reasons to believe that it is independent of the axioms of ZFC [9].

The construction goes as follows: an axiomatic theory such as ZFC is a recursively enumerable set of sentences. As such, there is a Turing machine M_{ZFC} over an adequate alphabet which recursively enumerates those sentences. If ξ is a sentence of L_{ZFC} which is undecidable w.r.t. the axioms of that theory, then neither ξ nor $\neg\xi$ can be proved in ZFC. If we suppose that ZFC is arithmetically consistent, and if m_ξ and $m_{\neg\xi}$ are adequate Gödel numbers for those sentences, then $M_{\mathrm{ZFC}}(m_\xi)$ and $M_{\mathrm{ZFC}}(m_{\neg\xi})$

do not stop over those inputs. Therefore we can algorithmically obtain an expression for a Diophantine polynomial $p_{\mathrm{ZFC}}(m, \ldots)$ such that neither $p_{\mathrm{ZFC}}(m_\xi, \ldots)$ nor $p_{\mathrm{ZFC}}(m_{\neg\xi}, \ldots)$ have roots on $\mathbf{M}$.

Therefore, using the same principles as in the previous constructions, we can conclude that statements such as '$\theta_{\mathrm{ZFC}}(m_\xi) = 0$' are undecidable in ZFC. For convenience, in the following section such functions $\theta_{\mathrm{ZFC}}(m_\xi)$ will be denoted simply by β.

3. Incompleteness of Theories of Noncooperative Games

We start from the usual mathematical definitions in game theory:

DEFINITION 3.16. A *noncooperative game* is given by the von Neumann triplet $\langle N, S_i, u_i\rangle$, with $i = 1, 2, \ldots, N$, where N is the number of players, S_i is the strategy set of player i and u_i is the real-valued utility function

$$u_i\colon \left(\prod_{j=1}^{N} S_j\right) \longrightarrow \mathsf{R},$$

for every $i = 1, 2, \ldots, N$, where each $s_i \in S_i$.

And,

DEFINITION 3.17. A strategy vector $s^* = \langle s_1^*, \ldots\rangle$, $s_k^* \in S_k$ is a *Nash equilibrium vector* for a finite noncooperative game Γ if for all strategies and all i,

$$u_i(s^*) = u_i(\langle s_1^*, \ldots, s_k^*, \ldots\rangle) \geq u_i(\langle s_1, \ldots, s_k, \ldots\rangle).$$

If we work within some strong theory T (like ZFC, for example), it is easy to transform the above two informal definitions into formal ones by means of set-theoretical concepts. The result is therefore a Suppes predicate (see [4]) for noncooperative games and a formalization of the relation between a game and its corresponding Nash equilibria.

Undecidability and Incompleteness Theorems

Here we work within a formal theory T strong enough to develop classical analysis. Thus, to begin with, let us consider the predicate $P_g(x, y)$ of L_T which states that x is a von Neumann triplet and y is its Nash equilibria. Let $\mathcal{G}$ be the metamathematical set $\mathcal{G} = \{\langle t_1, t_2\rangle : t_1, t_2$ are terms of $L_T \wedge \mathbf{M} \models$

$P_g(t_1, t_2)\}$ (with **M** as in Proposition 2.4). For notational convenience we label as Γ the elements of $\mathcal{G}$. In the next theorem we adopt the following convention: a predicate (or property) P 'applied' to $\Gamma = \langle t_1, t_2 \rangle$ means in fact $P(t_1, t_2)$. Then:

PROPOSITION 3.18. *If T is arithmetically consistent then:*

1. *Given any nontrivial property P of finite noncooperative games, there is an infinitely denumerable family of elements Γ_m of $\mathcal{G}$ such that for those m with $T \vdash P(\Gamma_m)$, for an arbitrary total recursive function $g\colon \omega \to \omega$, there is an infinite number of values for m such that $C_T(P(\Gamma_m)) > g(||P\Gamma_m||)$.*
2. *For any nontrivial property P of finite noncooperative games, there is one definite element Γ of $\mathcal{G}$ such that $T \vdash P(\Gamma)$ if and only if $T \vdash P = NP$.*
3. *There is an element Γ of $\mathcal{G}$ for which each strategy set S_i is finite but such that we cannot compute its Nash equilibria.*
4. *There is an element Γ of $\mathcal{G}$ for which each strategy set S_i is finite and such that the computation of its equilibria is T-arithmetically expressible as a Π_{n+1} problem, but not to any Σ_k problem, $k \leq n$.*
5. *There is an element Γ of $\mathcal{G}$ for which each strategy set S_i is finite and such that the computation of its equilibria is not arithmetically expressible.*

Proof. The first statement (1) says that, even if we can prove $P(\Gamma_m)$ in T, there will be infinitely many values of m such that the proof of $P(\Gamma_m)$ in T will be arbitrarily long. Follows from Proposition 2.9.

The second statement (2) is proved as follows: let Γ' and Γ'' be two elements of $\mathcal{G}$ which represent games with the same number of players but with different strategy sets and different equilibria; let us also assume that $T \vdash P(\Gamma')$. If u' and u'' are the expressions for the respective utility functions of the von Neumann triplets terms in Γ' and Γ'' and we take β as in Remark 2.15, the element Γ of $\mathcal{G}$ with the term $\iota_x\{x = u \wedge u = u' + \beta u''\}$ for the utility function of its von Neumann triplet is our desired element.

Statement (3) also follows from Proposition 2.9.

Statement (4) follows from similar constructions out of Proposition 2.12, while statement (5) follows (again using the same type of reasoning) from Proposition 2.13. $\square$

REMARK 3.19. We should clarify a bit more items (3), (4) and (5) above. Let us fix some decidability criterion D for sets of real numbers (Pour-el's one, for example) and let $P_D(t_1, t_2)$ be the predicate constructed in L_T saying that the term t_1 stands for a von Neumann triplet with D-decidable Nash equilibria term t_2. Then, what we mean when we say that the equilibria for an element $\langle t_1, t_2 \rangle$ of $\mathcal{G}$ is not computable is simply that $T \nvdash P_D(t_1, t_2)$. In other words, it is a result about the metamathematics of T which thus depends solely on T's nature. A more proper terminology would be to say that the equilibria for $\langle t_1, t_2 \rangle$ is not T-computable, stressing the fact that all such results must be taken relative to the formal theory T.

Incompleteness of Weak Theories of Noncooperative Games

We can take T barely beyond the simplest version of formalized arithmetic N and still find out that the computation of equilibria for finite games cannot always be made, even if we cannot prove most of the results about games in that weak theory, and even if the theory is only able to handle finite objects, or objects that can be inductively obtained out of those.

Now let us take N^* to be the theory whose axioms are a small fragment of ZFC that allows us to develop a theory of finite sets; that theory encompasses then a good portion of usual arithmetic. Explicitly one has:

DEFINITION 3.20. N^* is the theory whose underlying formal language is the classical first-order predicate calculus with equality to which we add a binary relation symbol $\in$ and a new constant $\emptyset$, plus the mathematical axioms:

1. **Equality.** $\forall x \forall y \forall z((z \in x \leftrightarrow z \in y) \rightarrow x = y)$.
2. **The empty set.** $\forall x(\neg x \in \emptyset)$.
3. **Pair.** $\forall x \forall y \exists z \forall w(w \in z \leftrightarrow w = x \vee w = y)$.
4. **Union.** $\forall x \forall y \exists z \forall w(w \in z \leftrightarrow w \in x \vee w \in y)$.
   Out of those axioms we define a predicate $I(x)$ which reads 'x is an integer'; we define the numeral '0', and we define the operation '$x + 1$.' We then state, for a formula $A(x, t_1, \ldots, t_n)$ on parameters $t_1, \ldots$ with at least one free variable:
5. **Induction.** $\forall x \forall y \forall t_1 \ldots \forall t_n[(A(0, t_1, \ldots) \wedge (I(y) \wedge A(y, t_1, \ldots) \rightarrow A(y + 1, t_1, \ldots))) \rightarrow (I(x) \rightarrow A(x, t_1, \ldots))]$.

We can easily define functions with finite domains and with values in the objects of N^*. We can also define $+$ and $\times$ for integers, and as a consequence we can form polynomials on the integers in N^*. We also have 'discontinuous' operations such as the truncated difference $\dot{-}$ in that formal system.

We have that:

LEMMA 3.21. $\mathsf{N}^* \vdash \exists z(z = \langle\langle 1, f(1)\rangle, \ldots, \langle n, f(n)\rangle\rangle)$, *where* f *is a function whose domain includes the set* $\{1, 2, \ldots, n\}$.

Proof. The Pair axiom implies the existence of ordered n-ples. Since we can construct functions in N^*, the lemma follows. □

We rework the definition of strategies and utility functions given above in Definitions 3.16 and 3.17 to restrict them to the objects of N^*. With the help of the underlying first-order predicate calculus and description operator ι we can also define a set $\mathcal{G}^*$ in L_{N^*} analogous to our previous set $\mathcal{G}$ constructed in L_T. Then, with the help of Lemma 3.21 we can use an 'undecidable' polynomial q (see [7] for its construction) and take the element Γ^* of $\mathcal{G}^*$ with the term $\iota_x\{x = u^* \wedge u^* = u' + qu''\}$ for the utility function of its von Neumann triplet (where u' and u'' are the expressions for the utility functions of two given elements of $\mathcal{G}^*$ with and without equilibria, respectively). The remainder of the argument goes as usual. Thus:

PROPOSITION 3.22. *If* N^* *is consistent then there is an element* Γ^* *of* $\mathcal{G}^*$ *such that the assertion* $\iota_x\{x = u^* \wedge u^* = u' + u''\} \leftrightarrow \iota_y\{y = u'\}$ *is undecidable in* N^*.

REMARK 3.23. Lemma 3.21 implies the existence of sets of strategies. We can therefore construct integer-valued functions, and in particular utility functions on those sets of strategies.

Finite or Infinite Games?

We go back to our extended theory T and to our set $\mathcal{G}$. We say that an element $\Gamma = \langle t_1, t_2\rangle$ of $\mathcal{G}$ is finite or infinite if t_1 stands for a finite or infinite von Neumann triplet, respectively.

PROPOSITION 3.24. *Let* Γ', Γ'' *be such that* $T \vdash \Gamma'$ is finite, *and* $T \vdash \Gamma''$ is infinite. *Then there is an element* Γ *in* $\mathcal{G}$ *such that the assertion* Γ is finite *is undecidable in* T.

Proof. For β as in Remark 2.15, we can write (with an obvious abuse of language) $\Gamma = \Gamma' + \beta\Gamma''$. □

REMARK 3.25. We can obviously obtain here in Proposition 3.24 statements similar to (and as nasty as) those in Proposition 3.18.

A Blocked Backroute or a Problem to be Found Everywhere?

Gödel-like incompleteness theorems are warning posts that indicate blocked routes; they mean that one cannot go farther along some direction in mathematical theories. Incompleteness is known to appear in lost back-alleys within the domains of mathematics; yet we have always strongly believed that the incompleteness phenomenon is part of the actual practice in any axiomatized science. The authors' endeavor proved a fruitful one when it was shown that simple questions in chaos theory ('given a dynamical system, can we check whether it is chaotic? Can we prove that it is chaotic?') led to undecidability and incompleteness [5].

Finite Games: From Informal Theories to Axiomatic Ones

When it comes to the theory of finite games the situation again looks very neat at first. If we can extensionally describe the game by tables, it should have computable equilibria; if not, its description within a formal language may have some undecidable properties which are dependent on the axioms of this same formal language (of course we refer to properties which are relevant from the point of view of game theory).

The axiomatic background is made explicit for clarity. The incompleteness phenomenon means that within a (consistent) prescribed axiomatic framework certain facts cannot be proved. Assuredly, if we add stronger axioms to our system, those facts may become provable. Yet the stronger axioms may also be debatable on philosophical grounds, so that the proof of a desired fact from the enriched system turns out to be technically correct but philosophically (and perhaps empirically) doubtful. (For a recent reflection on the import of incompleteness to foundational issues and on some new techniques for the construction of undecidable sentences see [13].)

4. Conclusion

The main point of the present paper was to stress the gap between an extensional approach to game theory (or to any other microeconomic theory in general) and its intensional counterpart. Our results have thus an unappealingly abstract look. Yet they have an immediate consequence for a question of great practical and historical importance: the controversy on economic planning between L. von Mises and O. Lange (see on that controversy [19, I, p. 115ff]). We have dealt with decision problems about finite games. Intuitively everything should be computable; however when we try to clarify our intuitions we see that they break down and an essential noncomputability appears.

The central problem of economic planning is an allocation problem; very frequently allocation is to be done on the basis of maximizing (or minimizing) simple functions on finite sets. We have shown that trouble is to be expected even when the problem of planning is reduced to the problem of determining equilibria for a set of expressions of finite noncooperative Nash games. So, the main argument by Lange in favor of a planned economy (by the way, an argument also shared by von Mises) is not as forceful as it appears to be. Lange thought that given the possibly many explicit equations defining an economy, a huge and immensely powerful computer would *always* be able to figure out the equilibrium prices, therefore allowing (at least theoretically) the existence of an efficient global policy maker (note the intensional flavor of Lange's formulation). However the present paper (or the previous results by Lewis [16, 17]) disproves Lange's conjecture: there will be an explicit set of equations, even in a finite number, describing a market economy whose equilibria will not be computable (if, as we insist, our computations are thought as being performed within an adequate formal theory).

The rather coarse hierarchy of problems in Proposition 3.18 easily extends to a competitive market situation. This means that we can go from computable but extremely hard to calculate equilibrium prices (or equilibrium prices whose computation is equivalent to the proof of problems like the $P?NP$) to prices in all levels of the arithmetical hierarchy, and even beyond.

However we do not mean to say that our whole mathematical machinery has its value solely determined by a couple of interesting but local, restricted practical consequences. We would like to fit it into the historical framework of mathematical modeling in economics. We followed tradition here, since our techniques stemmed from physics and found their original applications in the natural sciences. It is no accidental feature that most methods in the mathematized social sciences have originated in the 'hard' scientific domains. So, limitations to our scientific knowledge found in the latter could reasonably be expected to emerge in the former. That fact points out to a still badly perceived common ground between the natural and social domains. The search for that common epistemological ground has been one of our motivations for the present work.

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