

Reyes, Reyes and Zolfaghari's book, "Generic Figures and Glueings", is centered on preheaves, and they use their notation for the Yoneda Lemma:

$$\begin{array}{c} \sigma: A \rightarrow P \\ B \dashrightarrow \text{Hom}(-, B) \\ f: B \xrightarrow{\text{id}_B} P \\ A \dashrightarrow F \\ \mathbb{C} \quad \text{Set}^{\mathbb{C}^{\text{op}}} \end{array}$$

$f: A \rightarrow B$ is a morphism in a category \mathbb{C} (note: we will look at a concrete example soon), and $\bar{\sigma}: \text{Hom}_{\mathbb{C}}(-, B) \rightarrow P$ is a morphism of " \mathbb{C} -sets" in the book's terminology; $\text{Hom}(-, B)$ and P that is, they are contravariant functors, more precisely, objects of $\text{Set}^{\mathbb{C}^{\text{op}}}$ and $\bar{\sigma}: \text{Hom}_{\mathbb{C}}(-, B) \rightarrow P$ is a natural transformation whose square condition is:

$$\begin{array}{c} y \xrightarrow{\text{y op}} \text{Hom}(y, B) \xrightarrow{\bar{\sigma}_y} P_y \\ g \xleftarrow{\text{g op}} \text{Hom}(g, B) \xrightarrow{\bar{\sigma}_g} P_g \\ x \xrightarrow{\text{x op}} \text{Hom}(x, B) \xrightarrow{\bar{\sigma}_x} P_x \\ \mathbb{C} \quad \mathbb{C}^{\text{op}} \xrightarrow{P} \text{Set} \\ \text{Hom}(-, B) \end{array}$$

$$\begin{array}{c} y \xrightarrow{\text{y op}} \text{Hom}(y, B) \xrightarrow{\bar{\sigma}_y} P_y \\ g \xleftarrow{\text{g op}} \text{Hom}(g, B) \xrightarrow{\bar{\sigma}_g} P_g \\ x \xrightarrow{\text{x op}} \text{Hom}(x, B) \xrightarrow{\bar{\sigma}_x} P_x \\ \mathbb{C} \quad \mathbb{C}^{\text{op}} \xrightarrow{P} \text{Set} \\ h_B: h_B y \rightarrow h_B x \end{array}$$

where $h_B: h_B y \rightarrow h_B x$ is the book's shorthand for $\text{Hom}(g, B)$.

The book
Remark: the book uses a (standard) shorthand, that we will not use much here:

$$h_B: h_B y \rightarrow h_B x$$

for: $\text{Hom}(g, B): \text{Hom}(y, B) \rightarrow \text{Hom}(x, B)$.
(Reason: $h_B^A = h_B^B = \text{Hom}(A, B)$, so there is an h -subscript and an h -supercript, and I never remember which is which.)

$$\begin{array}{c} \sigma: A \dashrightarrow P \\ B \dashrightarrow \text{Hom}(-, B) \\ f: B \xrightarrow{\text{id}_B} P \\ A \dashrightarrow F \\ \mathbb{C} \quad \text{Set}^{\mathbb{C}^{\text{op}}} \end{array}$$

are sets, not hom-sets. In $B \dashrightarrow P$, for example, $B \dashrightarrow P$ is the set P_B , and $\sigma: B \dashrightarrow P$ is a point of P_B .

The lower bijection, $(A \xrightarrow{f} B) \leftrightarrow (A \xrightarrow{f} \text{Hom}(-, B))$, is obvious, the map $f: A \rightarrow B$ corresponds to a point in $\text{Hom}(A, B)$.

The book uses these notations for "real compositions"; they are shorthands.

$$\begin{array}{c} \Phi(\sigma) \xrightarrow{\text{Q}} Q \\ B \dashrightarrow \sigma \dashrightarrow P \\ f \xrightarrow{\sigma} \sigma \circ f \\ A \end{array}$$

They are shorthands:

$$\begin{array}{c} \Phi: P \rightarrow Q \\ \sigma \in PB \quad \Phi_B: PB \rightarrow QB \\ \Phi(\sigma) := \Phi_B(\sigma) \in QB \end{array}$$

$$\begin{array}{c} f: A \rightarrow B \quad P \xrightarrow{\text{C}^{\text{op}}} \text{Set} \\ \sigma \in PB \quad P_f: PB \rightarrow PA \\ \sigma \circ f := (P_f(\sigma)) \in PA \end{array}$$

The upper bijection, $\sigma \leftrightarrow \bar{\sigma}$, is the Yoneda Lemma, in its dual version. The direction $\sigma \leftrightarrow \bar{\sigma}$ is easy - just take $\sigma := \bar{\sigma}(\text{id}_B) = \bar{\sigma}_B(\text{id}_B)$.

Here is a (proto-)proof for the bijection:

$$\begin{array}{c} \begin{pmatrix} 1 & \text{for } \sigma \\ \text{for } \bar{\sigma} \end{pmatrix} \leftrightarrow (X^{\text{op}} \xrightarrow{\text{Hom}(X, B)} \begin{pmatrix} B^{\text{op}} & \xrightarrow{\sigma_X} \begin{pmatrix} 1 & \\ & PB \end{pmatrix} \\ X^{\text{op}} \end{pmatrix}) \\ \sigma \in PB \quad X^{\text{op}} \xrightarrow{\text{Hom}(X, B)} \begin{pmatrix} P & \xrightarrow{\sigma_X} PB \\ \downarrow & \end{pmatrix} \\ \bar{\sigma}: \text{Hom}(-, B) \rightarrow P \end{array}$$

Now let's look at a concrete example. A directed multi-graph, P , is a set of vertices, P_V , plus a set of arrows, P_A , plus "source" and "target" maps, $P_S, P_T: P_A \rightarrow P_V$

$$\begin{array}{c} P_A \\ \Downarrow P_T \\ P_V \end{array}$$

Let's make this a bit more complex.

A directed symmetric multigraph, P , is a set of vertices, P_V , a set of arrows, P_A , a set of vertices, P_V , "source" and "target" maps $P_S, P_T: P_A \rightarrow P_V$, and a "dualizing" (or "mirror image") map $P_m: P_A \rightarrow P_A$ obeying:

$$\begin{array}{l} P_m \circ P_m = \text{id}_{P_A} \\ P_m \circ P_S = P_T \\ P_m \circ P_T = P_S \end{array}$$

This will be our favorite DSMG:

$$P \equiv \begin{array}{c} \alpha \dashrightarrow b \\ \alpha' \end{array} \beta \quad c$$

$$P_A = \{\alpha, \alpha', \beta\} \quad (\text{arrows})$$

$$P_V = \{a, b, c\}$$

$$\text{where: } (P_S)(\alpha) = a, (P_S)(\alpha') = b, (P_S)(\beta) = c$$

$$\begin{array}{lll} (P_T)(\alpha) = a & (P_T)(\alpha') = b & (P_m)(\alpha) = a' \\ (P_T)(\alpha') = b & (P_T)(\beta) = a & (P_m)(\alpha') = a \\ (P_T)(\beta) = c & (P_T)(\beta) = c & (P_m)(\beta) = \beta \end{array}$$

We can define a category \mathbb{S} with two objects and three non-identity morphisms, and DSMGs become ~~functors~~ (contravariant, for technical reasons).

$$\begin{array}{c} \mathbb{S} \xrightarrow{\text{op}} \mathbb{C}^{\text{op}} \xrightarrow{\text{Set}} PA = \{\alpha, \alpha', \beta\} \\ S \dashrightarrow t \quad s \dashrightarrow t \dashrightarrow P_S \dashrightarrow P_T \\ V \quad V^{\text{op}} \xrightarrow{\text{Set}} PV = \{a, b, c\} \\ S \quad S^{\text{op}} \end{array}$$

$\text{Hom}(-, A) \dashrightarrow : S^{\text{op}} \rightarrow \text{Set}$ is also a DSMG:

$$\begin{array}{c} \text{op} \\ \mathbb{C}^{\text{op}} \end{array} \quad \begin{array}{c} \text{op} \\ \mathbb{C}^{\text{op}} \end{array} \xrightarrow{\text{Set}} \text{Hom}(A, A) = \{1, m\}$$

$$\begin{array}{c} \text{op} \\ \mathbb{C}^{\text{op}} \end{array} \quad \begin{array}{c} \text{op} \\ \mathbb{C}^{\text{op}} \end{array} \xrightarrow{\text{Set}} \text{Hom}(B, A) \\ \text{op} \quad \text{op} \end{array}$$

$$\begin{array}{c} \text{op} \\ \mathbb{C}^{\text{op}} \end{array} \quad \begin{array}{c} \text{op} \\ \mathbb{C}^{\text{op}} \end{array} \xrightarrow{\text{Set}} \text{Hom}(V, A) = \{s, t\}$$

So:

$$\text{Hom}(-, A) \equiv \begin{array}{c} \text{id} \\ S \end{array} \dashrightarrow t$$

In this case we only have two arrows in $V \rightarrow A$, two dashed arrows (points) (the hom-set)

In (the set) $A \dashrightarrow \text{Hom}(-, A)$, and three dashed arrows in $A \dashrightarrow P$, so we can draw all arrows ~~possible~~ in the diagram explicitly:

$$\begin{array}{c} a \dashrightarrow P \\ \dashrightarrow \alpha \dashrightarrow \beta \dashrightarrow P \\ A \dashrightarrow \text{Hom}(-, A) \\ s \dashrightarrow t \end{array}$$

We still need to construct explicitly the natural transformations $\bar{\alpha}, \bar{\alpha}', \bar{\beta}$ and to check that the square conditions hold. We can see both P and $\text{Hom}(-, A)$ as diagrams, and we have this:

$$\begin{array}{c} P \dashrightarrow PA \xrightarrow{\text{Ps}} PV \\ n \in A \xrightarrow{s} V \quad \bar{\alpha}_A, \bar{\alpha}'_A, \bar{\beta}_A: PA \xrightarrow{\text{Pc}} PV \\ \text{Hom}(A, A) \xrightarrow{\text{Hm}(A, A)} \text{Hom}(V, A) \\ \text{Hom}(A, A) \xrightarrow{\text{Hm}(A, A)} \text{Hom}(V, A) \\ \text{Hm}(A, A) \end{array}$$

S

Set

Let's check explicitly the commutativity of one of the nine squares, we choose $\bar{\alpha}$ in the vertical and $\bar{\beta}$ in the horizontal.

$$\begin{array}{c} \text{The directed multi-graph } P \\ \text{can be seen as a functor:} \\ A \dashrightarrow \text{PA} \xrightarrow{\text{Ps}} PV \\ \bar{\alpha}_A \dashrightarrow \bar{\beta}_A: PA \dashrightarrow PV \\ \text{Hm}_A: \text{Hom}(A, A) \xrightarrow{\text{Hm}(A, A)} \text{Hom}(V, A) \\ \text{Hm}(A, A) \end{array}$$

its internal diagram is:

$$\begin{array}{c} \text{tobm} \quad \text{grb} \\ \begin{array}{c} \alpha \dashrightarrow \beta \\ \alpha' \end{array} \quad \begin{array}{c} \beta \dashrightarrow \gamma \\ \gamma \end{array} \\ \alpha \dashrightarrow \beta \dashrightarrow \gamma \\ \text{m} \quad \text{m} \quad \text{m} \\ \text{id} \dashrightarrow s \end{array}$$