Review of the typical point notation

$\mathbf{O}_a$ is the "object of 'a's", in a certain category; if this category is **Set** then this $\mathbf{O}_a$ can also be written as $\mathbf{E}_a$, the "space of 'a's".

An arrow $a \to b$ is a morphism from $\mathbf{O}_a$ to $\mathbf{O}_b$; sometimes an unspecified morphism, sometimes a specific morphism, the "unnamed morphism" from $\mathbf{O}_a$ to $\mathbf{O}_b$; we have a function that converts names of morphisms to morphisms; $a \stackrel{f}{\to} b$, for example, another name for a specific morphism from $\mathbf{O}_a \to \mathbf{O}_b$. The '$f$' is the "decoration" of the (arrow of the) morphism, and "unnamed" means "with an empty decoration".

We have chosen this function that interpret names in a way that it behaves as naturally as possible, in an informal sense, that is even allowed to change with the context; for example, we will often impose that the result of composing two unnamed morphisms $a \to b$ and $b \to c$ is the unnamed morphism $a \to c$.

$\mathbf{Cat}_a$ is the category where the object $\mathbf{O}_a$ lives.

$a \Rightarrow a^F$ is a functor from $\mathbf{Cat}_a to \mathbf{Cat}_{a^F}$; when this is interpreted as a specific functor (i.e., most of the time) the notation implies that it takes each object $\mathbf{O}_a$ and takes it to $\mathbf{O}_{a^F}$ – i.e., it takes the name of an object and adds a decoration 'F' to it.

It is so common that there is a "natural" way to choose the images of morphisms once we are told how a functor acts on (names of) objects that we will generally denote a functor only by its action on objects; and these functors will take unnamed morphisms to unnamed morphisms.

An arrow with a curly tail, like $a \hookrightarrow b$, denotes a monic, or an injection. An arrow like '$a \leftrightarrow b$' denotes an isomorphism or bijection. Having an identity arrow between two objects with different names, like $a' \stackrel{\mathrm{id}}{\to} a''$, implies that we are using two different names for the same object (generally to let us write more arrows as unnamed arrows).

A natural transformation between two unnamed functors $a \Rightarrow a^F$ and $a \Rightarrow a^G$ (unnamed in the sense that their arrows have no decorations) is denoted by $a \stackrel{\bullet}{\to} (a^F \to a^G)$.

We write (a, b) for the product of $\mathbf{O}_a$ and $\mathbf{O}_b$, as in **Set** an (a, b) (i.e., an element of $\mathbf{E}_{(a,b)}$) can be seen as a pair made of an a and a b.

We use ';'s for more formal pairs; an $\mathbf{O}_{(a;b)}$ is a pair $(\mathbf{O}_a, \mathbf{O}_b)$, an object of the bicategory $\mathbf{Cat}_a \times \mathbf{Cat}_b$ (also called $\mathbf{A} \times \mathbf{B}$ or $\mathbf{Cat}_{(a;b)}$).

The projection functors $(a; b) \Rightarrow a$ and $(a; b) \Rightarrow b$ are typical cases of a functor where we want the target object to have a simpler name than the original object; likewise with some care we can allow functors like $a^F \Rightarrow a^G$, that syntactically work by removing an F from the name of an object and placing a G in its place, without even requiring that we have a meaning for $\mathbf{O}_a$ and $\mathbf{Cat}_a$.

Adjunctions are written as square diagrams (like the ones below) showing the bijection between hom-sets. The two most common examples are $(s \to m^U) \leftrightarrow (s^{[\,]} \to m)$, between the "free monoid" and "underlying set" functors between **Set** and **Mon**, and $(a \to (b \to c)) \leftrightarrow (a, b \to c)$ between the product and the exponential by $\mathbf{O}_b$ in a cartesial closed category (e.g. **Set**).

$$\begin{array}{ccc} s & \Rightarrow & s^{[\,]} \\ \downarrow & \leftrightarrow & \downarrow \\ m^U & \Leftarrow & m \end{array} \qquad \begin{array}{ccc} a & \Rightarrow & a,b \\ \downarrow & \leftrightarrow & \downarrow \\ b \to c & \Leftarrow & c \end{array}$$

We also have a "less precise" notation for categories, and some extra symbols to help converting between our notations and the usual ones. *This part of the notation has many rough edges and won't be used much, but, anyway:* instead of saying that $\mathbf{O}_a$ is an object of $\mathbf{A}$ and that $a \to b$ is a morphism of $\mathbf{A}$, we can say that $\mathbf{O}_a$ is an $\mathbf{o_A}$ and that $a \to b$ is a $\mathbf{m_A}$; an element of a space $\mathbf{E}_a = A$ is an $\mathbf{e}_A$; a functor F from $\mathbf{A}$ to $\mathbf{B}$ is a functor $F : \mathbf{A} \Rightarrow \mathbf{B}$, or just "a functor $\mathbf{A} \Rightarrow \mathbf{B}$"; $\mathbf{A} \Rightarrow \mathbf{B}$ is also used as the name of the category that has these functors as its objects, and natural transformations between them as its morphisms; $\mathbf{A} \times \mathbf{B}$ is the bicategory that is the product of $\mathbf{A}$ and $\mathbf{B}$.

We will sometimes use $\mathbf{O}[a]$, $\mathbf{E}[a]$, $\mathbf{Cat}[a]$, etc, for $\mathbf{O}_a$, $\mathbf{E}_a$, $\mathbf{Cat}_a$.

Notation for limits and colimits: let's start with pullbacks. A diagram $\begin{smallmatrix} & b \\ & \downarrow \\ c \to & d \end{smallmatrix}$ in a category $\mathbf{A}$ can be seen as a functor $\begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \to & \bullet \end{smallmatrix} \Rightarrow \mathbf{A}$; it will be convenient to write this functor just as $\begin{smallmatrix} & b \\ & \downarrow \\ c \to & d \end{smallmatrix}$, and a natural transformation between two such functors will be written like $\begin{smallmatrix} & b \\ & \downarrow \\ c \to & d \end{smallmatrix} \to \begin{smallmatrix} & b' \\ & \downarrow \\ c' \to & d' \end{smallmatrix}$. Inspired by how things work in $\mathbf{Set}$, we write the object of the pullback of $\begin{smallmatrix} & b \\ & \downarrow \\ c \to & d \end{smallmatrix}$ as $(b,c)|_d$ — $\mathbf{E}_{(b,c)|_d}$ is the "space of pairs (b,c) such that d is well-defined".

A category $\mathbf{A}$ has pullbacks if the constant functor $K : \begin{smallmatrix} & \bullet \\ & \downarrow \\ \bullet \to & \bullet \end{smallmatrix} \Rightarrow \mathbf{A}$ has a right adjoint:

$$\begin{array}{ccc} x & \Rightarrow & \begin{smallmatrix} & x \\ & \downarrow \\ x \to & x \end{smallmatrix} \\ \downarrow & \leftrightarrow & \downarrow \\ (b,c)|_d & \Leftarrow & \begin{smallmatrix} & b \\ & \downarrow \\ c \to & d \end{smallmatrix} \end{array}$$

We can write this as $K \dashv \lim$, or as, say,

$$\left(x \Rightarrow \begin{smallmatrix} & x \\ & \downarrow \\ x \to & x \end{smallmatrix} \right) \dashv \left(\begin{smallmatrix} & b \\ & \downarrow \\ c \to & d \end{smallmatrix} \Rightarrow (b,c)|_d \right)$$

$$\text{or} \qquad \left(x \Rightarrow \begin{smallmatrix} & x \\ & \downarrow \\ x \to & x \end{smallmatrix} \right) \dashv \left((b,c)|_d \Leftarrow \begin{smallmatrix} & b \\ & \downarrow \\ c \to & d \end{smallmatrix} \right) \quad .$$

Take a functor $R : \mathbf{A} \Rightarrow \mathbf{B}$. If $\mathbf{A}$ has pullbacks and R has a left adjoint $L : \mathbf{B} \Rightarrow \mathbf{A}$ then $\mathbf{B}$ has pullbacks, and R preserves pullbacks, i.e., for any pullback diagram we have an isomorphism ${(b,c)|_d}^R \leftrightarrow (b^R, c^R)|_{d^R}$.

The preservation of limits: (MacLane, p.114)

$\Delta \dashv \mathrm{Lim}$; $\mathrm{Colim} \dashv \Delta$; $F \dashv U$; $F \dashv G$.
$T : J \to A \qquad j \Rightarrow t_j$

$$\dfrac{a \in A \quad \Delta : A \to A^J}{\Delta a : J \to A} \qquad \dfrac{\mathbf{O}_a \quad a \Rightarrow (j \Rightarrow a)}{j \Rightarrow a}$$

$$\dfrac{\dfrac{h : Fx \to a}{h^\sharp : x \to Ga} \quad \dfrac{\tau : \Delta a \overset{\bullet}{\to} T \quad G : A \to X}{G\tau : \Delta Ga \overset{\bullet}{\to} GT}}{G\tau \cdot h^\sharp : \Delta x \overset{\bullet}{\to} GT} \qquad \dfrac{\dfrac{x^F \to a}{x \to a^G} \quad \dfrac{j \overset{\bullet}{\to} (a \to t_j) \quad a \Rightarrow a^G}{j \overset{\bullet}{\to} (a^G \to {t_j}^G)}}{j \overset{\bullet}{\to} (x \to {t_j}^G)}$$

$$\dfrac{\dfrac{\dfrac{[i \in J]^1 \quad \sigma : x \overset{\bullet}{\to} GT}{\sigma_i : x \to GTi}}{(\sigma_i)^\flat : Fx \to Ti}}{\sigma^\flat : \Delta Fx \overset{\bullet}{\to} T}\,1 \qquad \dfrac{\dfrac{\dfrac{[\mathbf{O}_i]^1 \quad j \overset{\bullet}{\to} (x \to {t_j}^G)}{x \to {t_i}^G}}{x^F \to t_i}}{j \overset{\bullet}{\to} (x^F \to t_j)}\,1$$

$$\dfrac{\dfrac{h : Fx \to a \quad \tau : \Delta a \overset{\bullet}{\to} T}{\tau h : \Delta Fx \overset{\bullet}{\to} T}}{(\tau h)^\sharp : \Delta x \overset{\bullet}{\to} GT} \qquad \dfrac{\dfrac{x^F \to a \quad j \overset{\bullet}{\to} (a \to t_j)}{j \overset{\bullet}{\to} (x^F \to t_j)}}{j \overset{\bullet}{\to} (x \to {t_j}^G)}$$

Theorem 1. *If the functor $G : A \to X$ has a left adjoint, while the functor $T : J \to A$ has a limiting cone $\tau : a \overset{\bullet}{\to} T$ in A, then GT has the limiting cone $G\tau : Ga \overset{\bullet}{\to} GT$ in X.*

Proof. By composition, $G\tau$ is indeed a cone from the vertex Ga in X. If F is a left adjoint to G, and if we apply the adjunction isomorphism to every arrow of a cone $\sigma : x \overset{\bullet}{\to} GT$, we get arrows $(\sigma_i)^\flat : Fx \to Ti$ for $i \in J$ which form a cone $\sigma^\flat : Fx \overset{\bullet}{\to} T$ in A. But $\tau : a \overset{\bullet}{\to} T$ is universal among cones to T in A, so there is a unique arrow $h : Fx \to a$ with $\tau h = \sigma^\flat$. Taking adjuncts again, this gives a unique arrow $h^\sharp : x \to Ga$ with $(\tau h)^\sharp = G\tau \cdot h^\sharp = (\sigma^\flat)^\sharp = \sigma$. The uniqueness of the arrow $h^\flat$ states precisely that $G\tau : Ga \overset{\bullet}{\to} T$ is universal, q.e.d.

Theorem 1. *If the functor $a \Rightarrow a^G$ has a left adjoint, while the functor $j \Rightarrow t_j$ has a limiting cone $j \overset{\bullet}{\to} (a \to t_j)$ in $\mathbf{A}$, then $j \Rightarrow {t_j}^G$ has the limiting cone $j \overset{\bullet}{\to} (a^G \to {t_j}^G)$ in $\mathbf{X}$.*

Proof. By composition, $j \overset{\bullet}{\to} (a^G \to {t_j}^G)$ is indeed a cone from the vertex $\mathbf{O}_{a^G}$ in $\mathbf{X}$. If $x \Rightarrow x^F$ is a left adjoint to $a \Rightarrow a^G$, and if we apply the adjunction isomorphism to every arrow of a cone $j \overset{\bullet}{\to} (x \to {t_j}^G)$, we get arrows $x^F \to t_i$ for $\mathbf{O}_i \in J$ which form a cone $j \overset{\bullet}{\to} (x \to {t_j}^G)$ in $\mathbf{A}$. But $j \overset{\bullet}{\to} (a \to t_j)$ is universal among cones to $j \Rightarrow t_j$ in $\mathbf{A}$, so there is a unique arrow $x^F \to a$ with [$j \overset{\bullet}{\to} (x^F \to t_j)$ well-defined]. Taking adjuncts again, this gives a unique arrow $x \to a^G$ with [$j \overset{\bullet}{\to} (a \to {t_j}^G)$ well-defined]. The uniqueness of the arrow $x \to a^G$ states precisely that $j \overset{\bullet}{\to} (a^G \to {t_j}^G)$ is universal, q.e.d.

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{j \overset{\bullet}{\to} (x \to {t_j}^G)}{j \overset{\bullet}{\to} (x^F \to t_j)}}{x^F \to a}}{x \to a^G}
\end{array}
\qquad
\begin{array}{ccccccc}
x^F & \to & a & \to & t_j & & \mathbf{A} \\
\Uparrow & & \Downarrow & & \Downarrow & & \\
x & \to & a^G & \to & {t_j}^G & & \mathbf{X}
\end{array}
$$

$$
\begin{array}{ccccccc}
\begin{array}{ccc} & & x \\ & & \downarrow \\ x & \to & x \end{array} & \Rightarrow & \begin{array}{ccc} & & x^F \\ & & \downarrow \\ x^F & \to & x^F \end{array} & \Leftarrow & x^F & \Leftarrow & x \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\begin{array}{ccc} & & b^G \\ & & \downarrow \\ c^G & \to & d^G \end{array} & \Leftarrow & \begin{array}{ccc} & & b \\ & & \downarrow \\ c & \to & d \end{array} & \Rightarrow & (b,c)|_d & \Rightarrow & {(b,c)|_d}^G
\end{array}
$$

And the dual of this:

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{j \overset{\bullet}{\to} ({s_j}^F \to x)}{j \overset{\bullet}{\to} (s_j \to x^G)}}{a \to x^G}}{a^F \to x}
\end{array}
\qquad
\begin{array}{ccccc}
{s_j}^F & \to & a^F & \to & x \\
\Uparrow & & \Uparrow & & \Downarrow \\
s_j & \to & a & \to & x^G
\end{array}
$$

$$
\begin{array}{ccccccc}
\begin{array}{ccc} b^F & \to & c^F \\ \downarrow & & \\ c^F & & \end{array} & \Leftarrow & \begin{array}{ccc} b & \to & c \\ \downarrow & & \\ c & & \end{array} & \Rightarrow & (c \amalg d)_{/b} & \Rightarrow & {(c \amalg d)_{/b}}^F \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\begin{array}{ccc} x & \to & x \\ \downarrow & & \\ x & & \end{array} & \Rightarrow & \begin{array}{ccc} x^G & \to & x^G \\ \downarrow & & \\ x^G & & \end{array} & \Leftarrow & x^G & \Leftarrow & x
\end{array}
$$

Concept trees

The bars of the following trees will not have the usual meanings, where was to be interte

"if the things over the bar are true then the things below will also be" or, in, say, $\frac{\alpha \quad \beta}{\gamma}$, "from a point of type α and one of type β we obtain one of type γ"; our bars here will always have a flavour of *building a concept from other concepts that are more basic.* A simple example of this would be:

$$\frac{P \quad Q}{P \to Q}$$

to be interpreted (very roughly) as "once we understand P and Q we can understand $P \to Q$", or, in more practical terms, and specifically for this case, "from the truth values (the "meanings") of P and Q we can apply a known logical operation and obtain the truth value of $P \to Q$"; the semantics for the term below the bar is constructed from the semantics of the terms above.

We will use a '$\lambda\alpha$' at the right of a bar to point that when passing to the lower side of the bar the "variable" α got free. Let's take the functor 'Hom' (i.e., $a^{\mathrm{op}} \Rightarrow (b \Rightarrow a \to b)$) as a first realistic example:

$$\dfrac{\dfrac{\dfrac{a \xrightarrow{f} b}{\mathbf{E}[a \to b]}\;\lambda f}{b \Rightarrow a \to b}\;\lambda b}{a^{\mathrm{op}} \Rightarrow (b \Rightarrow a \to b)}\;\lambda a, (\Rightarrow) \qquad \dfrac{\dfrac{\dfrac{a \xrightarrow{f} b}{\mathbf{E}[a \to b]}\;\lambda f}{b \Rightarrow a \to b}\;\lambda b}{a^{\mathrm{op}} \Rightarrow (b \Rightarrow a \to b)}\;\lambda a, (\Rightarrow)$$

The first thing we can do with such trees is to use them as "dictionaries" to help us understand concepts written in conventional notations; compare the two trees above.

Test case: the Yoneda embedding. Its concept tree, first in TP notation, then in rougher (i.e., types only) TP notation, then in MM's notation (p. 26):

$$\dfrac{\mathbf{O}[a] \qquad \dfrac{\dfrac{\mathbf{O}[x]}{\mathbf{O}[x]^{\mathrm{op}}} \quad \dfrac{\mathbf{O}[x] \quad \mathbf{O}[a]}{\mathbf{E}[x \to a]}}{x^{\mathrm{op}} \Rightarrow x \to a}}{a \Rightarrow (x^{\mathrm{op}} \Rightarrow x \to a)} \qquad \dfrac{\mathbf{o}[\mathbf{C}] \qquad \dfrac{\dfrac{\mathbf{o}[\mathbf{C}]}{\mathbf{o}[\mathbf{C}^{\mathrm{op}}]} \quad \dfrac{\mathbf{o}[\mathbf{C}] \quad \mathbf{o}[\mathbf{C}]}{\mathbf{o}[\mathbf{Set}]}}{\mathbf{C}^{\mathrm{op}} \Rightarrow \mathbf{Set}}}{\mathbf{C} \Rightarrow (\mathbf{C}^{\mathrm{op}} \Rightarrow \mathbf{Set})}$$

$$\dfrac{C \qquad \dfrac{\dfrac{D}{D^{\mathrm{op}}} \quad \dfrac{D \quad C}{\mathrm{Hom}_{\mathbf{C}}(D, C) = \mathbf{y}(C)(D)}}{\mathrm{Hom}_{\mathbf{C}}(-, C) = \mathbf{y}(C)}}{\mathbf{y} : \mathbf{C} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}$$

The arrow of the Yoneda lemma: concept tree in MM notation (p. 26), then it in TP notation:

$$\dfrac{\dfrac{\dfrac{C \quad \mathbf{y}}{\mathbf{y}(C)} \quad P}{\mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathbf{y}(C), P)} \quad \dfrac{C \quad P}{P(C)}}{\theta : \mathrm{Hom}_{\widehat{\mathbf{C}}}(\mathbf{y}(C), P) \xrightarrow{\sim} P(C)}$$

$$\dfrac{\dfrac{\dfrac{\mathbf{O}[a] \quad a \Rightarrow (x^{\mathrm{op}} \Rightarrow x \to a)}{x^{\mathrm{op}} \Rightarrow x \to a} \qquad x^{\mathrm{op}} \Rightarrow x^P}{\mathbf{E}[x^{\mathrm{op}} \xrightarrow{\bullet} ((x \to a) \to x^P)]} \quad \dfrac{\mathbf{O}[a] \quad x^{\mathrm{op}} \Rightarrow x^P}{\mathbf{O}[a^P] = \mathbf{E}[a^P]}}{(x^{\mathrm{op}} \xrightarrow{\bullet} ((x \to a) \to x^P)) \leftrightarrow a^P}$$

How θ works: we define $\theta(\alpha) := \alpha_C(1_C)$,

$$\dfrac{\alpha : \mathbf{y}(C) \to P \quad \theta}{\theta(\alpha)} \qquad \dfrac{x^{\mathrm{op}} \xrightarrow{\bullet} ((x \to a) \to x^P) \quad (x^{\mathrm{op}} \xrightarrow{\bullet} ((x \to a) \to x^P)) \leftrightarrow a^P}{a^P}$$

$$\dfrac{\dfrac{C}{1_C} \quad \dfrac{C \quad \alpha}{\alpha_C}}{\alpha_C(1_C)} \qquad \dfrac{\dfrac{\mathbf{O}[a]}{a \xrightarrow{\mathrm{id}} a} \quad \dfrac{\mathbf{O}[a] \quad x^{\mathrm{op}} \xrightarrow{\bullet} ((x \to a) \to x^P)}{(a \to a) \to a^P}}{a^P}$$

that is, we can use the following tree to define θ using a discharge:

$$\dfrac{\dfrac{\dfrac{C}{1_C} \quad \dfrac{C \quad [\alpha]^1}{\alpha_C}}{\theta(\alpha) := \alpha_C(1_C)}}{\theta}\,1 \qquad \dfrac{\dfrac{\dfrac{\mathbf{O}[a]}{a \overset{\mathrm{id}}{\to} a} \quad \dfrac{\mathbf{O}[a] \quad [x^{\mathrm{op}} \overset{\bullet}{\to} ((x \to a) \to x^P)]^1}{(a \to a) \to a^P}}{a^P}}{(x^{\mathrm{op}} \overset{\bullet}{\to} ((x \to a) \to x^P)) \to a^P}\,1$$

Of course we still need to prove that this is functorial on $\mathbf{O}_a$ and P; in MM the objects C and C^{op} are the same, and these trees are following this; they need corrections to be made into valid constructions in our Natural deduction system.

Comma Categories: BW, pp.13–14
$\mathbf{O}_{(\underline{c'}^F \to \underline{d'}^G)} = (\mathbf{O}_{c'}; c'^F \to d'^G; \mathbf{O}_{d'})$

$$\dfrac{(C, f, D) \quad (C', f', D')}{(h, k) : (C, f, D) \to (C', f', D')}$$

$$\dfrac{(\underline{c'}^F \to \underline{d'}^G) \quad (\underline{c''}^F \to \underline{d''}^G)}{(c' \to c''; d' \to d'') = (\underline{c'}^F \to \underline{d'}^G) \to (\underline{c''}^F \to \underline{d''}^G)}$$

$$\dfrac{\dfrac{h \quad F}{Fh} \quad f'}{f' \circ Fh} = \dfrac{f \quad \dfrac{k \quad G}{Gk}}{Gk \circ f}$$

$$\dfrac{\dfrac{c' \to c'' \quad c \Rightarrow c^F}{c'^F \to c''^F} \quad c''^F \to d''^G}{c'^F \to d''^G} = \dfrac{c'^F \to d'^G \quad \dfrac{d' \to d'' \quad d \Rightarrow d^G}{d'^G \to d''^G}}{c'^F \to d''^G}$$

Test case: exponentials in $\mathbf{Set}^{\mathbf{C}}$
First we need the definition of a category of arrows over an object; using concept trees, and comparing with Goldblatt (p. 208, and also p.36),

$$\dfrac{\dfrac{\dfrac{\mathbf{O}[a] \quad a \overset{f}{\to} b}{\mathbf{O}[(\underline{a} \to b_f)]}}{(\underline{a} \to b_f) \Rightarrow b}\,\lambda b, f \quad b \Rightarrow b^F}{(\underline{a} \to b_f) \Rightarrow b^F} \qquad \dfrac{\dfrac{\dfrac{a \in \mathcal{C} \quad f : a \to b}{(f, a) \in \mathcal{C}{\uparrow}a}}{... : \mathcal{C}{\uparrow}a \to \mathcal{C}} \quad F : \mathcal{C} \to \mathbf{Set}}{F_a : \mathcal{C}{\uparrow}a \to \mathbf{Set}}$$

$$\dfrac{\dfrac{(\underline{a} \to b_f) \Rightarrow b^F \quad (\underline{a} \to b_f) \Rightarrow b^G}{(\underline{a} \to b_f) \overset{\bullet}{\to} (b^F \to b^G)}}{\mathbf{E}[(\underline{a} \to b_f) \overset{\bullet}{\to} (b^F \to b^G)]}\,\lambda \overset{\bullet}{\to} \qquad \dfrac{\dfrac{F_a : \mathcal{C}{\uparrow}a \to \mathbf{Set} \quad G_a : \mathcal{C}{\uparrow}a \to \mathbf{Set}}{... : F_a \to G_a}}{G^F(a) = \mathrm{Nat}[F_a, G_a]}$$

Now the definition of the exponential in $\mathbf{Set}^{\mathbf{C}}$. suppose that we know $a \Rightarrow a^P$, $a \Rightarrow a^Q$, and $a \Rightarrow a^R$, three functors $\mathbf{C} \to \mathbf{Set}$.

Here's how to construct the functor $a \Rightarrow a^{Q\to R}$:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[a'\to a'']^3 \quad [a''\to b]^1}{\mathbf{O}[\underline{a}'\to b]}}{(\underline{a}''\to b)\Rightarrow(\underline{a}'\to b)}\;1 \quad [(\underline{a}'\to b)\overset{\bullet}{\to}(b^F\to b^G)]^2}{(\underline{a}''\to b)\overset{\bullet}{\to}(b^F\to b^G)}}{((\underline{a}'\to b)\overset{\bullet}{\to}(b^F\to b^G))\to((\underline{a}''\to b)\overset{\bullet}{\to}(b^F\to b^G))}\;2}{a\Rightarrow((\underline{a}\to b)\overset{\bullet}{\to}(b^F\to b^G))}\;3$$

This is how to construct ev, i.e., $a \overset{\bullet}{\to} (a^{Q\times(Q\to R)} \to a^R)$, from nothing (?).

$$\cfrac{\cfrac{\cfrac{[a^Q]^1 \quad \cfrac{\cfrac{[\mathbf{O}[a]]^2}{\mathbf{O}[(\underline{a}\to a)]} \quad [(\underline{a}\to b)\overset{\bullet}{\to}(b^Q\to b^R)]^1}{a^Q\to a^R}}{a^R}}{(a^Q,((\underline{a}\to b)\overset{\bullet}{\to}(b^Q\to b^R)))\to a^R}\;1}{a\overset{\bullet}{\to}((a^Q,((\underline{a}\to b)\overset{\bullet}{\to}(b^Q\to b^R)))\to a^R)}\;2$$

Here's the easy direction of the transposition: $a \overset{\bullet}{\to} (a^P \to a^{Q\to R})$ apparently carries more information than $a \overset{\bullet}{\to} (a^{P\times Q} \to a^R)$.

$$\cfrac{\cfrac{\cfrac{[a^Q]^1 \quad \cfrac{\cfrac{[\mathbf{O}[a]]^2}{\mathbf{O}[(\underline{a}\to a)]} \quad \cfrac{[a^P]^1 \quad \cfrac{[\mathbf{O}[a]]^2 \quad a\overset{\bullet}{\to}(a^P\to((\underline{a}\to b)\overset{\bullet}{\to}(b^Q\to b^R)))}{a^P\to((\underline{a}\to b)\overset{\bullet}{\to}(b^Q\to b^R))}}{(\underline{a}\to b)\overset{\bullet}{\to}(b^Q\to b^R)}}{a^Q\to a^R}}{a^R}}{(a^P,a^Q)\to a^R}\;1}{a\overset{\bullet}{\to}((a^P,a^Q)\to a^R)}\;2$$

And here is the hard direction:

$$\cfrac{\cfrac{\cfrac{\cfrac{[a^P]^2 \quad \cfrac{\cfrac{\mathbf{O}[(\underline{a}\to b)]^1}{a\to b} \quad a\Rightarrow a^P}{a^P\to b^P}}{b^P} \quad \cfrac{\mathbf{O}[b] \quad a\overset{\bullet}{\to}((a^P,a^Q)\to a^R)}{(b^P,b^Q)\to b^R}}{b^Q\to b^R}}{(\underline{a}\to b)\overset{\bullet}{\to}(b^Q\to b^R)}\;1}{a^P\to((\underline{a}\to b)\overset{\bullet}{\to}(b^Q\to b^R))}$$

There must be something wrong here, if we think in terms of ops and not ops; the $a \Rightarrow a^P$ seems to be appearing with the wrong sign... or maybe that's because be we are in a transposition, and the signs must be derived from $(a^{\mathrm{op}}; b) \overset{\bullet}{\to} ((a \to a^R) \leftrightarrow (b^L \to b))$...

Notes about natural deduction in categories
(To do: compare with `http://www.disi.unige.it/person/MoggiE/ftp/plilp96.ps.gz`).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\mathbf{O}[(\underline{a} \to b)]]^1}{\mathbf{O}[(\underline{a} \to \underline{b})]}}{\mathbf{O}[(a^{\mathrm{op}}; b)]}}{\mathbf{O}[b]} \qquad b \Rightarrow b^F}{\mathbf{O}[b^F]}}{(\underline{a} \to b) \Rightarrow b^F}$$

Both $b \Rightarrow b^F$ and $(\underline{a} \to b) \Rightarrow b^F$ can be seen as objects of categories of morphisms.

Choosing an arrow and then packing all arrows of that kind in a set:

$$\dfrac{\dfrac{\dfrac{\mathbf{O}[b \Rightarrow b^F]^{\mathrm{op}}}{\mathbf{O}[(\underline{a} \to b) \Rightarrow b^F]^{\mathrm{op}}} \quad [\to]^2 \quad \dfrac{\mathbf{O}[b \Rightarrow b^G]}{\mathbf{O}[(\underline{a} \to b) \Rightarrow b^G]}}{(\underline{a} \to b) \overset{\bullet}{\to} (b^F \to b^G)}}{\mathbf{E}[(\underline{a} \to b) \overset{\bullet}{\to} (b^F \to b^G)]}\; 2$$

Another way to get to same result is to apply the Hom functor.

$$\dfrac{\dfrac{\dfrac{\mathbf{O}[b \Rightarrow b^F]^{\mathrm{op}}}{\mathbf{O}[(\underline{a} \to b) \Rightarrow b^F]^{\mathrm{op}}} \quad \dfrac{\mathbf{O}[b \Rightarrow b^G]}{\mathbf{O}[(\underline{a} \to b) \Rightarrow b^G]}}{\mathbf{O}[(((\underline{a} \to b) \Rightarrow b^F)^{\mathrm{op}}; ((\underline{a} \to b) \Rightarrow b^G))]}}{\mathbf{E}[(\underline{a} \to b) \overset{\bullet}{\to} (b^F \to b^G)]}$$

Now we want to discharge $\mathbf{O}[a]$; it seems that its behaviour concerning sign (op/not op) is not the same as common objects, maybe because it is used only to build a category from another. Note the we build categories in one direction but we have an inclusion functor between them going in the opposite direction.

$$\dfrac{\mathbf{O}[(\underline{a} \to b)] \quad \in \quad \dfrac{\mathbf{O}[a]^{\mathrm{op}} \quad \mathbf{Cat}[(\underline{a} \to \underline{b})]}{\mathbf{Cat}[(\underline{a} \to b)]}}{\mathbf{O}[(a^{\mathrm{op}}; b)]}$$

$$\dfrac{\dfrac{\dfrac{[\mathbf{O}[a]^{\mathrm{op}}]^1 \quad \mathbf{O}[b \Rightarrow b^F]^{\mathrm{op}}}{\mathbf{O}[(\underline{a} \to b) \Rightarrow b^F]^{\mathrm{op}}} \quad \dfrac{[\mathbf{O}[a]^{\mathrm{op}}]^1 \quad \mathbf{O}[b \Rightarrow b^G]}{\mathbf{O}[(\underline{a} \to b) \Rightarrow b^G]}}{\mathbf{E}[(\underline{a} \to b) \overset{\bullet}{\to} (b^F \to b^G)]}}{a^{\mathrm{op}} \Rightarrow ((\underline{a} \to b) \overset{\bullet}{\to} (b^F \to b^G))}$$

Natural deduction in a topos

(This corresponds to the Mitchell-Bénabou language, but with a twist to make the usual restriction on implication ([J76], p.xx; ...) unnecessary).

1. Propositional Calculus. Take for example

$$\dfrac{A\lor B \qquad \dfrac{[A]^1}{A\lor C} \qquad \dfrac{\dfrac{[B]^1 \quad B\to C}{C}}{A\lor C}}{A\lor C}\,1$$

that becomes

$$\dfrac{A\lor B\vdash A\lor B \qquad \dfrac{A\vdash A}{A\vdash A\lor C} \qquad \dfrac{\dfrac{B\vdash B \quad B\to C\vdash B\to C}{B,(B\to C)\vdash C}}{B,(B\to C)\vdash A\lor C}}{(A\lor B),(B\to C)\vdash A\lor C} \qquad ;$$

to convert from DN to sequents we use the standard trick of making all the non-discarged hypotheses explicit in the left side of the turnstile; the ','s are to be interpreted as '$\land$'s.

Now suppose that these A, B and C stand for arrows $x\to\omega_A$, $x\to\omega_B$ and $x\to\omega_C$. Then the sequents represent natural inclusions between subobjects of $\mathbf{O}_x$, and the trees above become

$$\dfrac{x|_{A\lor B}\hookrightarrow x|_{A\lor B} \qquad \dfrac{x|_A\hookrightarrow x|_A}{x|_A\hookrightarrow x|_{A\lor C}} \qquad \dfrac{\dfrac{x|_B\hookrightarrow x|_B \quad x|_{B\to C}\hookrightarrow x|_{B\to C}}{x|_{B\land(B\to C)}\hookrightarrow x|_C}}{x|_{B\land(B\to C)}\hookrightarrow x|_{A\lor C}}}{x|_{(A\lor B)\land(B\to C)}\hookrightarrow x|_{A\lor C}}$$

that is, we are deducing the existence of certain natural inclusions from the existence of others. But we could have regarded this in a different way:

$$\dfrac{x|_{A\lor B} \qquad \dfrac{[x|_A]^1}{x|_{A\lor C}} \qquad \dfrac{\dfrac{[x|_B]^1 \quad x|_{B\to C}}{x|_C}}{x|_{A\lor C}}}{x|_{A\lor C}}\,1$$

can be seen as a recipe to build a (p-)point $x|_{A\lor C}$ from a (p-)point $x|_{A\lor B}$ and a (p-)point $x|_{A\to B}$. This works without further restrictions if $\mathbf{O}_x$ is the terminal object, but for the general case we must impose certain compatibility conditions on the original p-points, or else some of the natural deduction rules will not make sense; these conditions turn out to be that all our original p-point arrows, $p\to x|_P$, $p\to x|_Q$, ..., must be factorable through the same subobject of $\mathbf{O}_x$, say $\mathbf{O}_{x'}$, as $p\to x|_P = p\to x'\hookrightarrow x|_P$, ..., using the same arrow $p\to x'$ and taking the '$x'\hookrightarrow x|_P$'s as being natural inclusions. It is easy to check that the universal way to do that is to choose $\mathbf{O}_{x'}$ as $\mathbf{O}_{x|_{P\land Q\land\dots}}$.

2. First order logic (typed). What we are trying to do is to make sense of deductions like this one[1]; P and Q are functions of two variables, that for us would corresponds to $x, y \to \omega_P$ and $y, x \to \omega_Q$; Prawitz distinguishes between variables and constants, and at some points of the deduction we have to suppose that a constant a has been chosen as a value for the variable x, or a constant b was chosen as a value for y, or both.

$$\dfrac{\dfrac{\dfrac{\forall x.\exists y.Pxy}{\exists y.Pay}(\forall E) \qquad \dfrac{\dfrac{[Pab]^1 \qquad \dfrac{[Pab]^1 \qquad \dfrac{\dfrac{\forall x.\forall y.(Pxy \to Qyx)}{\forall y.(Pay \to Qya)}(\forall E)}{Pab \to Qba}(\forall E)}{Qba}}{Pab \wedge Qba}}{\exists y.(Pay \wedge Qya)}(\exists I)}{\exists y.(Pay \wedge Qya)}1(\exists E)}{\forall x.\exists y.(Pxy \wedge Qyx)}(\forall I)$$

In a topos we can have "spaces without points", and the system used in the tree above doesn't deal with that; in it there is always an a to play the role of an x, for example. Our translation will require "witnesses" at some rules to cope with that, and the tree above will become:

$$\dfrac{\dfrac{\dfrac{[x]^2 \quad \top|_{\forall x.\exists y.Pxy}}{\top|_{\exists y.Pxy}}(\forall E) \qquad \dfrac{\dfrac{[\top|_{Pxy}]^1 \qquad \dfrac{[\top|_{Pxy}]^1 \qquad \dfrac{[y]^1 \qquad \dfrac{[x]^2 \quad \top|_{\forall x.\forall y.(Pxy \to Qyx)}}{\top|_{\forall y.(Pxy \to Qyx)}}(\forall E)}{\top|_{Pxy \to Qyx}}(\forall E)}{\top|_{Qyx}}}{\top|_{Pxy \wedge Qyx}}}{\top|_{\exists y.(Pxy \wedge Qyx)}}(\exists I)}{\top|_{\exists y.(Pxy \wedge Qyx)}}1(\exists E)}{\top|_{\forall x.\exists y.(Pxy \wedge Qyx)}}2(\forall I)$$

and the compatibility conditions are a bit stranger; let's discuss them before showing the rules of deduction for the quantifiers.

Points named x, y and $\top|_{Pxy}$ cannot be considered as independent; there's a natural inclusion $\top|_{Pxy} \hookrightarrow \top$, that induces $(x, y, \top|_{Pxy}) \hookrightarrow (x, y, \top)$; we have the projection $(x, y, \top) \to (x, t)$ (where this arrow is actually an isomorphism), and we know an arrow $(x, y) \to \omega_{P(x,y)}$; combining these we have an arrow $(x, y, \top|_{Pxy}) \to \omega_{P(x,y)}$, and we also have another natural way to obtain a $\omega_{P(x,y)}$ from $(x, y, \top|_{Pxy})$, namely by $(x, y, \top|_{Pxy}) \to \top|_{Pxy} \hookrightarrow \top_{Pxy} \hookrightarrow \omega_{P(x,y)}$; the two must give the result, or else we wouldn't be able to speak so freely of natural arrows, as some of them wouldn't be well-defined... well, it turns out that the "natural" way of defining $\mathbf{O}_{(x,y,\top|_{Pxy})}$ is to take it as the

[1] inspired by the one in p. 19 of [Pr65]; the main difference is that Prawitz makes P and Q the same function.

pullback $\mathbf{O}_{(x,y)|_{Pxy}}$, which means forcing ω_{Pxy} to be well-defined; in other cases, like $\mathbf{O}_{(x,(y,z),(z,w),\top|_{P(x)},\top|_{Q(y,w)})}$, we would have to force z, $\omega_{P(x)}$ and $\omega_{Q(y,w)}$ to be well-defined, and so on.

[It is easy to establish sets of rules that will define the "universal 'p's" for trees (like the $\mathbf{O}_{(x,y,\top|_{Pxy})}$ above), in ways that will cover the simpler cases; but if we want to allow richer systems where more constructions are "natural" then things can get out of control very quickly, as we can reach systems that are inconsistent or undecidable.]

[Describe the rules for $(\forall/\exists)$-introduction/elimination, and how to understand them in sequents and categorically.]

$$
\dfrac{\top|_{\exists x.Px} \qquad \dfrac{[x]^1 \quad [Px]^1 \quad ...}{\top|_\varphi}}{\top_\varphi}\,1(\exists\mathrm{E}) \qquad \dfrac{x \quad \top|_\varphi}{\top|_{\exists x.\varphi}}\,(\exists\mathrm{I})
$$

$$
\dfrac{x \quad \top|_{\forall x.\varphi}}{\top|_\varphi}\,(\forall\mathrm{E}) \qquad \dfrac{\dfrac{[x]^1 \quad ...}{\top|_\varphi}}{\top|_{\forall x.\varphi}}\,1(\forall\mathrm{I})
$$

Natural deduction in SSA: the missing restriction bars

[Why are we being so loose about restrictions?]

We are using the "restriction bars", like in $n|_{n\text{ is prime}}$ so little explicitly because they are going to be reintroduced in the language later, in a much more convenient way. The most convincing example I have at this moment is this one (in a topos): let $x_0, y_0, \ldots$ denote standard numbers, and $x \to y$ and $y \to z$ standard smooth functions; x_1 will be a number infinitely close to x_0, y_1 a number infinitely close to y_0, and so on. We have:

$$
\dfrac{\dfrac{x_0 \quad x \to y}{y_0} \qquad y \to z}{z_0} \qquad \dfrac{\dfrac{x_1 \quad x \to y}{y_1} \qquad y \to z}{z_1}
$$

Where the second tree makes sense (i.e., it derives objects that are "compatible with their names") because the rules implicit in its bars are in fact just shorthand for rules with more hypotheses: writing "f SiSNo w_0" for "f is smooth in some neighbourhhod of w_0 we have, expanding the tree a little,

$$
\dfrac{\dfrac{x_1, \top[x_1{\sim}x_0] \quad (x \to y), \top[x \to y \;\;\text{SiSNo}\;\; x_0]}{y_1, \top[y_1{\sim}y_0]} \qquad (y \to z), \top[y \to z \;\;\text{SiSNo}\;\; y_0]}{z_1, \top[z_1{\sim}z_0]}
$$

where the '$\top[P]$' are the points that attest that the propositions 'P's are true, as in page ... and on.

Spaces like $\mathbf{E}_n$ and $\mathbf{E}_{\top[n\text{ is prime}]}$ are not independent of one another; indeed, when both n and $\top[n$ is prime$]$ appear we should consider that they are obtained from $\mathbf{E}_{n,\top[n\text{ is prime}]}$ by projections.

When we work in such demonstrations — like the one given by the tree above, that shows that $z_1 \sim z_0$ — we may start by supposing that $x \to y$ and $y \to z$ are specific arrows of a topos, or, equivalently, that we were given "points" of $\mathbf{O}_{x\to y}$ and $\mathbf{O}_{y\to z}$, by arrows $1 \to (x \to y)$ and $1 \to (y \to z)$; or we may try to work in several "possible worlds" at once, either by using another object in place of $\mathbf{O}_1$, or by using a topos that is not **Set**, or by doing both things.

Imagem direta: comparar com MM (pags. 171–173). Diagramas adaptados dos que eu fiz em 00jan09.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[b\to\omega_P]^1}{b|_P\hookrightarrow b}\quad b\hookrightarrow c}{b|_P\hookrightarrow c}}{c\to\omega}}{(b\to\omega|_P)\to(c\to\omega)}\;1$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(b,(b\to\omega))\to\omega}}{(b,(b\to\omega))|_\top\hookrightarrow(b,(b\to\omega))}\quad\dfrac{b\hookrightarrow c\quad\dfrac{}{(b\to\omega)\hookrightarrow(b\to\omega)}}{(b,(b\to\omega))\hookrightarrow(c,(b\to\omega))}}{(b,(b\to\omega))|_\top\hookrightarrow(c,(b\to\omega))}}{(c,(b\to\omega))\to\omega}}{(b\to\omega)\to(c\to\omega)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(b,(b\to\omega_P))\to\omega_P}}{(b,(b\to\omega_P))|_P\hookrightarrow(b,(b\to\omega_P))}\quad\dfrac{b\hookrightarrow c\quad\dfrac{}{(b\to\omega_P)\hookrightarrow(b\to\omega_P)}}{(b,(b\to\omega_P))\hookrightarrow(c,(b\to\omega_P))}}{(b,(b\to\omega_P))|_P\hookrightarrow(c,(b\to\omega_P))}}{(c,(b\to\omega_P))\to\omega_{\exists b.P}}}{(b\to\omega_P)\to(c\to\omega_{\exists b.P})}$$

Eu tenho que verificar que essas duas operações são equivalentes:

$$\dfrac{\dfrac{\dfrac{b\to\omega_P}{1\to(b\to\omega_P)}\quad\dfrac{b\hookrightarrow c}{(b\to\omega_P)\to(c\to\omega)}}{1\to(c\to\omega)}}{c\to\omega}\qquad\dfrac{\dfrac{\dfrac{b\to\omega_P}{b|_P\hookrightarrow b}\quad b\hookrightarrow c}{b|_P\hookrightarrow c}}{c\to\omega}$$

MM, p. 109: topologias de Grothendieck.
$\mathbf{y}$ é $c \Rightarrow (x^{\mathrm{op}} \Rightarrow x \to c)$
C é $\mathbf{O}_c$
$\mathbf{y}(C)$ é $(x^{\mathrm{op}} \Rightarrow x \to c)$
S é $x^{\mathrm{op}} \Rightarrow (x \to c)|_{S_c}$
E temos $x^{\mathrm{op}} \stackrel{\bullet}{\to} ((x \to c)|_{S_c} \hookrightarrow (x \to c))$ inclusões naturais.

Tome uma seta $d \to c$ (o $h : D \to C$ do MM).

Defina $\mathbf{E}[(x \to d)|_{S_d}]$ como $\{\, x \to d \mid x \to d \to c \in \mathbf{E}[(x \to c)|_{S_c}] \,\}$. Temos

$$\dfrac{d \to c \quad x^{\mathrm{op}} \stackrel{\bullet}{\to} ((x \to c)|_{S_c} \hookrightarrow (x \to c))}{x^{\mathrm{op}} \stackrel{\bullet}{\to} ((x \to d)|_{S_d} \hookrightarrow (x \to d))}$$

e definindo $\mathbf{E}[S_a]$ como sendo o conjunto (ou classe) dos S_a sieves sobre $\mathbf{O}_a$, temos

$$\dfrac{\dfrac{\dfrac{[d \to c]^2 \quad [S_c]^1}{S_d}}{S_c \to S_d}\;1}{c^{\mathrm{op}} \Rightarrow S_c} \qquad ,$$

ou seja, S é um funtor contravariante, que talvez mereça uma notação melhor... por enquanto vou usar $c^{\mathrm{op}} \Rightarrow S_c$.

Uma topologia de Grothendieck J é um funtor $c^{\mathrm{op}} \Rightarrow J_c$, que vem com uma TN $c^{\mathrm{op}} \stackrel{\bullet}{\to} (J_c \hookrightarrow S_c)$ (i.e., para cada $\mathbf{O}_c$ o J responde um subconjunto de todos os sieves sobre c), tal que para todo $\mathbf{O}_c$ o $\mathbf{E}[J_c]$ contém o sieve maximal, e a condição de transitividade vale; a condição de estabilidade já fica embutida em considerarmos o J como funtor. Repara que o S, que para todo $\mathbf{O}_c$ responde o conjunto de todos os sieves sobre c, é uma topologia de Grothendieck.

A condição de transitividade é a seguinte: $\forall (d \to c)|_{J_c}.(S_d \in \mathbf{E}[J_d])$ implica em $S_c \in J_c$.

$$\dfrac{\dfrac{\dfrac{R \qquad \dfrac{\dfrac{[h]^1 \quad \in \quad \dfrac{S \quad \in \quad \dfrac{C \quad J}{J(C)}}{S}}{h}}{h^*}}{h^*(R)} \qquad \dfrac{[D]^1 \quad J}{J(D)}}{h^*(R) \in J(D)}}{\forall h\ h^*(R) \in J(D)}\;1
\qquad
\dfrac{\dfrac{\dfrac{S_c \qquad \dfrac{\dfrac{[d \to c]^1 \quad \in \quad \dfrac{J_c \quad \in \quad \dfrac{\mathbf{O}[c] \quad a^{\mathrm{op}} \Rightarrow J_a}{\mathbf{E}[J_c]}}{J_c}}{(d \to c)|_{J_c}}}{S_c \to S_d}}{S_d} \qquad \dfrac{[\mathbf{O}[d]]^1 \quad a^{\mathrm{op}} \Rightarrow J_a}{\mathbf{E}[J_d]}}{\omega[S_d \in \mathbf{E}[J_d]]}}{\omega[\forall (d \to c)|_{J_c}.(S_d \in \mathbf{E}[J_d])]}\;1$$

Matching families (MM, p.121):

$$\dfrac{\dfrac{(d \to c)|_{S_c} \quad (d \to c)|_{S_c} \to d^P}{d^P} \qquad \dfrac{e \to d \quad x^{\mathrm{op}} \Rightarrow x^P}{d^P \to e^P}}{e^P}
\qquad
\dfrac{\dfrac{(f : D \to C) \in S \quad M(D)}{x_f \in P(D)} \qquad \dfrac{g : E \to D \quad P}{P(g)}}{x_f \cdot g = P(g)(x_f)}$$

$$\dfrac{\dfrac{e \to d \quad (d \to c)|_{S_c}}{(e \to c)|_{S_c}} \qquad (e \to c)|_{S_c} \to e^P}{e^P}
\qquad
\dfrac{\dfrac{e \to d \quad (d \to c)|_{S_c}}{(e \to c)|_{S_c}} \qquad (e \to c)|_{S_c} \to e^P}{e^P}$$

Uma matching family é um M em que os dois jeitos de obter o e^P dão o mesmo resultado.

Uma "amalgamation" é um c^P tal que o processo abaixo acaba recuperando todos os pontos d^P:

$$\dfrac{c^P \qquad \dfrac{(d \to c)|_{S_c} \quad x^{\mathrm{op}} \Rightarrow x^P}{c^P \to d^P}}{d^P} \qquad\qquad \dfrac{x \in P(C) \qquad \dfrac{(f : D \to C) \in S \quad P}{P(f)}}{x \cdot f = P(f)(x)}$$

De algum modo esse c^P corresponde à seção grande que bate com todas as seçõezinhas, quando tudo está sobre espaços topológicos.

MacLane/Moerdijk, p.41:

$$\dfrac{\dfrac{\dfrac{\dfrac{[C \in \mathbf{C}]^1 \quad A : \mathbf{C} \to \mathcal{E}}{A(C) \in \mathcal{E}} \qquad [E \in \mathcal{E}]^2}{\mathrm{Hom}_{\mathcal{E}}(A(C), E) \in \mathbf{Sets}}}{R(E) = (C \mapsto \mathrm{Hom}_{\mathcal{E}}(A(C), E)) \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}}\;1}{R : \mathcal{E} \to \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\mathbf{O}_c]^1 \quad c \Rightarrow c^A}{\mathbf{O}_{c^A}} \qquad [\mathbf{O}_e]^2}{\mathbf{E}[c^A \to e]}}{c^{\mathrm{op}} \Rightarrow (c^A \to e)}\;1}{e \Rightarrow (c^{\mathrm{op}} \Rightarrow (c^A \to e))}\;2$$

$$\dfrac{\dfrac{\dfrac{\pi_P : \int P \to \mathbf{C} \quad A : \mathbf{C} \to \mathcal{E}}{\int P \to \mathbf{C} \to \mathcal{E}}}{L(P) = \mathrm{Colim}(\int P \to \mathbf{C} \to \mathcal{E}) \in \mathcal{E}}}{L : \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}} \to \mathcal{E}} \qquad \dfrac{\dfrac{\dfrac{(\mathbf{O}_c, c^P) \Rightarrow c \quad c \Rightarrow c^A}{(\mathbf{O}_c, c^P) \Rightarrow c^A}}{\mathbf{O}[\mathrm{colim}((\mathbf{O}_c, c^P) \Rightarrow c^A)]}}{(c^{\mathrm{op}} \Rightarrow c^P) \Rightarrow \mathrm{colim}((\mathbf{O}_c, c^P) \Rightarrow c^A)}$$

Como fazer corresponder um $a \to b^R$ (um $\mathbf{m}[\mathbf{C}^{\mathrm{op}} \Rightarrow \mathbf{Set}]$, i.e., uma certa TN) a um $a^L \to b$ (um morfismo de $\mathcal{E}$)?
Notação melhor: $P \to e^R$ e $P^L \to e$.

$P \to e^R = c^{\mathrm{op}} \overset{\bullet}{\to} (c^P \to (c^A \to e))$
$P^L \to e = \mathrm{colim}((\mathbf{O}_c, c^P) \Rightarrow c^A) \to e$

Terms and types for ND in categories
Below there are some of the rules of the system of natural deduction for categories. The list is by no means complete, and neither the reduction rules nor the compatibility conditions for the hypotheses of some rules are being shown.

At this point I'm using this system because it is a very nice representation of its semantics, and I'm not properly "studying it"; my priority right now is to use it to understand sheafification, sythetic differential geometry and classifying topoi (esp. the "universal models" described in pages 568–571 of MacLane and Moerdijk's book).

$$\dfrac{\mathbf{O}_a}{a \to a}\;\mathrm{id} \qquad \dfrac{a \to b}{\mathbf{O}_a}\;\mathrm{msrc} \qquad \dfrac{a \to b}{\mathbf{O}_b}\;\mathrm{mtgt}$$

$$\dfrac{a \to b \quad b \to c}{a \to c}\;\mathrm{mm}$$

$$\dfrac{\mathbf{O}_a \quad a \Rightarrow a^F}{\mathbf{O}_{a^F}}\ \mathrm{OF} \qquad \dfrac{a \to b \quad c \Rightarrow c^F}{a^F \to b^F}\ \mathrm{mF}$$

$$\dfrac{\mathbf{O}_a \quad c \stackrel{\bullet}{\to} (c^F \to c^G)}{a^F \to a^G}\ \mathrm{OTN} \qquad \dfrac{a \to b \quad c \stackrel{\bullet}{\to} (c^F \to c^G)}{a^F \to b^G}\ \mathrm{mTN}$$

$$\dfrac{a \Rightarrow a^F \quad b \stackrel{\bullet}{\to} (b^H \to b^K)}{a \stackrel{\bullet}{\to} (a^{FH} \to a^{FK}}\ \mathrm{FTN} \qquad \dfrac{a \stackrel{\bullet}{\to} (a^F \to a^G) \quad b \Rightarrow b^H}{a \stackrel{\bullet}{\to} (a^{FH} \to b^{GH})}\ \mathrm{TNF}$$

$$\dfrac{\mathbf{O}_a}{\mathbf{O}_a^{\mathrm{op}}}\ \mathrm{Oop} \qquad \dfrac{a \to b}{b^{\mathrm{op}} \to a^{\mathrm{op}}}\ \mathrm{mop}$$

$$\dfrac{\mathbf{O}_a \quad \mathbf{O}_b}{\mathbf{O}_{(a;b)}}\ \mathrm{OO}\times \qquad \dfrac{\mathbf{O}_{(a;b)}}{\mathbf{O}_a}\ \mathrm{O}\pi_0 \qquad \dfrac{\mathbf{O}_{(a;b)}}{\mathbf{O}_b}\ \mathrm{O}\pi_1$$

$$\dfrac{a \to a' \quad b' \to b'}{(a;b) \to (a';b')}\ \mathrm{mm}\times \qquad \dfrac{(a;b) \to (a';b')}{a \to a'}\ \mathrm{m}\pi_0 \qquad \dfrac{(a;b) \to (a';b')}{b \to b'}\ \mathrm{m}\pi_1$$

$$\dfrac{}{(a^{\mathrm{op}};b) \Rightarrow (a \to b)}\ \mathrm{Hom}$$

$$\dfrac{a \to b^R}{a^L \to b}\ \flat_{LR} \qquad \dfrac{a^L \to b}{a \to b^R}\ \sharp_{LR}$$

$$\dfrac{}{a \stackrel{\bullet}{\to} (a \to a^{RL})}\ \eta_{LR} \qquad \dfrac{}{b \stackrel{\bullet}{\to} (b^{LR} \to b)}\ \epsilon_{LR}$$

There are also rules that discharge objects to form functors. For example:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\mathbf{O}_a]^2 \quad [\mathbf{O}_b]^1}{\mathbf{O}_{(a;b)}}}{b \Rightarrow (a;b)}\ 1}{\mathbf{O}_{b \Rightarrow (a;b)}}}{a \Rightarrow (b \Rightarrow (a;b))}\ 2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\mathbf{O}_a]^1 \quad [\mathbf{O}_b]^2}{[\mathrm{OO}\times](\mathbf{O}_a, \mathbf{O}_b)}}{\Lambda\mathbf{O}_b.[\mathrm{OO}\times](\mathbf{O}_a, \mathbf{O}_b)}}{[\mathrm{F}{\to}\mathrm{O}](\Lambda\mathbf{O}_b.[\mathrm{OO}\times](\mathbf{O}_a, \mathbf{O}_b))}}{\Lambda\mathbf{O}_a.[\mathrm{F}{\to}\mathrm{O}](\Lambda\mathbf{O}_b.\mathrm{OO}\times(\mathbf{O}_a, \mathbf{O}_b))}$$

We are only allowed to discharge an object $\mathbf{O}_a$ to form a functor $a \Rightarrow a^F$ when we can check that the tree encodes an operation on objects that is really functorial: it must be possible to modify the tree to make it take morphisms $a' \to a''$ and produce morphisms ${a'}^F \to {a''}^F$, and this operation on morphisms must preserve identities and respect composition; something similar can be done to discharge the '$\mathbf{O}_a$'s of a tree that produces an '$a^F \to a^G$'; after the discharge we will obtain a natural transformation $a \stackrel{\bullet}{\to} (a^F \to a^G)$.

Terms for t.p.s, example:

$$\dfrac{\dfrac{\mathbf{O}_a \quad \eta}{a \to a^{LR}} \quad \dfrac{\dfrac{a \to b^R}{a^L \to b} \quad b \Rightarrow b^R}{a^{LR} \to b^R}}{a \to b^R} \qquad = \qquad a \to b^R$$

$$\mathrm{mm}(\mathrm{OTN}(\mathbf{O}_a, \eta_{LR}), \mathrm{mF}(\flat_{LR}(a \to b^R), (b \Rightarrow b^R))) \quad = \quad a \to b^R$$